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Non-linear approximation of a hyperbolic system in the neighbourhood of a bicharacteristic curve and stability of two-dimensional and axi-symmetric steady transonic flows

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ABSTRACT

In this paper we discuss the propagation of plane non-linear waves governed by a system of quasi-linear partial differential equations of first order with $m+1$ independent variables $(t, x_1, x_2, \ldots, x_m)$. We create a disturbance on a given steady solution in the neighbourhood of an arbitrary point $(x_\alpha{}^*)$. Following the motion of the disturbance, we have shown that the wave propagation is governed by a single quasi-linear differential equation of first order which can be solved analytically and which is an approximation of the original system of equations in the neighbourhood of a bicharacteristic curve. We can easily follow the change in the shape of the wave form and thus discuss the formation of a shock wave in the interior of the wave. The approximate equation assumes a simpler form when the wave front coincides with the tangent plane of a characteristic surface of the corresponding steady equations and the disturbance moves in the characteristic surface. When there are only two spatial coordinates (x_1, x_2) and the steady equations change their nature from an elliptic system to a hyperbolic system while crossing a curve P, we have shown that a part of any disturbance created in the neighbourhood of a point on the curve P will stay in the small neighbourhood of the point for a time interval of the order of unity. This part of the disturbance is governed by a simple equation $\dfrac{\partial \omega}{\partial t} + (c_\xi{}^{(1)}\xi + c_\omega{}^{(1)}\omega)\dfrac{\partial \omega}{\partial \xi} = K\omega$ where $c_\xi{}^{(1)}$, $c_\omega{}^{(1)}$ and K are constants depending on the steady solution. With the help of this equation we can discuss non-linear wave propagation in the transonic region of a two dimensional or axi-symmetric flow of a compressible fluid. In the last section we have discussed an example and shown that the two-dimensional steady spiral flow of a polytropic gas is unstable if the sonic transition takes from a supersonic state to a subsonic state and it is stable when the sonic transition is from a subsonic state to a supersonic state.

1. INTRODUCTION

The fact that the direction of propagation of a disturbance in two and three dimensions may be different from the normal direction of the wave front

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which bounds the region of the disturbance makes the theory of wave propagation in multidimensional space much more complicated and interesting than that in one-dimension. In the latter case, we get only two independent variables: t-time and x_1-one spatial coordinate, the wave front is always perpendicular to the x_1 axis and the disturbance is carried along the characteristic curves in (t, x_1) plane. In multidimensional space we need to prescribe the normal direction of the wave front in $(x_1, x_2, \ldots, x_m)$ space and this determines the speed V of the wave front [i.e. the orientation of the characteristic surface in $(t, x_1, x_2 \ldots, x_m)$ space] and also the velocity of propagation of disturbances along the bicharacteristic curve lying in the characteristic surface [Courant & Hilbert (1962), Varley & Cumberbatch (1965)]. In this paper we shall discuss the propagation of plane non-linear waves with a given direction of the normal of the wave front and governed by the system of quasi-linear equations

$$\sum_{j=1}^{n} A_{ij}\frac{\partial u_j}{\partial t} + \sum_{j=1}^{n}\sum_{\alpha=1}^{m} B_{ij}{}^{(\alpha)}\frac{\partial u_j}{\partial x_\alpha} + C_i = 0 \quad (i = 1, 2, \ldots, n) \tag{1.1}$$

where the elements of the matrices $[A_{ij}]$, $[B_{ij}{}^{(\alpha)}]$ and the column vector $[C_i]$ do not depend on time t explicitly. We shall assume the existence of a steady solution $u_i = u_{io}\ (x_\alpha)$ of (1.1), create a disturbance on the steady state in the neighbourhood of an arbitrary point $(x_\alpha{}^*)$ and then determine a single and very simple equation governing the motion of the disturbance. Whitham's (1959) approximate equation in the form

$$\frac{\partial \varphi}{\partial t} + V\frac{\partial \varphi}{\partial x} = K\varphi \tag{1.2}$$

represents such an approximation for the case $m = 1$ in the frame work of small amplitude linear waves assuming the steady solution to be a constant solution. The approximate equation

$$\frac{\partial c}{\partial t} + c\frac{\partial c}{\partial x} = \alpha c + \beta x \tag{1.3}$$

of Kulikovskii and Slobodkina (1967) and its generalisation by Bhatnagar and Prasad (1971) also represents such an approximation when $m = 1$ and when the steady solution is an arbitrary steady solution.

In the above approximations (1.2) and (1.3), it has been possible to express all dependent variables u_i in terms of a single variable φ or c and the equation represents the time rate of change of the variable as we move along a characteristic curve of the original system of equations. Naturally, in multi-dimensional space, we still like to follow a disturbance and try to approximate the system (1.1) by a single partial differential equation in the neighbourhood of a bicharacteristic curve. Along with the effect of the terms of the first order, the approximate equation, thus obtained, will also include the effect of the lower order terms [i.e. the third term in (1.1)] on the waves governed by some of the first two higher order terms. We find here that the effect of the lower order terms is to introduce an exponential decay or amplification, a result already known for one-dimensional waves [Whitham (1959), Prasad (1967)].

Following exactly the same procedure as discussed in the following sections we can also obtain an approximate equation determining the effect of the higher order terms. These approximate equations will be generalisations of Burger and *KdV* type of equations [Taniuti & Wei (1968), Prasad & Tagare (1971)] to two or three dimensions and they will be useful for discussing internal structure of the waves [Lighthill (1956)]. However, we shall not discuss these approximations here.

The system of equations (1.1) is sufficiently genearl to include the equations of motion of fluid mechanics and elasticity as its particular cases. Throughout this paper we use the convention that a repeated suffix in any term will represent sum over the spectrum of the suffix. We assume the spectrum of the suffixes i, j and k to be $1, 2, \ldots, n$; of α and β to be $1, 2, \ldots, m$ and of p, q to be $1, 2, \ldots, m\text{-}1$. According to this convention, the system (1.1) of n equations with n dependent variables and $m+1$ independent variables can be simply written as

$$A_{ij}\,(u_k, x_\beta)\frac{\partial u_j}{\partial t} + B_{ij}^{(\alpha)}\,(u_k, x_\beta)\,\frac{\partial u_j}{\partial x_\alpha} + C_i\,(u_k, x_\beta) = 0. \qquad (1.4)$$

2. Formulation of the problem and the approximate equation

Let us consider a known solution

$$u_i = u_{i0}\,(x_\alpha) \qquad (2.1)$$

of the system of steady equations

$$B_{ij}^{(\alpha)}\,(u_k,x_\beta)\frac{\partial u_j}{\partial x_\alpha}+C_i\,(u_k,x_\beta)=0. \tag{2.2}$$

Let $(x_\alpha{}^*)$ be a fixed point in $(x_1,x_2,\ldots,x_m)$ space. We define the value of a quantity Q in the steady solution (2.1) at the point $(x_\alpha{}^*)$ by Q^*. Thus

$$u_i{}^*=u_{io}\,(x_\alpha{}^*) \text{ and } A_{ij}{}^*=A_{ij}\,\{u_{ko}\,(x_\alpha{}^*),\,x_\beta{}^*\}\ etc. \tag{2.3}$$

Let us create a disturbance of sufficiently small amplitude and bounded by a plane wave front in the neighbourhood of the point $(x_\alpha{}^*)$ and we assume that at $t=0$ the point $(x_\alpha{}^*)$ lies on the wave front the unit normal of which is given by the vector $(n_1,n_2,\ldots,n_m)$. The velocity V of the wave front is a solution of the characteristic equation.

$$|\,n_\beta\,B_{ij}{}^{(\beta)}-\lambda A_{ij}\,|=0. \tag{2.4}$$

The equation (2.4) gives, in general, n-values of λ which may be real or complex. However, for physically realistic non-stationary processes governed by the system (1.4), all the roots of the equation (2.4) are generally real. In this case a real root may be zero, finite or infinite, Thus the system (1.4) is generally hyperbolic or mixed parabolic and hyperbolic for the solution $u_{io}\,(x_\alpha)$. On the contrary the system of equations (2.2), governing the steady solutions of (1.4), may be either hyperbolic or elliptic or even mixed type for the solution $u_{io}\,(x_\alpha)$. As an example of such a situation, we may take the equations of two-dimensional motion of a compressible fluid without any disipative mechanism. In this case, we do get steady flows for which the steady equations are elliptic in a region where the flow is subsonic and hyperbolic in a region where the flow is supersonic and these two regions are separated by a sonic curve where the equations are parabolic. In such cases the point $(x_\alpha{}^*)$, where we wish to approximate the equations (1.4), may lie either in the hyperbolic domain or in the elliptic domain or even on the surface separating these domains.

To proceed further we assume that *the root* $\lambda=V\,(u_i,\,x_\alpha)$ *of the equation* (2.4) *is simple and real.*

This implies that the rank of the matrix

$$[n_\beta\,B_{ij}{}^{(\beta)}-VA_{ij}] \tag{2.5}$$

is $n-1$ and there exist unique left and right eigen vectors $l\equiv[l_1,l_2,\ldots,l_n]$ and $r=[r_1,\,r_2,\ldots,r_n]$ satisfying

$$l_i\, n_\beta\, B_{ij}^{(\beta)} = V\, l_i\, A_{ij} \quad \text{and} \quad n_\beta\, B_{ij}^{(\beta)}\, r_j = V A_{ij}\, r_j. \tag{2.6}$$

No other assumption has been made about the other roots of (2.4), They may be real or complex. They may be zero or infinite. The assumption that V is a simple root has been made in order to avoid the complications of the general theory. In the case V is a multiple root of multiplicity s, we can easily extend these results [see Bhatnagar & Prasad (1971)] provided the rank of the matrix in (2.5) is $n-s$.

We introduce a new set of independent variables $(t', \xi, \eta_1, \eta_2, \ldots, \eta_{m-1})$ given by

$$t' = t,\ \xi = n_\beta\,(x_\beta - x_\beta^*) - V^*\, t,\ \eta_p = a_\beta^{(p)}\,(x_\beta - x_\beta^*) \tag{2.7}$$

where the matrix

$$N = \begin{bmatrix} n_1 & n_2 \ldots\ldots\ldots\ldots\ldots\ldots n_m \\ a_1^{(1)} & a_2^{(1)} \ldots\ldots\ldots\ldots a_m^{(1)} \\ a_1^{(2)} & a_2^{(2)} \ldots\ldots\ldots\ldots a_m^{(2)} \\ \ldots\ldots\ldots & \ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ a_1^{(m-1)} & a_2^{(m-1)} \ldots\ldots a_m^{(m-1)} \end{bmatrix} \tag{2.8}$$

is orthogonal. At $t=0$ the surface $\xi=0$ coincides with the wave front and the η_p axis lies in the wave front. Also $\xi=0$ is the tangent plane at the point $(t=0,\ x_\alpha = x_\alpha^*)$ to a characteristic surface [in (t, x_α)—space] of the system (1.4). In terms of the new independent variables, the system (1.4) reduces to

$$A_{ij}\frac{\partial u_j}{\partial t'} + B_{ij}\frac{\partial u_j}{\partial \xi} + D_{ij}^{(p)}\frac{\partial u_j}{\partial \eta_p} + C_i = 0 \tag{2.9}$$

where $$B_{ij} = n_\alpha\, B_{ij}^{(\alpha)} - V^*\, A_{ij} \quad \text{and} \quad D_{ij}^{(p)} = a_\alpha^{(p)}\, B_{ij}^{(\alpha)}. \tag{2.10}$$

The coefficients A_{ij}, B_{ij}, $D_{ij}^{(p)}$ and C_i are no longer independent of t'. However, they depend on ξ and t' only in the combination $\xi + V^* t'$.

The velocity $c(u_i, x_\alpha)$ of the wave front in new coordinate system is given by

$$c\,(u_i,\, x_\alpha) = V\,(u_i,\, x_\alpha) - V^*, \tag{2.11}$$

c is a simple root of the equation

$$|B_{ij}-cA_{ij}|=0$$

and $$c\{u_{io}(x_\alpha^*),\ x_\alpha^*\}=0 \tag{2.12}$$

The behaviour of perturbations

$$v_k=u_k(x_\alpha,t)-u_{ko}(x_\alpha) \tag{2.13}$$

of the steady solution is described by the system

$$A_{ij}(u_k,x_\alpha)\frac{\partial v_j}{\partial t'}+B_{ij}(u_k,x_\alpha)\frac{\partial v_j}{\partial \xi}+D_{ij}^{(p)}(u_k,x_\alpha)\frac{\partial v_j}{\partial \eta_p}+F_i=0 \tag{2.14}$$

where

$$F_i=\{B_{ij}(u_k,x_\alpha)+V^*A_{ij}(u_k,x_\alpha)\}\frac{\partial u_{jo}}{\partial \xi}+D_{ij}^{(p)}(u_k,x_\alpha)\frac{\partial u_{jo}}{\partial \eta_p}+C_i(u_k,x_\alpha) \tag{2.15}$$

and we assume v_k to be a small quantity of the order of δ.

The steady equations (2.2) give us

$$\{B_{ij}(u_{ko},x_\alpha)+V^*A_{ij}(u_{ko};x_\alpha)\}\frac{\partial u_{jo}}{\partial \xi}D_{ij}^{(p)}u_{ko},x_\alpha)\frac{\partial u_{jo}}{\partial \eta_p}+C_i(u_{ko},x_\alpha)=0. \tag{2.16}$$

With the help of (2.16), we can write (2.15) in the form

$$F_i=F_{iv_k}v_k+0(\delta^2) \tag{2.17}$$

where

$$F_{iv_k}{}^*=\left\{\left(\frac{\partial B_{ij}}{\partial u_k}\right)^*+V^*\left(\frac{\partial A_{ij}}{\partial u_k}\right)^*\right\}\left(\frac{\partial u_j}{\partial \xi}\right)^*+\left(\frac{\partial D_{ij}^{(p)}}{\partial u_k}\right)^*\left(\frac{\partial u_j}{\partial \eta_p}\right)^*+\left(\frac{\partial C_i}{\partial u_k}\right)^*. \tag{2.18}$$

With a given normal direction (n_β) of the wave front, the velocity of propagation is the whole set of values given by the equation (2.4). However, we are interested only in a part of the wave for which the velocity of the wave front is V. Thus we wish to retain the most dominant terms in (2.14) keeping in view that we wish to study only those waves remain in the neighbourhood of the plane $\xi=0$ in (ξ,η_p)-space for a time interval of the order of unity Therefore, if we consider v_i to be a small quantity of the order of δ, we need to approximate the system (2.I4) over a domain in which $\xi=0\ (\delta)$ and each of t', η_p is of the order of unity.

We write (2.14) in the form

$$A_{ij}\frac{\partial v_j}{\partial t'}+\frac{1}{\delta}B_{ij}\frac{\partial v_j}{\partial \xi'}+D_{ij}^{(p)}\frac{\partial v_j}{\partial \eta_p}+F_i=0, \tag{2.19}$$

where

$$\xi'=\frac{1}{\delta}\xi. \tag{2.20}$$

Substituting

$$v_k=v_k^{(0)}+\delta\, v_k^{(1)}+\ldots\ldots\ldots\ldots \tag{2.21}$$

in (2.19), expanding various coefficients and equating the terms of the first two orders in δ we get,

$$B_{ij}^*\frac{\partial v_j^{(0)}}{\partial \xi'}=0 \tag{2.22}$$

$$B_{ij}^*\frac{\partial v_j^{(1)}}{\partial \xi'}+A_{ij}^*\frac{\partial v_j^{(0)}}{\partial t'}+\frac{1}{\delta}\triangle B_{ij}\frac{\partial v_j^{(0)}}{\partial \xi'}+D_{ij}^{(p)*}\frac{\partial v_j^{(0)}}{\partial \eta_p}+F_{iv_j}^*\, v_j=0 \tag{2.23}$$

where

$$B_{ij}(u_{ko}+v_k,x_\alpha)-B_{ij}^*=\triangle B_{ij}+0\,(\delta^2) \tag{2.24}$$

The velocity c vanishes at $(t'=0,\ \xi=0,\ \eta_p=0)$ so that the expansion for c is of the form

$$c=c^{(1)}+c^{(2)}\,\delta+\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ . \tag{2.25}$$

Expanding both sides of the relation

$$l_i B_{ij}=c l_i A_{ij} \tag{2.26}$$

and multiplying the resultant by r_j^* we can easily show that

$$c^{(1)}=\frac{l_i^*(\triangle B_{ij})\,r_j^*}{l_i^*A_{ij}^*\,r_j^*}. \tag{2.27}$$

The general solution of the equations (2.22) is

$$v_j^{(0)}=\omega\,(t',\xi,\eta_p)\,r_j^*+g_j\,(t',\eta_p) \tag{2.28}$$

where ω and g_j are arbitrary functions of their arguments. Substituting (2.28) in (2.23), multiplying the resultant by l_i^* and dividing by $l_i^* A_{ij}^* r_j^*$, we get

$$\frac{\partial \omega}{\partial t'} + c^{(1)} \frac{\partial \omega}{\partial \xi} + V_p^* \frac{\partial \omega}{\partial \eta_p} = K\omega + f(t', \eta_p) \tag{2.29}$$

where
$$V_p^* = \frac{l_i^* D_{ij}^{(p)*} r_j^*}{l_i^* A_{ij}^* r_j^*}, \qquad K = -\frac{l_i^* F_{iv_j}{}^* r_j^*}{l_i^* A_{ij}^* r_j^*} \tag{2.33}$$

$$f(t', \eta_p) = -\frac{l_i^* A_{ij}^* \dfrac{\partial g_j}{\partial t'} + l_i^* D_{ij}^{(p)*} \dfrac{\partial g_j}{\partial \eta_p} + l_i^* F_{iv_j}{}^* g_j(t)}{l_i^* A_{ij}^* r_j^*}. \tag{2.31}$$

From (2.24) and (2.27) we get

$$c^{(1)} = c_\xi^{(1)} (\xi + V^* t') + c_{\eta_p}^{(1)} \eta_p + c_\omega^{(1)} \omega + \varphi(t', \eta_p) \tag{2.32}$$

where

$$c_\xi^{(1)} = \frac{1}{l_i^* A_{ij}^* r_j^*} \left[l_i^* \left\{ \left(\frac{\partial B_{ij}}{\partial \xi}\right)^* + \left(\frac{\partial B_{ij}}{\partial u_k}\right)^* \left(\frac{\partial u_k}{\partial \xi}\right)^* \right\} r_j^* \right], \tag{2.33}$$

$$c_{\eta_p}^{(1)} = \frac{1}{l_i^* A_{ij}^* r_j^*} \left[l_i^* \left\{ \left(\frac{\partial B_{ij}}{\partial \eta_p}\right)^* + \left(\frac{\partial_i B_j}{\partial u_k}\right)^* \left(\frac{\partial u_k}{\partial \eta_p}\right)^* \right\} r_j^* \right], \tag{2.34}$$

$$c_\omega^{(1)} = \frac{1}{l_i^* A_{ij}^* r_j^*} \left[l_i^* \left(\frac{\partial B_{ij}}{\partial u_k}\right)^* r_k^* r_j^* \right] \tag{2.35}$$

$$\varphi(t', \eta_p) = \frac{1}{l_i^* A_{ij}^* r_j^*} \left[l_i^* \left(\frac{\partial B_{ij}}{\partial u_k}\right)^* g_k(t', \eta_p) r_j^* \right]. \tag{2.36}$$

The equation (2.28) shows that in the neighbourhood of the n-dimensional characteristic surface whose tangent plane is $\xi = 0$, the n-dependent variables can be expressed in terms of the arbitrary functions ω and g_i of which only ω varies significantly with ξ. The values of the functions g_i can be determined by the knowledge of the unsteady solution lying outside the small neighbourhood of the plane $\xi = 0$. The history of the wave which remains in the neighbourhood of the characteristic surface is given, essentially, by a single variable ω which satisfies a simple first order quasi-linear partial differential equation (2.29) depending on two arbitrary functions φ and f of t' and η_p. These two arbitrary functions represent the influence on the wave of the disturbance outside the neighbourhood of the characteristic manifold. If we are interested only in those waves for which the disturbance outside the neighbourhood of $\xi=0$ does not depend on t' and η_p (this is true in the particular case when the disturbance is present only in the neighbourhood of $\xi=0$), the functions $\varphi(t', \eta_p)$ and $f(t', \eta_p)$ can be taken to zero. For such waves, the equation (2.29) reduces to

$$\frac{\partial \omega}{\partial t'}+\{c_{\xi}^{(1)}(\xi+V^* t')+c_{\eta_p}^{(1)} \eta_p+c_{\omega}^{(1)} \omega\} \frac{\partial \omega}{\partial \xi}+V_p^* \frac{\partial \omega}{\partial \eta_p} K \omega . \tag{2.37}$$

To explain the significance of putting the functions φ and f equal to zero, we point out one simple example in a slightly different situation. While discussing the effects of second order viscous terms on the sound waves, we can get Burgers' equation by substituting the arbitrary functions g_i and f_i equal to zero in the equation (6) of Taniuti & Wei (1968). This step is valid if we wish to study the structure of the wave when one weak shock is overtaking another weak shock, both moving approximately with the velocity of sound in the undisturbed medium ahead of the shocks [Lighthill (1956)]. In this case, the state ahead of and behind the two shocks is constant so that the disturbance is independent of η for points where $|\xi|$ is large.

By the lemma on bicharacteristic directions [Courant & Hilbert (1962), page 597], the components χ_a of the velocity in x_a direction, obtained by moving along a bicharacteristic lying in the n-dimensional characteristic manifold whose tangent plane is given by $\xi=0$, are given by

$$\chi_a=\frac{l_i B_{ij}^{(a)} r_j}{l_i A_{ij} r_j} . \tag{2.38}$$

The equation (2.37) expresses the fact that the time rate of change of the quantity ω when we move with velocity $(c^{(1)}, V_p^*)$ in (ξ, η_p) space is K times the quantity ω. The component of this velocity in x_a direction in (x_a) space is

$$n_a(V^*+c^{(1)})+a_a^{(p)} V_p^*=\chi_a^*+n_a c^{(1)} \tag{2.39}$$

and the differential equation (2.37) reduces to

$$\frac{\partial \omega}{\partial t}+(\chi_a^*+n_a c^{(1)}) \frac{\partial \omega}{\partial x_a}=K \omega . \tag{2.40}$$

If we are interested only in the qualitative behaviour of the wave front and not in its finer structure over a length scale of the order of δ in the direction of the normal of the front, we may neglect $n_\alpha c^{(1)}$ in comparison with $\chi_\alpha{}^*$ and we get a linear equation

$$\frac{\partial \omega}{\partial t} + \chi_\alpha{}^* \frac{\partial \omega}{\partial x_\alpha} = K\,\omega . \tag{2.41}$$

The equation (2.41), giving the time rate of change of ω as we move along a bicharacteristic curve, is a generalisation of Whitham's result (1.2) for one-dimensional waves. The quantity $c^{(1)}$ is the component of the excess $\triangle \chi_\alpha$ of the bicharacteristic velocity in dircetion (n_α) i.e.

$$c^{(1)} = n_\alpha . \triangle \chi_\alpha \quad . \tag{2.42}$$

We can easily verify that, in general,

$$\triangle \chi_\alpha \neq n_\alpha \, c^{(1)} \quad . \tag{2.43}$$

The equation (2.37) is valid only in a small neighbourhood of the point $(t=0,\ x_\alpha = x_\alpha{}^*)$ where we can use the expansion of the coefficients of (1.4) in power series in $\xi + V^* t'$ and η_p.

The time rate of change of the position as well as the amplitude of any wave governed by the equation (2.37) is given by the characteristic equations

$$\frac{d\omega}{dt'} = K\omega \ , \tag{2.44}$$

$$\frac{d\xi}{dt'} = c_\xi{}^{(1)} (\xi + V^* t') + c_{\eta_p}{}^{(1)} \eta_p + c_\omega{}^{(1)} \omega , \tag{2.45}$$

$$\frac{d\eta_p}{dt'} = V_p{}^* . \tag{2.46}$$

The general solution of equations (2.44)—(2.46) is

$$\eta_p = V_p{}^* t' + \eta_{po} \ , \tag{2.47}$$

$$\omega = \omega_o \ exp\,(Kt') \tag{2.48}$$

$$\xi = \xi_o \ exp\ (c_\xi{}^{(1)} t') - \frac{c_\xi{}^{(1)} V^* + c_{\eta_p}{}^{(1)} V_p{}^*}{c_\xi{}^{(1)}} t' + \frac{c_\xi{}^{(1)} V^* + c_{\eta_p}{}^{(1)} V_p{}^* + c_{\eta_p}{}^{(1)} c_\xi{}^{(1)} . \eta_{po}}{(c_\xi{}^{(1)})^2}$$

$$\times (exp\,(c_\xi{}^{(1)} t') - 1) + \frac{c_\omega{}^{(1)} \omega_o}{K - c_\xi{}^{(1)}} (exp\,(Kt') - exp\,(c_\xi{}^{(1)} t')) \tag{2.49}$$

where ω_o, ξ_o and η_{po} are the initial values of ω, ξ and η_p. The equations (2.47)—(2.49) give the magnitude ω of the disturbance at the position (ξ, η_p) at any time t'. The disturbance moves with a constant velocity in η_p direction and its amplitude increases or decreases according as $K>0$ or $K<0$. Due to the presence of the linear as well as non-linear terms, its behaviour with respect to ξ coordinate is complicated. To discuss the change in the shape of the wave, we first notice that its initial shape is given by $\omega_o=\omega_o(\xi_o)$. At any fixed point t' and for fixed values of η_{po}, we may regard the shape to be given by $\omega=\omega(\xi)$ in the parametric form $\omega=\omega(\xi_o)$ and $\xi=\xi(\xi_o)$ from the equations (2.48) and (2.49). Assuming η_{po} to be constants we can determine the gradient $d\omega/d\xi$ at any given instant (i. e. t' being taken to be a parameter giving different profiles at different times) in terms of $d\omega_o/d\xi_o$.

We get

$$\frac{d\omega}{d\xi}=\frac{exp\,(Kt')\dfrac{d\omega_o}{d\xi_o}}{exp\,(c_t^{(1)}\,t')+\dfrac{c_\bullet^{(1)}}{(K-c_t^{(1)})}\,\dfrac{d\omega_o}{d\xi_o}(exp(Kt')-exp\,(c_t^{(1)}\,t'))}\,. \tag{2.50}$$

Due to the non-zero velocity components $V_p{}^*$ in η_p directions the disturbance stays in the neighbourhood of $(\xi=0,\ \eta_p=0)$ only for a time interval of the order δ. Also the equation (2.37) is valid only in the neighbourhood of this point $(\xi=0,\ \eta_p=0)$ as we have expanded the coefficients of (2.9) about $\xi+V^*t'=0,\ \eta_p=0,\ u_i=u_i{}^*$. Therefore, the equation (2.50) is valid only for sufficiently small values of t' and we get

$$\frac{d\omega}{d\xi}\sim\frac{d\omega_o}{d\xi_o}\left\{1+\left(K-c_t^{(1)}-c_\bullet^{(1)}\frac{d\omega_o}{d\xi_o}\right)t'\right\} \tag{2.51}$$

which shows that the wave form tends to become more steep or less steep according as $K-c_t^{(1)}-c_\bullet^{(1)}\,(d\omega_o/d\xi_o)\gtrless 0$.

In the case the coefficients A_{ij}, $B_{ij}^{(\alpha)}$ and C_i are independent of the spatial coordinates x_α (which is generally the case in fluid mechanics) and the steady solution is a constant solution satisfying $C_i\,(u_{ko}, x)=0$, we have

$$c_t^{(1)}=0,\quad c_{\eta_p}^{(1)}=0 \tag{2.52}$$

and equation (2.37) reduces to

$$\frac{\partial \omega}{\partial t'} + c_*^{(1)}\,\omega \frac{\partial \omega}{\partial \xi} + V_p^* \frac{\partial \omega}{\partial \eta_p} = K\omega. \tag{2.53}$$

The equation (2.53), valid in the neighbourhood of the plane wave front, gives the history of the disturbance over a time interval of the order of unity and during this period it moves through a distance of the order of unity. In this case K is given by a simple expression

$$K = \frac{l_i^* \left(\dfrac{\partial C_i}{\partial u_j}\right)^* r_j^*}{l_i^* A_{ij}^* r_j^*}. \tag{2.54}$$

The general solution of the characteristic equations of (2.53) is

$$\omega = \omega_o \, exp\,(Kt')$$

$$\eta_p = \eta_{po} + V_p^* \, t'$$

$$\xi = \xi_o + \frac{\omega_o \, c_*^{(1)}}{K}(exp\,(Kt') - 1) \tag{2.55}$$

where t', can take any finite value. The disturbance ultimately dies if $K<0$ and grows without bounds as time increases if $K>0$.

3. Approximation valid in the neighbourhood of a characteristic manifold of the steady equations

In this section we discuss a particular case by assuming that the steady equation (2.2) is hyperbolic or more precisely, it has at least one real characteristic surface S of multiplicity one for the solution $u_{io}(x_\alpha)$. If (m_α) represent the direction cosines of the normal at an arbitrary point of the characteristic manifold S, the matrix $(m_\alpha B_{ij}^{(\alpha)})$ is singular and of rank (n-1) at every point of S. Let, (x_α^*) be a point on S. We consider a disturbance for which the normal to the wave front is (m_α^*). Therefore, we substitute $n_\alpha = m_\alpha^*$ in the previous results. The velocity V_o given by

$$| m_\alpha^* \, B_{ij}^{(\alpha)}\,(u_{ko}, x_\beta) - V_o \, A_{ij}\,(u_{ko}, x_\beta) | = 0 \tag{3.1}$$

vanishes at the point (x_α^*). Thus : *"the point where the characteristic velocity V vanishes in the steady solution, the normal to the wave front coincides with the normal to a characteristic surface S of the steady equations"*. Since a characteristic manifold S can be regarded as a singular surface of the steady equations in the sense that different steady solutions can be joined along this surface, we can regard the above result as a generalisation of the following result of Kulikovskii and Slobodkina for $m=1$: "*the point where a characteristic velocity vanishes is a singular point of the ordinary differential equations governing the steady solutions.*"

Since $V^*=0$, time does not appear in the transformation

$$\xi = n_\beta \,(x_\beta - x_\beta{}^*), \qquad \eta_p = a_\beta{}^{(p)} \,(x_\beta - x_\beta{}^*) \tag{3.2}$$

of the spatial coordinates. Now we may replace t' by t. B_{ij} is defined by

$$B_{ij} = n_\alpha \, B_{ij}{}^{(\alpha)}, \tag{3.3}$$

$D_{ij}{}^{(p)}$ by the same expression as that in (2.10) and equation (2.37) reduces to

$$\frac{\partial \omega}{\partial t} + c_\xi{}^{(1)}\,\xi + c_{\eta_p}{}^{(1)}\,\eta_p + c_\omega{}^{(1)}\,\omega)\frac{\partial \omega}{\partial \xi} + V_p{}^*\frac{\partial \omega}{\partial \eta_p} = K\omega. \tag{3.4}$$

We also have

$$c^{(1)} = c_\xi{}^{(1)}\,\xi + c_{\eta_p}{}^{(1)}\,\eta_p + c_\omega{}^{(1)}\,\omega. \tag{3.5}$$

The coefficients $c_\xi{}^{(1)}$, $c_{\eta_p}{}^{(1)}$, $c_\omega{}^{(1)}$, $V_p{}^*$ and K are given by the same expressions (2.33), (2.34), (2.35) and (2.30) with $V^*=0$ in the expression (2.18) for $F_{iv_k}{}^*$.

Since the velocity V^* given by (3.1) is zero at an arbitrary point $(x_\alpha{}^*)$ of S, it follows that if the wave front of an infinitely weak disturbance initially coincides with a characteristic surface S of the steady equations then, as the disturbance propagates, the wave front will remain coincident with S. However, if the initial wave front is not tangential to S, the disturbance initially at some point $(x_\alpha{}^*)$ of S will, in general, move away from S due to a non-zero component of the bicharacteristic velocity in a direction normal to S. To prove the latter statement, it is sufficient to give the following example in which we consider two-dimensional supersonic flow of a polytropic gas [Courant & Friedricks (1948)] without any dissipative mechanism. The equations of motion is given by (1.4) where $n=3$, $m=2$ and

$$U=\begin{vmatrix} u_1 \\ u_2 \\ \rho \end{vmatrix}, \quad A=\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}, \quad B^{(1)}=\begin{vmatrix} u_1 & 0 & \frac{a^2}{\rho} \\ 0 & u_1 & 0 \\ \rho & 0 & u_1 \end{vmatrix},$$

$$B^{(2)}=\begin{vmatrix} u_2 & 0 & 0 \\ 0 & u_2 & \frac{a^2}{\rho} \\ 0 & \rho & u_2 \end{vmatrix}, \quad \text{and} \quad C=\begin{vmatrix} 0 \\ 0 \\ 0 \end{vmatrix}, \tag{3.6}$$

u_1 and u_2 are the components of the particle velocity, ρ mass density and a the local sound speed related to the density by

$$a^2 = constant \times \rho^{\gamma-1}. \tag{3.7}$$

The three roots of (2.4) are

$$n_1u_1+n_2u_2 \quad , \quad n_1u_1+n_2u_2 \pm a$$

and we select

$$V=n_1\, u_1+n_2\, u_2-a. \tag{3.8}$$

The normal direction (m_1, m_2) of a characteristic curve S of the steady equations satisfies

$$|\, m_1\, B^{(1)}+m_2\, B^{(2)}\,|=0. \tag{3.9}$$

This gives three possible values

$$\frac{m_2}{m_1}=-\frac{u_1}{u_2}, \quad \frac{u_1\, u_2 \pm a\, \{u_1^2+u_2^2-a^2\}^{1/2}}{a^2-u_2^2}.$$

If we assume that the components u_1, u_2 of the particle velocity are positive, the values of m_1 and m_2 for which V vanishes at (x_1^*, x_2^*) when $n_1=m_1$ and $n_2=m_2$, should be taken to be

$$m_1=\frac{a^2-u_2^2}{au_1+u_2\,\{u_1^2+u_2^2-a^2\}^{1/2}}, \quad m_2=\frac{u_1\, u_2+a\,\{u_1^2+u_2^2-a^2\}^{1/2}}{au_1-u_2\,\{u_1^2+u_2^2-a_2\}^{1/2}}. \tag{3.10}$$

For a given normal (n_1, n_2) to the wave front, the components of the bicharacteristic velocity are (Varley & Cumberbatch (1965)

$$\chi_1 = u_1 - a n_1 \quad , \quad \chi_2 = u_2 - a n_2 \quad . \tag{3.11}$$

If we take $n_1 = m_2$, $n_2 = -m_1$ so that the wave front is normal to the characteristic curve S, the components of the bicharacteristic velocity (χ_1, χ_2) in (m_1, m_2) direction is

$$m_1 \chi_1 + m_2 \chi_2 = a \tag{3.12}$$

which is non-zero. This proves our assertion.

Since the characteristic equations of (3.4) form an autonomous system of ordinary differential equations in (ξ, η_p, ω) space, we may be tempted to discuss the wave propagation in the phase-plane of these equations [Kulikovskii & Slobodkina (1967), Bhatnagar & Prasad (1971)]. However, this method will not be very useful due to non-zero velocity component V_p^* in η_p direction. The disturbance moves away from the small neighbourhood of the point (x_α^*) in a time interval of the order of δ and the equation (3.4) ceases to be valid. Fortunately, there is one happy situation where these components vanish and the wave stays in the neighbourhood of (x_α^*) for a time interval of the order of unity. This forms the subject of our discussion in the next section.

4. Equilibrium of steady flows of transonic type $(m = 2)$

In this section, we wish to discuss the wave propagation in the neighbourhood of a m–1 dimensional manifold P across which the system of steady equations changes its nature from a hyperbolic system to an elliptic system. More precisely, we wish to approximate the system (1.4) in the neighbourhood of a point (x_α^*) of the manifold P such that one family of characteristic surfaces S is real on one side of P and complex on the other side. Due to a complicated nature of the characteristic equation $| m_\alpha B_{ij}^{(\alpha)} | = 0$, it seems necessary to restrict to the case of steady solutions in two-dimensional space i. e. $m = 2$. In this particular case, the characteristic equation reduces to a n^{th} degree algebraic equation

$$| B_{ij}^{(1)} (u_{ko}, x_\alpha) + k \, B_{ij}^{(2)} (u_{ko}, x_\alpha) | = 0 \tag{4.1}$$

in a single unknown $k = m_2/m_1$. For a given real steady solution, all the coefficients of this equation are real.

We assume that a root k of the equation (4.1) is real in some region H of (x_1, x_2) plane. We further assume that the region H is bounded by a curve P which separates H from another region E at every point of which the root k is complex. Let (x_α^*) be an arbitrary point of the curve P. In the neighbourhood of the point (x_α^*), the direction ratios of the normal to the characteristic curve S, which meets the curve P at (x_α^*), can be written as $(1, k^*+\epsilon)$; ϵ is real at a point in H, complex at point in E and as (x_1, x_2) tends to (x_1^*, x_2^*), ϵ tends to zero. Since complex roots of an algebraic equation with real coefficients always occur in pairs, there exists another root $k^*+\bar{\epsilon}$ of (4.1) giving the direction ratios $(1, k^*+\bar{\epsilon})$ of another real characteristic curve S' in H meeting P at the point (x_1^*, x_2^*). Here $\bar{\epsilon}$ is the complex conjugate of ϵ. The two characteristic curves S and S' belong to two different families, both meet at the point (x_α^*) and have a common tangent at this point with direction ratios $(k^*, -1)$. The two roots $k^*+\epsilon$ and $k^*+\bar{\epsilon}$ of the equation (4.1) coincide at the points of P. This implies that $\theta = k^*$ is a double root of the equation

$$|B_{ij}^{(1)*} + \theta B_{ij}^{(2)*}| = 0. \tag{4.2}$$

The matrix $[B_{ij}^{(1)*} + k^* B_{ij}^{(2)*}]$ or $[B_{ij}^*]$ will be, in general, of rank $n-1$ and we assume this to be so. Then, the left and right eigen-vectors l_i^* and r_j^* are uniquely determined except for common multiplying factors.

From $|B_{ij}^*| = 0$, we get

$$\left.\begin{aligned} B_{ij}^* \, b_{ik}^* &= 0 \\ B_{ji}^* \, b_{ki}^* &= 0 \qquad \text{(for all values of } j \text{ and } k) \end{aligned}\right\} \tag{4.2}$$

where b_{ij}^* is the cofactor of the element B_{ij}^* in the determinant $|B_{ij}^*|$.

We can write equation (4.2) also in the form

$$|B_{ij}^* + m_1^* (\theta - k^*) B_{ij}^{(2)*}| = 0 \tag{4.4}$$

and $\theta - K^* = 0$ is a double root. This implies that

$$B_{ij}^{(2)*} b_{ij}^* = 0. \tag{4.5}$$

From (4.3) and (4.5) we get

$$B_{ij}^{(1)*} b_{ij}^* = 0. \tag{4.6}$$

From (4.3) it follows that

$$b_{ij}^* = z\, l_i^*\, l_j^* \tag{4.7}$$

where the non-zero constant z is independent of i and j.

From (4.5)—(4.7) we get two important results

$$l_i^*\, B_{ij}^{(1)*}\, r_j^* = l_i^*\, B_{ij}^{(2)*}\, r_j^* = 0, \tag{4.8}$$

Therefore, at any point (x_1^*, x_2^*) of the curve P across which the root k changes from real values to complex values, the components χ_1^* and χ_2^* of the velocity of a disturbance moving along the bicharacteristic curve vanish provided we select the wave front to be coincident with the tangent line to the characteristic curve at (x_1^*, x_2^*). Now $V_1^* = 0$ and the equation (3.4) reduces to

$$\frac{\partial \omega}{\partial t} + (c_\xi^{(1)}\, \xi + c_{\eta_1}^{(1)}\, \eta_1 + c_\omega^{(1)}\, \omega)\frac{\partial \omega}{\partial \xi} = K\omega. \tag{4.9}$$

If we create a small arbitrary disturbance in the neighbourhood of a point on the critical curve P, we can easily obtain the local ray cone which is the locus of the points reached by the disturbance in (t, x_1, x_2) space. However, a part of the disturbance will remain in the neighbourhood of the point (x_1^*, x_2^*) for a time interval of the order of unity. The propagation and modification in the wave form of this disturbance over a finite time will be governed by the equation (4.9). If the magnitude $|\,\omega\,|$ tends to infinity as t tends to infinity, the steady solution will be unstable due to the growth of the perturbations at the point (x_1^*, x_2^*). In the case $|\omega|$ tends to zero as t tends to infinity, we get only local stability. The steady solution may still be unstable due to a source of instability away from the points of the critical curve P.

The function ω, appearing in the equation (4.9) depends also on the variable η_1. However, one of the characteristic equations is

$$\frac{d\eta_1}{dt} = 0 \tag{4.10}$$

which shows that as the disturbance propagates, η_1 remains constant. In this case we can always assume η_1 to be zero by a suitable choice of the origin in (ξ, η_1) plane. Thus, it is sufficient to discuss the equation

$$\frac{\partial \omega}{\partial t}-(c_\xi^{(1)}\,\xi+c_\omega^{(1)}\,\omega)\,\frac{\partial \omega}{\partial \xi}=K\omega \tag{4.11}$$

where ω is regarded as a function of two variables ξ and t only. We have reduced the whole problem to the discussion of an equation (4.11) governing one-dimensional waves. In fact, the equation (4.11) is exactly the same as the equation (9) of Kulikovskil & Slobodkina (1967). We may be tempted to eliminate the values $(\partial u_i/\partial x_\alpha)^*$ of the derivatives of the dependent variables appearing in the coefficients $c_\xi^{(1)}$ and K by a suitable transformation [Bhatnagar & Prasad (1971)] in order to get an approximate equation valid for the perturbations of an arbitrary steady solution. However, such a transformation seems to be difficult as K depends not only on $(\partial u_i/\partial \xi)^*$ but also on $(\partial u_i/\partial \eta_1)^*$.

The characteristic equations of (4.11) are

$$\frac{d\omega}{dt}=K\omega \tag{4.12}$$

$$\frac{d\xi}{dt}=c_\xi^{(1)}\,\xi+c_\omega^{(1)}\,\omega. \tag{4.13}$$

For given values of $c_\xi^{(1)}/K$ and $c_\omega^{(1)}/K$, the phase-plane of (4.12) and (4.13) remains unchanged under a transformation $\bar{\omega}=(1/\delta)\,\omega$ and $\bar{\xi}=(1/\delta)\,\xi$, which shows that while drawing the phase–plane we may assume ξ and ω to be of the order of unity. The propagation of waves governed by (4.11) can be easily discussed in this phase-plane [Kulikovskii & Slobodkina (1967), Prasad (1971)]. We shall not go into detail of drawing the phase-plane and discuss the individual particular cases for different vlaues of the constants $c_\xi^{(1)}/K$ and $c_\omega^{(1)}/K$, as we just need to follow the method of Kulikovskii and Slobodkina for a different system of equations (4.12) and (4.13) with the exception that the steady solution is $\omega=0$. At a given time the gradient $d\omega/d\xi$ of the wave form is related to its initial value $d\omega_o/d\xi_o$ by the same relation (2.50), which can also be written in the form

$$\frac{d\omega}{d\xi}=\frac{\dfrac{d\omega_o}{d\xi_o}}{exp\,((c_\xi^{(1)}-K)t)+\dfrac{c_\omega^{(1)}}{K-c_\xi^{(1)}}\{1-exp\,((c_\xi^{(1)}-K)t)\}\dfrac{d\omega_o}{d\xi_o}} \tag{4.14}$$

where the constants appear only in two combinations $c_\omega^{(1)}/(K-c_\xi^{(1)})$ and $(K-c_\xi^{(1)})$. A discontinuity in the wave form appears whenever the value of t given by

$$t=\frac{1}{c_\xi^{(1)}-K}\,ln\left\{\frac{\dfrac{c_\omega^{(1)}}{c_\xi^{(1)}-K}\dfrac{d\omega_o}{d\xi_o}}{1+\dfrac{c_\omega^{(1)}}{c_\xi^{(1)}-K}\dfrac{d\omega_o}{d\xi_o}}\right\} \tag{4.15}$$

is real and positive. If a shock wave does not appear, the limiting value of $d\omega/d\xi$ is zero when $(c_\xi^{(1)}-K)>0$ and $(K-c_\xi^{(1)})/c_\omega^{(1)}$ when $c_\xi^{(1)}-K<0$.

5. Example: Stability of a spiral flow in a compressible fluid in the neighbourhood of the sonic circle

The theory presented in the previous section will be useful for discussing the non-linear stability of any two-dimensional or axi-symmetric transonic flow, calculated either by a numerical solution of the equations of gas dynamics or by approximate solution of the equations. In this case, we just need to evaluate the two constants $c_\xi^{(1)}$ and K approximately. The value of the third constant $c_\omega^{(1)}$ can always be obtained exactly from the coefficients of the equations of motion. The method provides us not only with stability criteria but also gives us the full history of an arbitrary disturbance which stays in the transonic region. We consider here a two dimensional steady flow of a polytropic gas with a sonic curve P in the flow. The equations of motion are (1.4) where U, A, $B^{(\alpha)}$ and C are given by (3.6). If we denote the fluid speed by q, then $q^2=u_1{}^2+u_2{}^2$ and at every point of the sonic line

$$q=a=a^*. \tag{5.1}$$

The two families of characteristic curves, having a common tangent at the points of the sonic, are given by

$$\frac{m_2}{m_1}=\frac{u_1\,u_2\pm a\,\{q^2-a^2\}^{1/2}}{a^2-u_2{}^2}$$

and at the point (x^*, y^*) on the sonic line we have

$$(m_1{}^*,m_2{}^*)=\left(\frac{u_1{}^*}{a^*}\,,\,\frac{u_2{}^*}{a^*}\right). \tag{5.2}$$

The expressions for $c_\xi^{(1)}$, $c_\omega^{(1)}$ and K are given by

$$c_\xi^{(1)}=\frac{u_1{}^*}{a^*}\left(\frac{\partial u_1}{\partial\xi}\right)^*+\frac{u_2{}^*}{a^*}\left(\frac{\partial u_2}{\partial\xi}\right)^*-\frac{\gamma-1}{2}\frac{a^*}{\rho^*}\left(\frac{\partial\rho}{\partial\xi}\right)^*, \tag{5.3}$$

$$c_{\bullet}^{(1)}=\frac{\gamma+1}{2}a^* \tag{5.4}$$

$$K=-\frac{(\gamma+1)\,a^*}{2\rho^*}\left(\frac{\partial\rho}{\partial\xi}\right)^*. \tag{5.5}$$

When we transform the derivatives with respect to ξ into the derivatives with respect to x_1 and x_2 and use the steady equations, we get

$$K=-c_\xi^{(1)}=\frac{\gamma+1}{2\rho^*}\left[u_1^*\left(\frac{\partial\rho}{\partial x_1}\right)^*+u_2^*\left(\frac{\partial\rho}{\partial x_2}\right)^*\right]. \tag{5.6}$$

The expressions (5.4) and (5.6) are valid for any steady flow which may or may not be irrotational, an assumption generally made while calculating a transonic flow.

In order to discuss the first application of the theory, we consider a steady spiral flow [Courant & Friedrichs (1948), § 104] where we can get a simple exact solution. The flow is supersonic in the interior of two circles $r_m<r=(x_1^2+x_2^2)^{1/2}<r_c$ and subsonic for $r>r_c$. With the help of hodograph transformation, we get the solution of the problem in the form

$$x_1=(-k_1\,u_{20}+k_2\,\rho_0^{-1}\,u_{10})\,q^{-2},\quad x_2=(k_1\,u_{10}+k_2\,\rho_0^{-1}\,u_{20})\,q_0^{-2}. \tag{5.7}$$

With the help of the relations

$$\frac{\partial u_{10}}{\partial x_1}=j_0\frac{\partial x_2}{\partial u_{20}},\quad \frac{\partial u_{10}}{\partial x_2}=-j_0\frac{\partial x_1}{\partial u_{20}},\quad \frac{\partial u_{20}}{\partial x_1}=-j_0\frac{\partial x_2}{\partial u_{10}},\quad \frac{\partial u_{20}}{\partial x_2}=j_0\frac{\partial x_1}{\partial u_{10}} \tag{5.8}$$

where
$$j_0=\frac{\partial u_{10}}{\partial x_1}\frac{\partial u_{20}}{\partial x_2}-\frac{\partial u_{10}}{\partial x_2}\frac{\partial u_{20}}{\partial x_1}$$

the relations (5.7) give us

$$\frac{\partial u_{10}}{\partial x_1}=j_0\left[\frac{k_2}{\rho_0\,q_0^2}\left\{1+u_{20}^2\left(\frac{1}{a_0^2}-\frac{2}{q_0^2}\right)\right\}-\frac{2k_1\,u_{10}\,u_{20}}{q_0^4}\right], \tag{5.9}$$

$$\frac{\partial u_{10}}{\partial x_2}=j_0\left[\frac{k_1}{q_0^2}\left(1-\frac{2u_{20}^2}{q_0^2}\right)-\frac{k_2\,u_{10}\,u_{20}}{\rho_0\,q_0^2}\left(\frac{1}{a_0^2}-\frac{2}{q_0^2}\right)\right], \tag{5.10}$$

$$\frac{\partial u_{20}}{\partial x_1}=-j_0\left[\frac{k_1}{q_0^2}\left(1-\frac{2u_{10}^2}{q_0^2}\right)+\frac{k_2\,u_{10}\,u_{20}}{\rho_0\,q_0^2}\left(\frac{1}{a_0^2}-\frac{2}{q_0^2}\right)\right], \tag{5.11}$$

$$\frac{\partial u_{20}}{\partial x_2}=j_0\left[\frac{K_2}{\rho_0\,q_0^2}\left\{1+u_{10}^2\left(\frac{1}{a_0^2}-\frac{2}{q_0^2}\right)\right\}+\frac{2k_1\,u_{10}\,u_{20}}{q_0^4}\right]. \tag{5.12}$$

From Bernoulli equation we get

$$\frac{\partial \rho_0}{\partial x_1} = -\frac{j_0}{q_0^2 a_0^2}\Big[\rho_0\, u_{20}\, k_1 - u_{10}\, k_2\Big], \tag{5.13}$$

$$\frac{\partial \rho_0}{\partial x_2} = -\frac{j_0}{q_0^2 a_0^2}\Big[\rho_0\, u_{10}\, k_1 + u_{20}\, k_2\Big]. \tag{5.14}$$

We can easily verify that the expressions (5.9)—(5.14) giving the derivatives of the dependent variables satisfy the equations of motion. (x^*, y^*) is any point on the sonic circle $r = r_c$. From the relation

$$j_0 = [-k_1^2\, q_0^{-4} + k_2^2\, \rho_0^{-2}\, q_0^{-2}\, (a_0^2 - q_0^{-2})]^{-1} \tag{5.15}$$

[Courant & Friedriches (1948), §104] we get

$$j_0^* = -\frac{a^{*4}}{k_1^2}. \tag{5.16}$$

Equations (5.6), (5.13) and (5.14) give us

$$K = -c_\xi^{(1)} = \frac{(\gamma+1)\, a^{*2}\, k_2}{2\, \rho^*\, k_1^2}. \tag{5.17}$$

The component of the fluid velocity in radial direction at any point of the flow is

$$u_1 \frac{x_1}{r} + u_2 \frac{x_2}{r} = \frac{k_2}{\rho r} \tag{5.18}$$

showing that when $k_2 > 0$ the fluid particles move outwards i. e. from the supersonic region to the subsonic region and when $k_2 < 0$ they move from the subsonic to the supersonic region.

The amplitude of the disturbance increases to infinity as t tends to infinity if $K > 0$ and it ultimately tends to zero if $K < 0$. The spiral flow is, therefore, unstable if the sonic transition takes from a supersonic state to a subsonic state. In the case the sonic transition is from a subsonic state to a supersonic state, the flow is stable in the neighbourhood of the sonic line.

The perturbations in the two components of velocity and the density can be expressed in terms of ω as

$$v_1^{(0)} = u_1^*\, \omega, \quad v_2^{(0)} = u_2^*\, \omega \quad \text{and} \quad v_3^{(0)} = -\rho^*\, \omega \tag{5.19}$$

and ξ increasing direction coincides with the vector $(u_1^*/a^*, u_2^*/a^*)$, i. e. the direction of the tangent to the streamline. Thus the wave front is normal to the stream lines. $d\omega_o/d\xi_0>0$ (or $d\omega_o/d\xi_o<0$) corresponds to that part of the disturbance where the excess density or pressure due to the disturbance decreases (or increases) in the direction of motion of the fluid particles. To discuss the change in the shape of the disturbance due to nonlinearity, we use the equation (4.14) which reduces to

$$\frac{d\omega}{d\xi}=\frac{\dfrac{d\omega_o}{d\xi_o}}{exp\,(-2Kt)\left\{1-\dfrac{\rho^* k_1{}^2}{2a^* k_2}\dfrac{d\omega_o}{d\xi_o}\right\}+\dfrac{\rho^* k_1{}^2}{2a^* k_2}\left(\dfrac{d\omega_o}{d\xi_o}\right)} \tag{5.20}$$

Case 1 : When the sonic transition is from a supersonic state to a subsonic state a discontinuity appears at a finite time T only in that part of the wave where $d\omega_o/d\xi_o<0$ and it is given by

$$T=-\frac{1}{2K}\,ln\left\{\frac{\dfrac{\rho^* k_1{}^2}{2k_2}\left|\left(\dfrac{d\omega_o}{d\xi_o}\right)_{min}\right|}{1+\dfrac{\rho^* k_1{}^2}{2k_2}\left|\left(\dfrac{d\omega_o}{d\xi_o}\right)_{min}\right|}\right\} \tag{5.21}$$

where $(d\omega_o/d\xi_o)_{min}$ represents the minimum value of $d\omega_o/d\xi_o$ in the given pulse. Thus a continuous disturbance always ends in a shock wave before its amplitude increases to infinity.

Case 2 : When the sonic transition is from a subsonic state to a supersonic state, a discontinuity appears at a finite time T again in that part of the wave where $d\omega_o/d\xi_o<0$ provided the initial shape is sufficiently steep so that the condition

$$\frac{4\rho^* k_1{}^2}{|k_2|}\left|\frac{d\omega_o}{d\xi_o}\right|>1 \tag{5.22}$$

can be satisfied. In this case

$$T=\frac{1}{2|K|}\,ln\left\{\frac{\dfrac{\rho^* k_1{}^2}{2a^*|k_2|}\left|\left(\dfrac{d\omega_o}{d\xi_o}\right)_{min}\right|}{\dfrac{\rho^* k_1{}^2}{2a^*|k_2|}\left|\left(\dfrac{d\omega_o}{d\xi_o}\right)_{min}\right|-1}\right\}. \tag{5.23}$$

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