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Numerical solution of hyperbolic equations by method of bicharacteristics

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ABSTRACT

Among the numerical procedures to solve a hyperbolic system of partial differential equation in 3-independent variables, method of bicharacteristics occupies an important position from the point of view of the accuracy of the solution. All bicharacteristic method developed so far employed the compatibility relations along, at the most, four bicharacteristics. We have presented in this paper a procedure for linear problems, taking as many bicharacteristics as poss ble and also have derived Butler's method as its particular case. Furthermore, stability criteria for these methods have been discussed. The present method is consistent and has second order accuracy at every time cycle and allows a timestep which is larger than that of Butler's method. A boundary method consistent with the present method has been derived. The present method has been illustrated by solving an initial-boundary value problem and a purely initial value problem, numerically and the results are compared with those of Butler's and Strang's schemes. Although Strang's scheme allows time step larger than those of bicharacteristic schemes, the bicharacteristic schemes are more accurate than Strang's scheme.

1. Introduction

The theory of characteristics for hyperbolic partial differential equations in two and more independent variables is well developed (Courant and Hilbert, 1962). The theory can also be used for the numerical solution of the equations. Massau gave the basic principles for such a numerical method for the case of 2 independent variables as early as 1900. Following the developments of the techniques by Busemann (1929) and Guderley (1940), the numerical method of solution following the compatibility conditions along the characteristic curves was extensively used in compressible flow problems (Meyer (1953), Shapiro (1954), Vol. I).

Unlike the case of two independent variables, where the compatibility condition along a characteristic curve contains only derivatives along the curve and from which finite-difference schemes for step-by-step integration can be easily written down, the compatibility conditions along a characteristic surface for three independent interior

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directions and hence do not lead quickly to a suitable finite-difference scheme. In the latter case however, one of the interior derivatives can be chosen in the direction of the bicharacteristic curves, which form an one parameter family of curves generating the characteristic surface. Attempts were made in 1947 (Coburn and Dolph (1949), Thornhill (1948)) to develop numerical schemes for the solution using the compatibility conditions along bicharacteristic curves. Out of all the methods developed so far, the one by Butler (1960) seems to be most natural, from the point of view of taking into account the boundary conditions (Cline and Hoffman (1972) and Zucrow and Hoffman (1977)), and also most accurate. Moreover, for a class of hyperbolic equations, Buter gives a theory for deriving a difference scheme using integration over the base curve (The curve of intersection of the characteristic cone and the initial space like surface where the Cauchy data is known). This way, he could, in principle, take into account the conditions of all the infinite bicharacteristic curves passing through the solution point i. e , the point where the solution is to be found (see also the method developed by Chu and his collaborators (1967, 1975, 1976)). However, in practical applications presented by him, he has used only four bicharacteristics to obtain the difference scheme.

The purpose of this paper is to develop a finite difference scheme taking all or as many bicharacteristic curves as possible, passing through the solution point by performing the integration over the characteristic conoid and to see whether inclusion of more bicharacteristics gives more accurate results We have shown that by employing a suitable bivariate interpolation formula for evaluating the values of the dependent variables on the base curve in the initial plane, it is possible to derive different difference schemes (the difference in the different schemes arises due to different number of bicharacteristic curves involved in the derivation of the schemes) in each of which the value at the solution point can be obtained in terms of the nine mesh points in the initial plane. Since nine points in the initial plane are also involved in Butler's method, the computational time involved in our method is approximately same as that in the Butler's method The numerical results show that our difference schemes are more accurate, because of relaxation of the stability requirement.

Our aim in this paper is to develop more accurate numerical schemes and compare our results with those obtained from the Butler's scheme and also with the exact solution. Hence we have taken only a first order system of three equations equivalent to the wave equation. It seems appropriate to comment here on the other finite-difference methods, such as those developed by Von Neumann and Richtmyer (1952), Lax and Wendroff (1960, 1964), and

Strang (1963) in which the derivatives are directly approximated by finite-differences. These methods are not so easily adoptable to the boundary value problems and they are far less accurate than bicharacteristic methods (Ravindran (1979)), also shown by our comparison with Strang's scheme.

2. Basic equations and integral (or integro-differential) equations formulation of the initial value problems

An initial value problem associated with a first order system of three equations equivalent to the wave equation in two-space dimensions is given by

$$\varphi_t - c(u_x + v_y) = 0 \tag{2.1}$$

$$u_t - c\varphi_x = 0 \tag{2.2}$$

$$v_t - c\varphi_y = 0 \tag{2.3}$$

with

$$\varphi(x, y, 0) = \varphi_0(x, y), \quad u(x, y, 0) = u_0(x, y), \quad v(x, y, 0) = v_0(x, y). \tag{2.4}$$

The parametric representation of the characteristic cone passing through a point (ξ, η, τ) in space-time is

$$\Gamma : x = \xi + c(\tau - t') \cos \theta, \; y = n + c(\tau - t') \sin \theta, \; t = t' \tag{2.5}$$

where t' and θ are the two parameters. For a constant value of θ, the equations (2.5), when t' varies, represent a straight line which is a bicharacteristic curve of the system (2.1) – (2.3). The bicharacteristic curves through P form a one parameter family and generate the characteristic cone (2.5).

Following the procedure given by Prasad and Ravindran (1980) (or multiplying (2.2) by $\cos \theta$ and (2.3) by $\sin \theta$ and adding them to (2.1), we can write the compatibility relation on the characteristic cone as

$$\frac{d\varphi}{d\sigma} + \cos \theta \frac{du}{d\sigma} + \sin \theta \frac{dv}{d\sigma} = S \tag{2.6}$$

where

$$S = c[u_x \sin^2\theta - (u_y + v_x) \sin\theta \cos\theta + v_y \cos^2\theta] \tag{2.7}$$

and $\frac{d}{d\sigma}$ denotes differentiation in the bicharacteristic direction and is given by

$$\frac{d}{d\sigma} = \frac{\partial}{\partial t} - \cos\theta - \frac{\partial}{\partial x} \sin\theta \; \frac{\partial}{\partial y} \tag{2.8}$$

The surface of the cone Γ is described by two independent parameters t' and θ. (see (2.5)). The relation (2.6) is valid on the cone, where all quantities

φ, u, v and S can be regarded as functions of t' and θ only. We also note that on Γ, the functions $u(t', \theta)$, $v(t', \theta)$, $\varphi(t', \theta)$ and their derivatives are all independent of when $t' = \tau$. To show that on Γ, S contains only interior derivatives of u and v, we transform the coordinate system from (x, y, t) to (θ, t', ψ) with the help of

$$x = \xi + [\tau = t') \tan \psi] \cos\theta,$$
$$y = \eta + [\tau - t') \tan \psi] \sin\theta, \; t' = t \tag{2.9}$$

and then set $\psi = \tan^{-1} c$. This gives

$$S = \frac{1}{\tau - t'} (-\sin\theta\, u_\theta + \cos\theta\, v_\theta). \tag{2.10}$$

Butler did not pay any attention to this form of S.

Multiplying (2.6) by unity, $\cos\theta$ and $\sin\theta$, separately and integrating with respect to the parameter $t'(=\sigma)$ we get the following set of integro-differencial equations

$$\varphi(P) = \frac{1}{2\pi} \int_0^{2\pi} [\varphi(Q) + u(Q) \cos\theta + v(Q) \sin\theta]\, d\theta$$
$$+ \frac{1}{2\pi} \int_{t=0}^{\tau} \int_0^{2\pi} S(t', \theta)\, d\theta\, dt' \tag{2.11}$$

$$u(P) = \frac{1}{\pi} \int_0^{2\pi} [\varphi(Q) \cos\theta + u(Q) \cos^2\theta + v(Q) \cos\theta \sin\theta]\, d\theta$$
$$+ \frac{1}{\pi} \int_{t'=0}^{t} \int_0^{2\pi} S(t', \theta) \cos\theta\, d\theta\, dt' \tag{2.12}$$

and

$$v(P) = \frac{1}{\pi} \int_0^{2\pi} [\varphi(Q) \sin\theta + u(Q) \cos\theta \sin\theta + v(Q) \sin^2\theta]\, d\theta$$
$$+ \frac{1}{\pi} \int_{t'=0}^{\tau} \int_0^{2\pi} S(t', \theta) \sin\theta\, d\theta\, dt' \tag{2.13}$$

The above system of integro-differential equations can be reduced to the following system of singular integral equations by making use of the expression (2.10) for S and performing integration by parts with respect to θ:

$$\varphi(P) = \frac{1}{2\pi}\int_0^{2\pi} [\varphi(Q) + u(Q)\cos\theta + v(Q)\sin\theta]\, d\theta$$

$$+ \frac{1}{2\pi}\int_{t'=0}^{\tau}\int_0^{2\pi} \frac{1}{\tau - t'}\,[u(t', \theta)\cos\theta + v(t', (Q)\sin\theta]\, d\theta\, dt' \qquad (2.14)$$

$$u(P) = \frac{1}{\pi}\int_0^{2\pi} [\varphi(Q)\cos\theta + u(Q)\cos^2\theta + v(Q)\cos\theta\sin\theta]\, d\theta$$

$$+ \frac{1}{\pi}\int_{t'=0}^{t}\int_0^{2\pi} \frac{1}{\tau - t'}\,[u(t', \theta)\cos\theta + v(t', \theta)\sin 2\theta]\, d\theta\, dt' \qquad (2.15)$$

and

$$v(P) = \frac{1}{\pi}\int_0^{2\pi} [\varphi(Q)\sin\theta + u(Q)\cos\theta\sin\theta + v(Q)\sin^2\theta]\, d\theta$$

$$+ \frac{1}{\pi}\int_{t'-0}^{\tau}\int_0^{2\pi} \frac{1}{\tau - t'}\,[u(t', \theta)\sin 2\theta - v(t', \theta)\cos 2\theta]\, d\theta\, dt' \qquad (2.16)$$

Care should be taken in the interpretation of the double integrals in (2.14) - (2.16) and also in the equations (2.11) – (2.13) when the expression (2.10) instead of (2.7) is used for S. All these singular integrals, with a singularity at $t' = \tau$, are interpreted as limits, as $\varepsilon \to 0$, of the corresponding integrals with the range of integration for t' given by $t' = 0$ to $t' = \tau - \varepsilon$.

Every solution of the initial value problem (2.1) – (2.4) is also a solution of the system of integro-differential equations (2.11) – (2.13) or its equivalent system of singular integral equations (2.14) – (2.16). However, unlike the case of two independent variables [see Courant and Hilbert (1962), 6 Chapter V] it is hard to prove the equivalence of the initial value problem (2.1) – (2.4) and the system (2.11) – (2.13) or (2.14) – (2.16). A proof of this equivalence is welcome. However, we shall, using integro-differential or integral equations, derive a finite-difference scheme to determine uniquely $\varphi(P)$, $u(P)$ and $v(P)$ in terms of the initial values at $t = 0$ for small τ. This will, therefore, give an approximate finite-difference solution of the initial value problem.

3. The Finite-Difference Schemes

First Method: Let us first take up the integro-differential equations (2.11) – (2.13), the last terms of which contain double integration with respect to the bicharacteristic variable t' and another interior variable θ. Following Butler, we take τ small and approximate the integration with respect to t' by the trapezoidal rule. For example, the equation (2.11) gives

$$\varphi(P) = \frac{1}{2\pi}\int_0^{2\pi} [\varphi(Q) + u(Q)\cos\theta + v(Q)\sin\theta]\, d\theta$$
$$+ \frac{\tau}{4\pi}\int_0^{2\pi} [S(P) + S(Q)]\, d\theta + o(\tau^3). \tag{3.1}$$

We note that the expression (2.10) does not define S at P since $t' = \tau$ at this point and hence it is only the expression (2.7) which is to be used for S in (3.1). When we do this note that the values of the u, v, φ and their derivatives at P do not depend on θ, we get

$$\varphi(P) = \frac{1}{2\pi}\int_0^{2\pi} [\varphi(Q) + u(Q)\cos\theta + v(Q)\sin\theta]\, d\theta$$
$$+ \frac{c\tau}{4}(u_x + v_y) + \frac{\tau}{4\pi}\int_0^{2\pi} S(Q)\, d\theta + o(\tau^3) \tag{3.2}$$

Similarly, we can use (2.12) and (2.13) to find expressions for $u(P)$ and $v(P)$. However, we note that the expressions for $\varphi(P)$ contains unknown derivations u_x and v_y at the solution point. Butler obtained a major success by eliminating these derivatives at P by integrating the equations along a time-like (of course non-bicharacteristic) curve. In this particular case, we integrate the equation (2.1) along the axis O' P of the cone (Fig. 1).

$$\varphi (P) = \varphi (0') = \frac{c\tau}{2} [u_x + v_y)_P + (u_x + v_y)_0'] + o (\tau^3). \tag{3.3}$$

Eliminating the derivatives at P from equations (3.2) and (3.3) we get

$$\varphi (P) = \frac{1}{\pi} \int_0^{2\pi} [\varphi (Q) + u (Q) \cos \theta + v (Q) \sin \theta] \, d\theta$$

$$+ \frac{\tau}{2\pi} \int_0^{2\pi} S(Q) \, d\theta - \varphi(0') - \frac{c\tau}{2} (u_x + v_y)_0' + o (\tau^3). \tag{3.4}$$

Now, we write the result which can be obtained from (2.12) and (2.13):

$$u (P) = \frac{1}{\pi} \int_0^{2\pi} [\varphi (Q) \cos \theta + u (Q) \cos^2\theta + v (Q) \sin \theta \cos \theta] \, d\theta$$

$$+ \frac{\tau}{2\pi} \int_0^{2\pi} S (Q) \cos \theta \, d\theta + o (\tau^3) \tag{3.5}$$

and

$$v (P) = \frac{1}{\pi} \int_0^{2\pi} [\varphi (Q) \sin \theta + u (Q) \cos \theta \sin \theta + v (Q) \sin^2\theta] \, d\theta$$

$$+ \frac{\tau}{2\pi} \int_0^{2\pi} S (Q) \sin \theta \, d\theta + o (\tau^3). \tag{3.6}$$

The equations (3.4) – (3.6) explicitly give the values of φ, u and v at a point $P (\xi, \eta, \tau)$ in terms of the initial values in the plane $t = 0$ and these formulae are correct upto $o(\tau^2)$, the error being $o(\tau^3)$ or less. These expressions can be deduced as a particular case of Butler's general theory given in the section 3 of this paper. We proceed now to derive from these equations a sequence of finite-difference schemes by replacing the integration with respect to by a suitable numerical quadrature formula and by using an appropriate bivariate interpolation formula to determine the values of φ, u, v and their derivatives at the points of the base curve in terms of the nine mesh points in the initial plane or in a parallel plane as shown in the Fig 2. For example, if we wish to replace the integration with respect to θ in (3.4) by Simpson's one-third quadrature formula by taking $N(=8)$ equal divisions of the interval $[0, 2\pi]$ by the points $\frac{2\pi r}{8}$ $(r = 0, 1, 2, \ldots, 7)$ and nothing that values of all quantities for $r = 0$ (i. e. $\theta = 0$) and $r = N$ (i. e. $\theta = 2\pi$) coincide, we get

$$\begin{aligned}\varphi(P) = \frac{1}{12}\Big[& 2\{Q_1) + \varphi(Q_2) + \varphi(Q_3) + \varphi(Q_4)\} + 4\{\varphi(Q_5) \\ &+ \varphi(Q_6) + \varphi(Q_7) + \varphi(Q_8)\} + 2\{u(Q_1) - u(Q_3)\} \\ &+ 4\left\{u(Q_5)\cos\frac{\pi}{4} + u(Q_6)\cos\frac{3\pi}{4} + u(Q_7)\cos\frac{5\pi}{4} + u(Q_8)\cos\frac{7\pi}{4}\right\} \\ &+ 2\{v(Q_2) - v(Q_4)\} + 4\left\{v(Q_5)\sin\frac{\pi}{4} + v(Q_6)\sin\frac{3\pi}{4}\right. \\ &\left.+ v(Q_7)\sin\frac{5\pi}{4} + v(Q_8)\sin\frac{7\pi}{4}\right\}\Big] \\ &+ \frac{\triangle t}{24}[2\{S(Q_1) + S(Q_2) + S(Q_3) + S(Q_4)\} \\ &+ 4\{S(Q_5) + S(Q_6) + S(Q_7) + S(Q_8)\}] - \varphi(0') \\ &- \frac{c\triangle t}{2}(u_x + v_y)_{0_1} + o((\triangle t)^3), \text{ with } N = 8. \end{aligned} \tag{3.7}$$

The formula (3.7) involves the values of the dependent variables and their derivatives at the non-grid points. We evaluate these by the use of the following bivariate interpolation formula involving only nine points in the in itia plane as shown in the Fig. 2

$$f(x, y) = \sum_{p=-1}^{1} \sum_{q=-1}^{1} \frac{\overline{P}_{i+p}(x)\,\overline{Q}_{j+q}(y)}{\overline{P}_{i+p}(x_{i+p})\,\overline{Q}_{j+q}(y_{j+q})} f(x_{i+p}, y_{j+q}) + o((\triangle x)^3) \tag{3.8}$$

where

$$\overline{P}_{i+p}(x) = \left\{\prod_{p=-1}^{1}(x - x_{i+p})\right\} \Big/ (x - x_{i+p}) \tag{3.9}$$

and

$$\overline{Q}_{j+q}(y) = \left\{ \prod_{q=-1}^{1} (y - y_{j+q}) \right\} \Big/ (y - y_{j+q}). \tag{3.10}$$

When we differentiate (3.8) to get the derivatives of $f(x, y)$, the result has an error $o(\triangle t)^2$. However, we shall notice that all the terms in (3.7) which contain derivatives are already multiplied by $\triangle t$. Therefore using (3.8) for the derivatives at all points and for the functions φ, u and v at the non-grid points, we get

$$\varphi^{n+1}{}_{i,j} = \sum_{p,q=-1}^{1} (a^{(1)}{}_{i+p,\, j+p}\, \varphi^n{}_{i+p,\, j+q} + b^{(1)}{}_{i+p,\, j+q}\, u^n{}_{i+p,\, j+q}$$
$$+ c^{(1)}{}_{i+p,\, j+q}\, v^n{}_{i+p,\, j+q}) + o((\triangle t)^3) \tag{3.11}$$

Similarly

$$u^{n+1}{}_{i,j} = \sum_{p,q=-1}^{1} (a^{(2)}{}_{i+p,\, j+q}\, \varphi^n{}_{i+p,\, j+q} + b^{(2)}{}_{i+p,\, j+q}\, u^n{}_{i+p,\, j+q}$$
$$+ c^{(2)}{}_{i+p,\, j+q}\, v^n{}_{i+p,\, j+q}) + o((\triangle t)^3) \tag{3.12}$$

and

$$v^{n+1}{}_{i,j} = \sum_{p,q=-1}^{1} (a^{(3)}{}_{i+p,\, j+q}\, \varphi^n{}_{i+p,\, j+q} + b^{(3)}{}_{i+p,\, j+q}\, u^n{}_{i+p,\, j+q}$$
$$+ c^{(3)}{}_{i+p,\, j+q}\, v^n{}_{i+p,\, j+q}) + o((\triangle t)^3). \tag{3.13}$$

where the coefficients in the case of $N = 8$ for the first method are given in Table 1.

The method described above for $N = 8$ is applicable to any even value of N. We call the differences schemes obtained from the integro-differential equations in this manner as Method 1. Table 1 gives the values of the coefficients corresponding to different values of N in terms of a parameter $R = \frac{c\triangle t}{\triangle x}$ introduced in the next section. It is also simple to prove that as $N \to \infty$ all the 81 coefficients tend to definite limits (see Appendix A).

We note that irrespective of number of bicharacteristics used, the error is of $o((\triangle t)^3)$. In what way does the number of bicharacteristics involved in the scheme affect the actual computation, is investigated.

A simple derivation of Butler's scheme: Butler's scheme can also be obtain- by the above method by approximating the integration with respect to θ by trapezoidal rule, taking the values on the base curve at $\theta = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}$ and 2π, which coincide with the grid points Q_1, Q_2, Q_3, Q_4 and Q_1 respectively. Equation (3.4) with the expression (2.7) for S, gives

$$\begin{aligned}\varphi(P) = & -\varphi(0') + 1/2\{\varphi(Q_1) + \varphi(Q_2) + \varphi(Q_3) + \varphi(Q_4)\} \\ & + 1/2\{u(Q_1) - u(Q_3) + v(Q_2) - v(Q_4)\} + c\triangle t/4 \left\{\left(\frac{\partial v}{\partial y}\right)_{Q_1}\right. \\ & \left. + \left(\frac{\partial u}{\partial x}\right)_{Q_2} + \left(\frac{\partial v}{\partial y}\right)_{Q_3} + \left(\frac{\partial u}{\partial x}\right)_{Q_4}\right\} - c\triangle t/2\,(u_x + v_y)_{0'} \\ & + o((\triangle t)^3) \end{aligned} \tag{3.14}$$

Similarly we can use (3.5) and (3.6) for $u(P)$ and $v(P)$ respectively. We note that the values of φ, u and v and their derivatives appearing here are only those at the grid points and are such that they can be approximated by a central difference formula with the help of the nine points shown in the Fig. 2. We get the difference scheme same as (3.11) – (3.13), with the values of the coefficients given in Table 1. One of the main points to be noted in Butler's scheme is that the term $(u_y + v_x)\sin\theta\cos\theta$ in the expression (2.7) for (2.7) for S does not contributed to the scheme.

However, Butler's scheme gives fairly good result since the four bicharacteristic relations used by Butler in his derivation are equivalent to the original equations (2.1) – (2.3).

Second Method : Our second set of difference schemes are derived from the singular integral equations (2.14) - (2.16). In the derivation of these, there is little more complication due to the presence of the singularity at $t' = \tau$. However, we get exactly the same scheme by starting from the equations (3.4) - (3.6) with the expression (2.10) for S and then by getting rid of the derivatives of u and v with respect to θ by integration by parts. This gives

$$\begin{aligned}\varphi(P) = & -\varphi(0') + \frac{1}{\pi}\int_0^{2\pi}\varphi(Q)\,d\theta + \frac{3}{2}\int_0^{2\pi}(u\cos\theta + v\sin\theta)\,d\theta \\ & - \frac{c\triangle t}{2}(u_x + v_y)_{0'} + o((\triangle t)^3)\end{aligned} \tag{3.15}$$

and similar expressions for $u(P)$ and $v(P)$. The only derivatives which appear here are those at $0'$, which can be immediately replaced by central finite-differences. The bivariate interpolation formula (3.8) is to be used for evaluating the values of φ, u and v at non-grid points. Therefore, this method (call it Method 2) also gives a finite-difference scheme of the form (3.11) - (3.13) with values of a's, b's and c's depending on N, the number of equal sub-intervals of $(0, 2\pi)$ used for numerical quadrature with respect to θ.

We note that in this case also as $N \to \infty$ all the coefficients tend to constant values and these limiting values are the same as the corresponding limiting values as $N \to \infty$ in the first method. Because of this reason, we have concentrated only one Method 1 in the rest of this paper.

4. Consistency of the difference scheme

We write the equations (2.1) - (2.3) in the form $U_t + B_1 U_x + B_2 U_y = 0$, where $U = (\varphi, u, v)'$ and B_1 and B_2 are 3×3 matrics, and the difference scheme in the form

$$U(x, y, t + \triangle t) = \sum_{i, j = -1}^{1} C_{ij}\, U(x+i \triangle x, y+j \triangle y, t) \tag{4.1}$$

where C_{ij} are 3×3 matrices independent of $\triangle t$. We assume that $\dfrac{\triangle t}{\triangle x}$ and $\dfrac{\triangle t}{\triangle y}$ are constants such that $\triangle x = \triangle y = \triangle t / R$.

In order that, in the limit as $t \to 0$, the solution of difference equation satisfy the differential equation, it is sufficient for us to show that the scheme is consistent and stable (Lax and Richtmyer (1956)). The stability will be discussed in the next section. Here we shall consider the consistency of the scheme, which means that the solution of the differential equation must be approximate solution of the difference equation with an error $o(\triangle t)^3$.

We expand both sides of (4.2) into Taylor series, one with respect to $\triangle t$ and the other with respect to $\triangle x$ and $\triangle y$ and get

$$U + \triangle t\, U_t + \frac{(\triangle t)^2}{2} U_{tt} + o(\triangle t)^3$$

$$= C_{ij} \Big[U + i \triangle x\, U_x + j \triangle y\, U_y + \frac{(i \triangle x)^2}{2} U_{xx}$$

$$+ ij \triangle x \triangle y\, U_{xy} + \frac{(j \triangle y)^2}{2} U_{yy} \Big] + o(\triangle t)^3$$

Substituting for U_t from the differential equation (4.2) and equating the coefficients of U, U_x, U_y, U_{xx}, U_{xy} and U_{yy}, we obtain the consistency conditions:

$$\sum C_{ij} = I, \ \sum iC_{ij} = -B_1 R, \ \sum jC_{ij} = -B_2 R$$

$$\sum i^2 C_{ij} = B_1 R^2, \ \sum ij = \frac{B_1 B_2 + B_2 B_1}{2} R^2,$$

$$\sum j^2 C_{ij} = B_2 R^2$$

where $\sum$ indicates summation over both i and j.

It is easy to verify that these relations are satisfied for all the three schemes indicating that all the three are consistent second order schemes.

5. Stability

To study the stability of the difference schemes, the dependent variables are written as a double Further series and its each component is examined as to its growth with time. i. e.,

$$U\ (j\triangle x,\ k\triangle y,\ n\triangle t) = U^n{}_0 \exp\{ i(\alpha j \triangle x + \beta k \triangle y)\} \tag{5.1}$$

and the same quantity at an advanced level $(n+1)\,t$,

$$U(j\triangle x, k\triangle y, (n+1)\triangle t) = U_0{}^{n+1} \exp\{ i(\alpha j \triangle x + \beta k \triangle y)\} \tag{5.2}$$

where $U = (\varphi, u, v)'$ and α and β are the wave numbers associated with x and y coordinates.

Substituting the values in a difference scheme, we get

$$U^{n+1} = A(\alpha\triangle x, \beta\triangle y, \triangle t)\, U_0{}^n \tag{5.3}$$

where A is amplification matrix of the scheme.

The amplification matrices corresponding to different schemes discussed earlier are given below. Here $R = \dfrac{c\triangle t}{\triangle x}$ $\triangle x = \triangle y$, $\xi = \alpha\triangle x$, $\eta = \beta\triangle y$.

Butler's scheme

$$A = \left\| \begin{array}{ccc} R^2(\cos\xi - 1) & iR\sin\xi\, X & iR\sin\eta\, X \\ R^2(\cos\eta - 1) + 1 & \left[\frac{R^2}{2}(\cos\eta - 1 + 1\right] & \left[\frac{R^2}{2}(\cos\xi - 1) + 1\right] \\ iR\sin\xi & [R^2(\cos\xi - 1) + 1 & -\frac{R^2}{2}\sin\xi\sin\eta \\ iR\sin\eta & -\frac{R^2}{2}\sin\xi\sin\eta & R^2(\cos\eta - 1) + 1 \end{array} \right\|$$

Method 1 when $N = 8$:

$$A \;=\; (A_{ij},\ i = 1, 2, 3;\ \ j = 1, 2, 3)$$

$$A_{11} = \frac{R^4}{3}(\cos\xi - 1)(\cos\eta - 1) + R^2(\cos\xi - 1) + R^2(\cos\eta - 1) + 1$$

$$A_{12} = iR\sin\xi\left[\frac{R^2(2\cos\eta - 3)}{6} + 1\right]$$

$$A_{13} = iR\sin\eta\left[\frac{R^2(2\cos\xi - 3)}{6} + 1\right]$$

$$A_{21} = iR\sin\xi\left[\frac{R^2}{3}(\cos\eta - 1) + 1\right]$$

$$A_{22} = \frac{R_4}{6}(\cos\xi - 1)(\cos\eta - 1) + R^2(\cos\xi - 1) + 1$$

$$A_{23} = -\frac{R^2}{2}\sin\xi\sin\eta$$

$$A_{31} = iR\sin\eta\left[\frac{R^2}{6}(\cos\xi - 1) + 1\right]$$

$$A_{32} = -\frac{R^2}{2}\sin\xi\sin\eta$$

$$A_{33} = \frac{R^4}{6}(\cos\xi - 1)(\cos\eta - 1) + R^2(\cos\eta - 1) + 1$$

Method 1 when $N = 16$:

$$A_{11} = \frac{R^4}{4}(\cos\xi - 1)(\cos\eta - 1)\ R^2(\cos\xi - 1) + R^2(\cos\eta - 1) + 1$$

$$A_{12} = iR\sin\xi\left[\frac{3R^2}{8}(\cos\eta - 1) + 1\right]$$

$$A_{13} = iR\sin\eta\left[\frac{3R^2}{8}(\cos\xi - 1) + 1\right]$$

$$A_{21} = iR\sin\xi\left[\frac{R^2}{4}(\cos\eta - 1) + 1\right]$$

$$A_{22} = \frac{R^4}{8}(\cos\xi - 1)(\cos\eta - 1) + R^2(\cos\xi - 1) + 1$$

$$A_{23} = -\frac{R^2}{2}\sin\xi\sin\eta$$

$$A_{31} = iR\sin\eta\left[\frac{R^2}{4}(\cos\xi - 1) + 1\right]$$

$$A_{32} = -\frac{R^2}{2}\sin\xi\,\sin\eta$$

$$A_{33} = \frac{R^4}{8}(\cos\xi - 1)(\cos\eta - 1) + R^2(\cos\eta - 1) + 1$$

Although it is very difficult to obtain the eigen-values of these matrics in their general form, the consideration of various particular choices of ξ and η revealed that the matrices possess the largest eigen-values in the absolute value, when $\xi = \eta = \pi$.

For this choice, the eigenvalues of the amplification matrix for Butler's scheme are

$$1 - 2R^2,\ 1 - 2R^2,\ 1 - 4R^2 \tag{5.4}$$

Therefore the stability criterion is

$$R \leqslant 1/\surd 2 = 0.7071068 \tag{5.5}$$

For the same choice of ξ and η, the eigenvalues of the amplification matrix for the scheme in Method 1 when $N = 8$ are

$$4/3\,R^4 - 4R^2 + 1, \quad 2/3\,R^4 - 2R^2 + 1, \quad 2/3\,R^4 - 2R^2 + 1 \tag{5.6}$$

The graphs of the eigenvalues for the three schemes have been shown in Fig. 3, 4 and 5 respectively and it follows that the stability criterion in the case, is

$$-4/3\,R^4 + 4R^2 - 1 \leqslant 1.$$

This is satisfied if $R^2 \geqslant 3/2(1 + 1/\surd 3)$ or $R^2 \leqslant 3/2(1 - 1/\surd 3)$. But CFL condition restricts that $R \leqslant 1$. Therefore the stability criterion is

$$R \leqslant \surd 3/2\,(1 - 1/\surd 3) = 0.7962252 \tag{5.7}$$

Again for $\xi = \eta = \pi$, the eigenvalues of the amplification matrix for the scheme in Method 1 when $N = 16$, are

$$R^4 - 4R^2 + 1,\ R^4/2 - 2R^2 + 1,\ R^4/2 - 2R^2 + 1 \tag{5.8}$$

By the same argument as above, the stability criterion is

$$R \leqslant \surd(2 - \surd 2) \simeq 0.7653669 \tag{5.9}$$

We observe from (5.5), (5.7) and (5.9) that, using the schemes in Method 1, we can march faster along t-axis than when Butler's scheme is used, without facing instabilities. We call the values of R in (5.5), (5.7) and (5.9) as the critical values of R for corresponding schemes.

6. Numerical Results and Discussion

The first problem that we consider is the *initial boundary value problem* for the system of equations (2.1) – (2.3) in the domain $0 \leqslant x, y \leqslant 1, t \geqslant 0$, with initial conditions given by

$$\varphi(x, y, 0) = 0, \quad u(x, y, 0) = \pi \cos \pi x \sin \pi y$$

$$v(x, y, 0) = \pi \sin \pi x \cos \pi y, \quad 0 \leqslant x, y \leqslant 1$$

and the boundary conditions given by

$$\varphi = 0, v = 0 \text{ on } x = 0 \text{ and } x = 1, 0 \leqslant y \leqslant 1, t \geqslant 0$$

and $\varphi = 0, u = 0$ on $y = 0$ and $y = 1, 0 \leqslant x \leqslant 1, t \geqslant 0$

On each of the boundaries only one dependent variable is not known. We first present a method of computing this variable.

Solution at boundary points

Butler's scheme as a boundary method has been discussed and compared with several boundary methods by Bramley and Sloan (1977). We present here a method for computing the values of the dependent variables on the boundaries suitable for Method 1.

The unknown dependent variable is calculated using the finite-difference

$$\varphi(P) - \varphi(Q) + \cos [u(P) - u(Q)] + \sin[v(P) - v(Q)]$$
$$= \frac{c \triangle t}{2} [S(P) + S(Q)] \tag{6.1}$$

of the compatibility relation along a bicharacteristic. This finite-difference form is integrated with respect to over an appropriate interval. Along the boundary $y = 0$, i. e. x-axis, the boundry conditions are $\varphi(x, 0, t) = 0 = u(x, 0, t)$ for $0 \leqslant x \leqslant 1, t \geqslant 0$. In this case the appropriate intervel for θ is $[0, \pi]$. Integrating (6.1) with respect to θ, from 0 to π, we get.

$$2v(P) = \int_0^{\pi} \left[\varphi(Q) + \cos \theta\, u(Q) + \sin \theta\, v(Q) + \frac{c \triangle t}{2} S(Q) \right] d\theta$$
$$- \pi/2\, \varphi(0')\, \pi\, c \triangle t/4\, (u_x + v_y)_{0}' \tag{6.2}$$

Note that we have used the relation (3.3) in eliminating the derivatives of the dependent variables at P.

Proceeding in a similar way, we get the following results on other boundaries :

on $x = 1$:

$$- 2v(P) = \int_{\pi}^{2\pi} [\varphi(Q) + \cos\theta\, u(Q) + \sin\theta\, v(Q) + c(\Delta t/2)\, S(Q)]\, d\theta$$
$$- \pi/2\, \varphi(0') - \pi c \Delta t/4\, (u_x + v_y)_{0'} \tag{6.3}$$

on $y = 0$:

$$2u(P) = \int_{-\pi/2}^{\pi/2} [\varphi(Q) + \cos\theta\, u(Q) + \sin\theta\, v(Q) + c(\Delta t/2)\, S(Q)]\, d\theta$$
$$- \pi/2\, \varphi(0') - c\Delta t/4\, (u_x + vy)_{0'} \tag{6.4}$$

on $y = 1$:

$$- 2u(P) = \int_{\pi/2}^{3\pi/2} [\varphi(Q) + \cos\theta\, u(Q) + \sin\theta\, v(Q) + c(\Delta t/2)\, S(Q)]\, d\theta$$
$$- \pi/2\, \varphi(0') - c\, \Delta t/4\, (u_x + vy)_{0'} \tag{6.5}$$

The properties at the points Q are interpolated using the bivariate interpolation formula (3.8) mentioned earlier. The only difference is that the nine points used here are not centered around the projection of the solution point in the initial plane, but they have been taken on one side of 0' in order to have a second order scheme.

The numerical solution is computed using the different schemes, taking 21×21 mesh points in the region $0 \leqslant x, y \leqslant 1$ and the maximum absolute error in the computed values are given in Table 2, using the exact solution, of problem, viz.,

$$\begin{aligned} \varphi(x, y, t) &= \sqrt{2}c\pi \sin \pi x \sin \pi y \sin(\sqrt{2}\pi c t) \\ u(x, y, t) &= \pi \cos \pi x \sin \pi y \cos(\sqrt{2}\pi c t) \\ v(x, y, t) &= \pi \sin \pi x \cos \pi y \cos(\sqrt{2}\pi c t) \end{aligned} \tag{6.6}$$

The value of c was chosen to be 1 in all the computations.

It was pointed out by Gourlay and Morris (1968) that the multistep formulation of Strang's scheme is superior to two step Lax-Wendroff scheme, Therefore, by way of comparison with the above three schemes, we also solved the problem using Strang's scheme, which makes use of one-dimensional two step Lax-Wendroff schemes. Multistep Strang's scheme for the problem $U_t + (F(U))_x + (G(U))_y = 0$ is given by

$$V^{n+1}{}_{(1)} = \mu_y U^n - (P/2)\delta_y G^n, W^{n+1}{}_{(1)} = \mu_x U^n - (P/2)\delta_x F^n$$

$$V^{n+1}{}_{(2)} = U^n - P\,\delta_y G^{n+1}{}_{(1)}, \quad W^{n+1}{}_{(2)} = U^n - P\,\delta_x F^{n+1}{}_{(1)}$$

$$V^{n+1}{}_{(3)} = \mu_x V^{n+1}{}_{(2)} - (P/2)\,\delta_x F^{n+1}, \quad W^{n+1}{}_{(3)} = \mu_y W^{n+1}{}_{(2)} - (P/2)\delta_y G^{n+1}{}_{(2)}$$

$$V^{n+1}{}_{(4)} = V^{n+1}{}_{(2)} - P\,\delta_x F^{n+1}{}_{(3)}, \quad W^{n+1}{}_{(4)} = W^{n+1}{}_{(2)} - P\,\delta_y G^{n+1}$$

$$U^{n+1} = (V^{n+1}{}_{(4)} + W^{n+1}{}_{(4)})/2$$

where $U = (U_1, U_2, U_3)$, $F^n = F(U^n)$, $F^{n+1}{}_{(1)} = F(U^{n+1}{}_{(1)})$, etc.

We note that we have Butler's boundary method for the Strang's scheme here.

Table 2.1 gives the max mum errors in the computed values for $R = 0\,6$ which is significantly less than the critical value of R required for all scheme. The computations were carried out upto 300 time steps. Both Butler's scheme as well as Method 1 give better results than those of Strang's scheme.

Table 2.2 gives the maximum absolute errors for $R = 0.8$. This value of R is greater than the critical value of Butler's scheme for which the error becomes very large for $n > 70$. Even for the scheme in Method 1 when $N = 16$, this value of R is greater than the critical value. However, error grows slowly and becomes very large only when $n > 175$. For the scheme in Method 1 when $N = 8$, the value of R, namely 0.8, is slightly greater than the critical value. The computations we carried out upto 400 time steps and no instability was observed For Strang's scheme, this value of R is less than he critical value. But at each step the maximum absolute error is greater than that in Method 1 when $N = 8$.

We have also done some computations with a mixed difference scheme in which the values at the interior points were calculated by our Method 1 ($N = 8$ and $N = 16$) and those at the boundaries, by Butler's boundary method. The numerical results obtained by this are not as good as reported in Tables 2.1 and 2 2, showing that Butler's boundary method is not consistent with our Method 1 for interior points.

The second problem that we consider is a *purely initial value problem*, namely, the linear propagation of initial pressure distribution in a medium at rest. Equations governing the motion are

$$\varphi_t + c(u_x + v_y) = 0$$
$$u_t + c\,\varphi_x = 0, \quad -\infty < x, y < \infty, \quad t \geqslant 0$$
$$v_t + c\,\varphi_y = 0$$

with initial conditions

$$\varphi(x, y, 0) = 1 - x^2 - y^2, \text{ if } x^2 + y^2 \leqslant 1$$
$$= 0 \qquad \text{otherwise}$$
$$u(x, y, 0) = v(x, y, 0) = 0 \quad -\infty < x, y < \infty$$

The exact integral solution of the initial value problem of the linear wave equation in two-dimensions can be obtained by the *Hadamard* method of descent. The solution of the initial value problem :

$$\varphi_{tt} = c^2(\varphi_{xx} + \varphi_{yy}) \quad -\infty < x, y < \infty \; t \geqslant 0$$
$$\varphi(x, y, 0) = \psi(x, y) \text{ and } \varphi_t(x, y, 0) = \rho(x, y)$$

is

$$\varphi(x, y, t) = \frac{\partial}{\partial t}\left\{\frac{1}{2\pi c}\iint\limits_{r \leqslant ct} \frac{\psi(\xi, \eta)\, d\xi\, d\eta}{\sqrt{(c^2 t^2 - r^2)}} + \frac{1}{2\pi c}\iint\limits_{r \leqslant ct} \frac{\rho(\xi, \eta)\, d\xi\, d\eta}{\sqrt{(c^2 t^2 - r^2)}}\right\}$$

where $r^2 = (x-\xi)^2 + (y-\eta)^2$. These integrals are calculated for the given initial pressure distribution using *Chebyshev* and *Gauss* formulae.

The maximum absolute errors in the numerical computation, using different schemes are given in Table 3. Butler's scheme is unstable for $R = 0.8$ and gives less accurate results even for small time steps. Both the schemes in Method 1 give better results than those of Strang's scheme. When $R = 0.6$ even Butler's gives better results than Strang's scheme.

We took this initial value problem with a view to compare our results with those of Ravindran (1979) who has used a first order bicharacteristic method to solve this problem. We find that here the scheme is stable for $R \leqslant 1$. However, when we plot our results in Fig. 5 in that paper for $t = 1.6$, we find that the second order accurate Method 1 when $N = 8$ gives much less error than the first order bicharacteristic schemes and can hardly be distinguished on the graph from the exact integral solution.

From the results of computations and stability analysis, we conclude that Method 1 when $N = 8$ shows some improvement from the point of view of accuracy and stability over the Butler's scheme using only 4 bicharacteristic. The computation times are approximately same for the four schemes, at least in the case of linear problems. But the bicharacteristic schemes are definitely more accurate than Strang's scheme.

TABLE 1

The following are the sets of 9 matrices which give the coefficients in different schemes for arbitrary $R = \dfrac{c\Delta t}{\Delta x}$

$$A_k = [a^{(k)}{}_{i+p,\ j+q}], \qquad B_k = [b^{(k)}{}_{i+p,\ j+q}], \qquad C_k = [c^{(k)}{}_{i+p,\ j+q}]$$

Coefficients in Butler's Scheme

$$A_1 = \begin{Bmatrix} 0 & R^2/2 & 0 \\ R^2/2 & 1-2R^2 & R^2/2 \\ 0 & R^2/2 & 0 \end{Bmatrix} \qquad B_1 = \begin{Bmatrix} -R^3/8 & 0 & R^3/8 \\ -R(2-R^2/4 & 0 & R(2-R^2)/4 \\ -R^3/8 & 0 & R^3/8 \end{Bmatrix}$$

$$C_1 = \begin{Bmatrix} R^3/8 & R(2-R^2)/4 & R^3/8 \\ 0 & 0 & 0 \\ -R^3/8 & -R(2-R^2)/4 & R^3/8 \end{Bmatrix} \qquad A_2 = \begin{Bmatrix} 0 & 0 & 0 \\ -R/2 & 0 & R/2 \\ 0 & 0 & 0 \end{Bmatrix}$$

$$B_2 = \begin{Bmatrix} 0 & 0 & 0 \\ R^2/2 & 1-R^2 & R^2/2 \\ 0 & 0 & 0 \end{Bmatrix} \qquad C_2 = \begin{Bmatrix} -R^2/8 & 0 & R^2/8 \\ 0 & 0 & 0 \\ R^2/8 & 0 & -R^2/8 \end{Bmatrix}$$

$$A_3 = \begin{Bmatrix} 0 & R/2 & 0 \\ 0 & 0 & 0 \\ 0 & -R/2 & 0 \end{Bmatrix} \qquad B_2 = \begin{Bmatrix} R^3/8 & 0 & R^2/8 \\ 0 & 0 & 0 \\ -R^2/8 & 0 & -R^2/8 \end{Bmatrix}$$

$$C_3 = \begin{Bmatrix} 0 & R^2/2 & 0 \\ 0 & 1-R^2 & 0 \\ 0 & R^2/2 & 0 \end{Bmatrix}$$

Coefficients in Method 1 *when* $N = 8$

$$A_1 = \left\{\begin{matrix} \frac{R^4}{12} & \frac{R^2(3-R^2)}{6} & \frac{R^4}{12} \\[2ex] \frac{R^2(3-R^2)}{6} & \frac{R^4}{3} - 2R^2+1 & \frac{R^2(3-R^2)}{6} \\[2ex] \frac{R^4}{12} & \frac{R^2(3-R^2)}{6} & \frac{R^4}{12} \end{matrix}\right\} \quad B_1 = \left\{\begin{matrix} \frac{-R^3}{12} & 0 & \frac{R^3}{12} \\[2ex] \frac{-R(2-R^2)}{4} & 0 & \frac{R(2-R^2)}{4} \\[2ex] \frac{-R^3}{12} & 0 & \frac{R^3}{12} \end{matrix}\right\}$$

$$C_1 = \left\{\begin{matrix} \frac{R^3}{12} & \frac{R(2-R^2)}{4} & \frac{R^3}{12} \\[2ex] 0 & 0 & 0 \\[2ex] \frac{-R^3}{12} & \frac{-R(2-R^2)}{4} & \frac{-R^3}{12} \end{matrix}\right\} \quad A_2 = \left\{\begin{matrix} \frac{-R^3}{12} & 0 & \frac{R^3}{12} \\[2ex] \frac{-R(3-R^2)}{6} & 0 & \frac{R(3-R^2)}{6} \\[2ex] \frac{-R^3}{12} & 0 & \frac{R^3}{12} \end{matrix}\right\}$$

$$B_2 = \left\{\begin{matrix} \frac{R^4}{24} & \frac{-R^4}{24} & \frac{R^4}{24} \\[2ex] \frac{R^2(6-R^2)}{12} & \frac{R^4}{6} - R^2+1 & \frac{R^2(6-R^2)}{12} \\[2ex] \frac{R^4}{24} & \frac{-R^4}{24} & \frac{R^4}{24} \end{matrix}\right\} \quad C_2 = \left\{\begin{matrix} \frac{-R^2}{8} & 0 & \frac{R^3}{12} \\[2ex] 0 & 0 & 0 \\[2ex] \frac{R^2}{8} & 0 & \frac{-R^2}{8} \end{matrix}\right\}$$

$$A_3 = \left\{\begin{matrix} \frac{R^3}{12} & \frac{R(3-R^2)}{6} & \frac{R^3}{12} \\[2ex] 0 & 0 & 0 \\[2ex] \frac{-R^3}{12} & \frac{-R(3-R^2)}{6} & \frac{-R^3}{12} \end{matrix}\right\} \quad B_3 = \left\{\begin{matrix} \frac{-R^2}{8} & 0 & \frac{R^2}{8} \\[2ex] 0 & 0 & 0 \\[2ex] \frac{R^2}{8} & 0 & \frac{-R^2}{8} \end{matrix}\right\}$$

$$C_3 = \left\{\begin{matrix} \frac{R^4}{24} & \frac{R^2(6-R^2)}{12} & \frac{R^4}{24} \\[2ex] \frac{-R^4}{12} & \frac{R^4}{6} - R^2+1 & \frac{-R}{12} \\[2ex] \frac{R^4}{24} & \frac{R^2(6-R^2)}{12} & \frac{R^4}{24} \end{matrix}\right\}$$

Coefficients in Method 1 when $N = 16$:

$$A_1 = \begin{bmatrix} \dfrac{R^4}{16} & \dfrac{R^2(4-R^2)}{8} & \dfrac{R^4}{16} \\[2ex] \dfrac{R^2(4-R^2)}{8} & \dfrac{R^4}{4} - 2R^2 + 1 & \dfrac{R^2(4-R^2)}{8} \\[2ex] \dfrac{R^4}{16} & \dfrac{R^2(4-R^2)}{8} & \dfrac{R^4}{16} \end{bmatrix} \qquad B_1 = \begin{bmatrix} \dfrac{-3R^3}{32} & 0 & \dfrac{3R^3}{32} \\[2ex] \dfrac{-R(8-3R^2)}{16} & 0 & \dfrac{R(8-3R^2)}{16} \\[2ex] \dfrac{-3R^3}{32} & 0 & \dfrac{3R^3}{32} \end{bmatrix}$$

$$C_1 = \begin{bmatrix} \dfrac{3R^3}{32} & \dfrac{R(8-3R^2)}{16} & \dfrac{3R^3}{32} \\[2ex] 0 & 0 & 0 \\[2ex] \dfrac{-3R^3}{32} & \dfrac{-R(8-3R^2)}{16} & \dfrac{-3R^3}{32} \end{bmatrix} \qquad A_2 = \begin{bmatrix} \dfrac{-R^3}{16} & 0 & \dfrac{R^3}{16} \\[2ex] \dfrac{-R(4-R^2)}{8} & 0 & \dfrac{R(4-R^2)}{8} \\[2ex] \dfrac{-R^3}{16} & 0 & \dfrac{R^3}{16} \end{bmatrix}$$

$$B_2 = \begin{bmatrix} \dfrac{R^4}{32} & \dfrac{-R^4}{16} & \dfrac{R^2}{32} \\[2ex] \dfrac{R^2(8-R^2)}{16} & \dfrac{R^4}{8} R^2 + 1 & \dfrac{R^2(8-R^2)}{16} \\[2ex] \dfrac{R^4}{32} & \dfrac{-R^4}{16} & \dfrac{R^4}{32} \end{bmatrix} \qquad C_2 = \begin{bmatrix} \dfrac{-R^2}{8} & 0 & \dfrac{R^2}{8} \\[2ex] 0 & 0 & 0 \\[2ex] \dfrac{R^2}{8} & 0 & \dfrac{-R^2}{8} \end{bmatrix}$$

$$A_3 = \begin{bmatrix} \dfrac{R^2}{16} & \dfrac{R(4-R^2)}{8} & \dfrac{R^3}{16} \\[2ex] 0 & 0 & 0 \\[2ex] \dfrac{-R^3}{16} & -\dfrac{R(4-R^2)}{1} & -\dfrac{R^3}{16} \end{bmatrix} \qquad B_3 = \begin{bmatrix} \dfrac{-R^2}{8} & 0 & \dfrac{R^2}{8} \\[2ex] 0 & 0 & 0 \\[2ex] \dfrac{R^2}{8} & 0 & \dfrac{-R^2}{8} \end{bmatrix}$$

$$C_3 = \begin{bmatrix} \dfrac{R^4}{32} & \dfrac{R^2(8-R^2)}{16} & \dfrac{R^4}{32} \\[2ex] \dfrac{-R^4}{16} & \dfrac{R^4}{8} R^2 + 1 & \dfrac{-R^4}{16} \\[2ex] \dfrac{R^4}{32} & \dfrac{R^2(8-R^2)}{16} & \dfrac{R^4}{32} \end{bmatrix}$$

TABLE 2

Maximum absolute error in computed values

Table 2.1 : $R = 0.6$

Time	Butler's Scheme		Method 1 when $N=8$		Method 1 when $N=16$		Strang's Scheme	
step n	Interior	Boundary	Interior	Boundary	Interior	Boundary	Interior	Boundary
10	0.00974	0.00464	0.04904	0.00965	0.00821	0.00708	0.01038	0.00851
50	0.06169	0.00641	0.03146	0.03399	0.06799	0.00841	0.08922	0.00960
100	0.10800	0.04375	0.03318	0.03564	0.11770	0.05059	0.15743	0.06481
200	0.16588	0.16650	0.20360	0.20540	0.18790	0.18680	0.24630	0.24737
300	0.23170	0.23340	0.27190	0.23920	0.24751	0.25167	0.34777	0.35030

Table 2.2 : $R = 0.8$

Time	Butler's Scheme		Method 1 when $N=8$		Method 1 when $N=16$		Strang's Scheme	
step n	Interior	Boundary	Interior	Boundary	Interior	Boundary	Interior	Boundary
10	0.00902	0.00307	0.00881	0.00625	0.00616	0.00532	0.01207	0.01009
50	0.01407	0.01378	0.05100	0.01375	0.03112	0.02169	0.06736	0.04833
100	–	–	0.06455	0.06295	0.06259	0.06200	0.13525	0.13507
200	–	–	0.13320	0.13670	19.50000	3.92400	0.30693	0.17617
300	–	–	0.32540	0.00274	–	–	0.54299	0.16887
400	–	–	0.27630	0.27330	–	–	0.55095	0.55581

TABLE 3

Maximum absolute errors in the computed values

Table 3.1 : R = 0.8

Time step n	Butler's scheme	Method 1 when $N = 8$	Method 1 when $N = 16$	Strang's Scheme
10	0.008998	0.007500	0.007505	0.007317
20	0.030200	0.012950	0.012707	0.020615
30	0.052859	0.037250	0.036450	0.055366
40	0.353449	0.006910	0.006670	0.013650
50	1.676160	0.006980	0.006931	0.018218
60	18.924250	0.005977	0.005901	0.012606
70	189.890000	0.004520	0.004385	0.015040

TABLE 3.2 : $R = 0\ 6$

Time step n	Butler's scheme	Method 1 when $N = 8$	Method 1 when $N = 16$	Strang's Scheme
10	0.00856	0.00849	0.00851	0.01050
20	0.01582	0.01623	0.01615	0.01735
30	0.05071	0.05256	0.05211	0.05566
40	0 07165	0 07595	0.07495	0.08375
50	0.01873	0.01973	0.01949	0.03487
60	0.00914	0.00853	0.00869	0.02090
70	0.00833	0.00798	0.00807	0.01953

Appendix — A

In the equations (3 4) - (3.6), if we replace integral by the Simpson's one-third rule taking N points on the base curve, and use the bivariate interpolation formula (3.8) to get the values of the dependent variables and their derivatives at the non-grid points on the base curve, we get the equations (3.11) - (3.13), in which the elements of the coefficient matrices depend on N.

Here we take a typical coefficient and show that it tend to a limit, as $N \to \infty$.

Consider $a^{(1)}{}_{i-1,\, j-1}$, which can be written as

$$a_{i-1,\, j-1} = \frac{4}{3N} \left[\sum_{l=1}^{N/2} A_{11}\,(2l-1) + 2 \sum_{l=1}^{N/2} A_{11}\,(2l) \right] \tag{1}$$

where

$$\begin{aligned} A_{11}\,(l) &= 1/2 \cos(x(l))\,[\cos(x(l)) - 1] \times 1/2 \sin(x(l)) \times [\sin(x(l)) - 1] \\ &= 1/32 - 1/32 \cos(4x(l)) - 1/16\,[\sin(3x(l)) + \sin(x(l))] \\ &\quad - 1/16\,[\cos(x(l)) - \cos(2x(l))], \text{ and} \end{aligned} \tag{2}$$

$x(l) = 2\pi\,(l-1)/N$

We note that all the arguments of cos or sin are multiples of $2\pi/N$. Using following formulae

$$\sum_{k=0}^{n-1} \begin{matrix}\sin\\ \cos\end{matrix}\,(\alpha + k\beta) = \frac{\sin n\beta/2}{\sin \beta/2} \times \begin{matrix}\sin\\ \cos\end{matrix}\,(\alpha + (n-1)\,\beta/2) \tag{3}$$

and substituting the result (2) in (1), we see after some simplification that

$$a^{(1)}{}_{i-1,\, j-1} = 1/16 \text{ for all even } N \geqslant 8.$$

Similar treatment applies to all the other coefficients and we can also show that for even $N \geqslant 12$, they are independent of N.

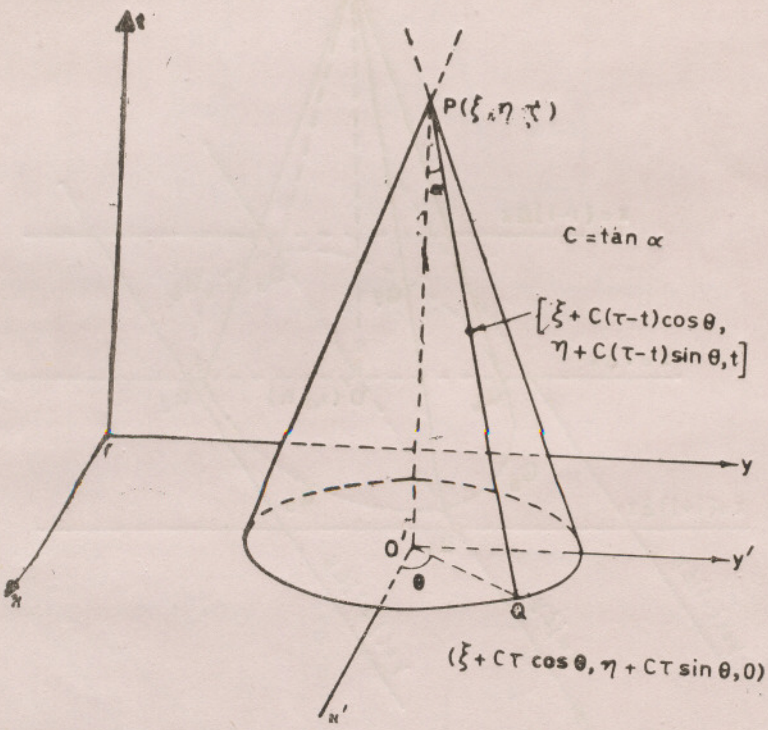

Fig. 1. Characteristic cone through (ξ, η, τ)

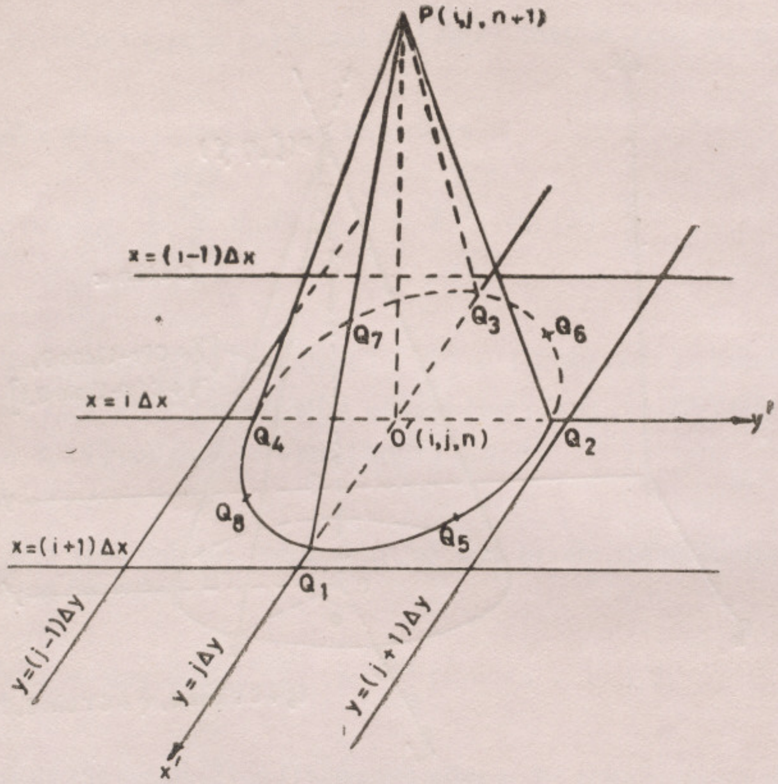

Fig. 2. Mesh points for a numerical scheme:
$\frac{C \Delta t}{\Delta x} < 1, \; \Delta x = \Delta y$

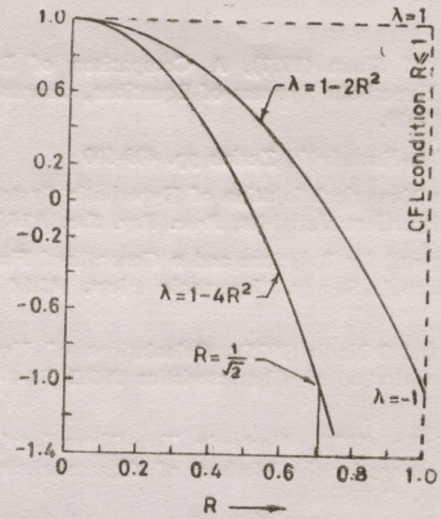

Fig. 3. Variation of the eigen values λ of the amplification matrix with R for Butler's scheme

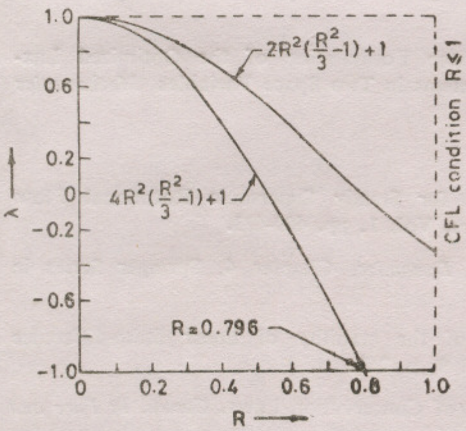

Fig. 4. Variation of the eigen values λ of the amplification matrix with R for method 1, $N = 8$

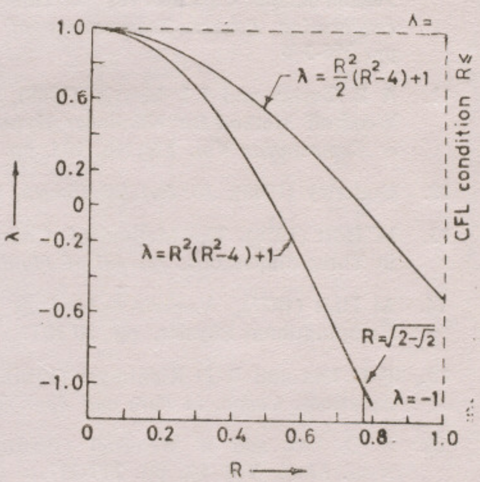

Fig. 5. Variation of the eigen values λ of the amplification with R for method 1, $N = 16$

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