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NON-LINEAR WAVES IN RADIATION-GAS-DYNAMICS

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Abstract

This work is a sequel to a previous paper where a new set of equations for one-dimensional motion in Radiation-Gas-Dynamics (RGD) has been derived. These equations are valid for an arbitrary but constant value of opacity and for all values of β, the ratio of the gas pressure to the total pressure, and they clearly show the existence of radiation induced waves, which have been called "precursor radiation" by Lick and Moore. In this paper non-linear waves, with special refeience to the formation of shock waves in stellar medium, are discussed by a general method developed by Whitham. § 2 contains a general discussion of the equations of motion. The interactions of waves of different orders are discussed and damping distances, decay times and diffusion coefficients are determined. The terms giving rise to the fifth, fourth and third order waves are found out and it is shown that the equations with third order terms can be used as approximate equations in RGD, when one is interested in changes in flow and physical parameters over distances which are large compared to the mean free path of radiation, *i.e.* "flow in large". The formation of shock waves from a given compression wave is discussed by the method of characteristics and it has been found that a discontinuous front is formed only if the initial disturbance is sufficiently strong. Simple waves and Rankine-Hugoniot conditions for shock waves are also considered. It is found that the Rankine-Hugoniot conditions, derived by Sachs apply only to shock waves in "flow in large". Formation of shock waves in spherical, cylindrical and plane motion is also considered and the results obtoined by Pack are rederived by a very simple alternative method.

1. Introduction

The present work is concerned with the waves in Radiation Gas-Dynamics (RGD) with special reference to formation of shock waves in very hot neutral gaseous medium. Due to the dependence on direction of the specific intensity of radiation, the problem of three dimensional waves in RGD is extremely complicated and not much progress has been made in this direction. But one-dimensional waves in RGD can be easily discussed and we shall limit ourselves only to the one-dimensional motion. It is true that "radiation-hydrostatics" attracted attention almost fifty years ago and the effect of radiation on the equilibrium of stars has been discussed in detail but

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only recently some work has been done in RGD. In "radiation-hydrostatics" various steady state approximations (*e g.* Eddington's approximation and Rosseland's diffusion approximation) to the radiative transfer equation were made and they are taken as basic equations even in RGD for discussing waves in a medium, where radiation pressure is comparable to the gas pressure. The expressions for the radiation pressure and radiation energy density, in terms of specific intensity, contain *c*, the speed of light in the medium, in denominators and hence when these quantities are comparable to the gas pressure and gas internal energy density we cannot neglect the time derivative in the radiative transfer equation through it comes with a factor $1/c$. The neglect of the time derivative in radiative transfer equation suppresses one mode of wave propagation excited by the radiation.

For the reasons given above, the exact nature of shock wave in RGD is not fully understood, as it is evident from the various assumptions made in the investigations of Sachs[6], Prokof'ev[7], Elliot[8], Marshak[9], Sen and Guess[10], Wang[11] and Bhatnagar aud Sachdev[16]. The successful attempts to analyse shock waves in more general terms with neglect of radiation pressure appear to have been initiated by Zel'dovich[12] who proved the existence of a sharp discontinuity in shock wave structure for strong shocks. This work is followed by another approximate but very interesting work by Raizer[13]. The papers by Vincenti and Baldwin[14] and Heaslet and Baldwin[15] are also worth mentioning. The first one contains a detailed discussion of small amplitude waves in RGD and in the second Zel'dovich's assertions are supported by theoretical work and numerical computations.

Based on Zel'dovich's qualitative picture of the structure of a strong shock, we have determined in a previous paper[3] the distributions of various flow and physical parameters with optical thickness measured from the sharp discontinuity The present work is a sequel to another paper[2] of ours, hereafter referred as paper I, containing a derivation of equations for one-dimensional motion in RGD and a discussion of small amplitude waves. The new set of equations, derived there, is valid for a medium with arbitrary but constant opacity and even when the radiation pressure is comparable to gas pressure. It is hyperbolic in nature with distinct characteristics and finite values of characteristic speeds.

2. Equations of motion and the acoustic equation

We shall reproduce here some of the equations which we have derived in Paper I for sake of ready reference. Under the assumptions:

(i) the volume absoprtion coefficient α is constant,

(ii) the medium is grey and the source function B is given by

$$B=(\sigma/\pi)\,T^4 \qquad [2.1]$$

where T is the temperature and σ is Stefan constant, and

(iii) the specific intensity I of radiation is taken according to the following scheme

$$I = I_+ \quad \text{for } 0 < \mu \leqslant 1$$

$$= I_- \quad \text{for } -1 \leqslant \mu < 0$$

where μ is the consine of the angle which I makes with positive direction of x-axis,

the equations for one dimentional radiation-gas-dynamics are

$$\frac{\partial \rho}{\partial t} + u \frac{\partial \rho}{\partial x} + \rho \frac{\partial u}{\partial x} = 0, \qquad [2.2]$$

$$\rho \left(\frac{\partial}{\partial t} + u \frac{\partial}{\partial x} \right) u + \frac{\partial}{\partial x}(p_G + p_R) = 0, \qquad [2.3]$$

$$\rho \left(\frac{\partial}{\partial t} + u \frac{\partial}{\partial x} \right)\left(E_G + \frac{E_R}{\rho} \right) + (p_G + p_R) \frac{\partial u}{\partial x} + \frac{\partial F}{\partial x} = 0, \qquad [2.4]$$

III

$$\frac{\partial^2 F}{\partial x^2} - \frac{3}{c^2} \cdot \frac{\partial^2 F}{\partial t^2} = 4\pi\alpha \frac{\partial B}{\partial x} + \frac{6\alpha}{c} \cdot \frac{\partial F}{\partial t} + 3\alpha^2 F, \qquad [2.5]$$

V IV

III

$$\frac{1}{c} \cdot \frac{\partial F}{\partial t} + c \frac{\partial p_R}{\partial x} + \alpha F = 0 \qquad [2.6]$$

IV

and

$$E_R = 3 p_R. \qquad [2.7]$$

Here p_R is radiation pressure, E_R radiation energy density, F radiation flux, ρ density, u particle velocity in positive direction of x-axis, p_G gas pressure and E_G the gas internal energy density. Under the assumption (ii), [2.6] is a general relation between p_R and F and assumptions (i) and (iii) are not necessary to derive it.

Using $E_G = [p_G/(\gamma - 1)\rho]$ and $p_G = R\rho T$, where R is the gas constant and γ the ratio of specific heats, we can derive from these equations

III

V

$$\left(\frac{\partial}{\partial t}+u\frac{\partial}{\partial x}\right)(p_G+p_R)-\frac{\gamma p_G+4(\gamma-1)p_R}{\rho}\left(\frac{\partial}{\partial t}+u\frac{\partial}{\partial x}\right)\rho+(3\gamma-4)\left(\frac{\partial}{\partial t}+u\frac{\partial}{\partial x}\right)p_R+(\gamma-1)\frac{\partial F}{\partial x}=0 .$$

IV

[2.8]

The equation for small perturbations, defined by

$$u'=u-0\ ,\ \rho'=\rho-\rho_0\ ,\ p_G'=p_G-p_{G0}\ ,\ F'=F-0\ ,\ p_R'=p_R-p_{R0}\ ,$$

about a constant equilibrium state

$$u=0\ ,\ p_G=p_{G0}\ ,\ T=T_0\ ,\ F=0\ ,\ p_R=p_{R0}=(4\sigma/3c)\,T_0^4\ ,$$ (*see* Paper I)

is
$$\left[\frac{3}{c^2}\left(\frac{\partial^2}{\partial t^2}-\frac{c^2}{3}\cdot\frac{\partial^2}{\partial x^2}\right)\left(\frac{\partial^2}{\partial t^2}-a_{S0}^2\frac{\partial^2}{\partial x^2}\right)\frac{\partial}{\partial t}\right]\phi$$

$$+\left[\left(\frac{3a_{10}^2}{c^2}+\frac{6\alpha}{c}\right)\left(\frac{\partial^2}{\partial t^2}-\alpha_{10}^2\frac{\partial^2}{\partial x^2}\right)\left(\frac{\partial^2}{\partial t^2}-\alpha_{20}^2\frac{\partial^2}{\partial x^2}\right)\right]\phi$$

$$+\left[3\left(\alpha^2+\frac{a_{10}^2\alpha}{c}\right)\left(\frac{\partial^2}{\partial t^2}-a_{S0}^2\frac{\partial^2}{\partial x^2}\right)\frac{\partial}{\partial t}\right]\phi=0, \qquad [2.9]$$

where ϕ is defined by

$$u'=\frac{\partial\phi}{\partial x}\ ,\quad p_G'+p_R'=-\rho_0\frac{\partial\phi}{\partial t} \qquad [2.10]$$

and $$a_1^2=12\alpha c(\gamma-1)\frac{1-\beta}{\beta}\equiv\frac{16(\gamma-1)\sigma\alpha T^3}{R\rho}, \qquad [2.11]$$

$$a_T^2=p_G/\rho\ , \qquad [2.12]$$

$$a_S^2=\{\gamma+4(\gamma-1)p_R/p_G\}\,a_T^2 \qquad [2.13]$$

$$a_S^2=\frac{(4-3\gamma)\beta^2-12(\gamma-1)\beta+16(\gamma-1)}{\{1-12(\gamma-1)\}\beta^2+12(\gamma-1)\beta}\,a_T^2 \qquad [2.14]$$

$$\alpha_{10}^2,\ \alpha_{20}^2=\frac{A_1\pm\sqrt{(A_1^2-4a_{10}^2a_{T0}^2B_1)}}{2B_1} \qquad [2.15]$$

$$\beta=p_G/[(4\sigma/3c)T^4+p_G)] \qquad [2.16]$$

$$A_1 = \frac{a_{10}^4 a_{T0}^2}{3(\gamma - 1)\alpha c^3} + 6\frac{a_{T0}^2 a_{10}^2}{c^2} + 6\frac{\gamma \alpha a_{T0}^2}{c} + a_{10}^2 \qquad [2.17]$$

$$B_1 = \frac{3a_{10}^2}{c^2} + \frac{6\alpha}{c}. \qquad [2.18]$$

In the equation [2.9] the quantities a_{10}, a_{50} and a_{s0} appear with a second suffix 0 to represent the values of a_1, a_5, and a_s in constant state.

The left hand expression of the hyperbolic differential equation [2.9] which is symmetric in space coordinate is grouped in three square brackets, each containing a homogeneous differential operator of orders five, four and three respectively. If we denote these operators by P_5, P_4 P_3 (2.9) can be written as

$$P_5\phi + P_4\phi + P_3\phi = 0. \qquad [2.19]$$

As in paper I, we define the solutions $\{\phi\}$ satisfying $P_n\phi = 0$ $(n = 5, 4, 3)$ as n-th order waves.

The characteristics of [2 9] are

$$dx/dt = \pm c/\sqrt{3}, \qquad [2.20]$$

$$dx/dt = \pm a_{50} \qquad [2.21]$$

and

$$dx/dt = 0 \qquad [2.22]$$

and thus these are the only curves in x-t plane across which discontinuities in the flow quantities and their derivatives can exist. The range of influence and domain of dependence are bounded by the outermost characteristics [2.20]. A disturbance, created in a region, is initially divided into three groups. The first group corresponding to characteristics [2.20] travels with speeds comparable to c and forms " radiation induced wave." The second group, corresponding to [2.21] travels with speed a_{50} and forms " modified gas-dynamic waves." The third group, corresponding to [2.22] may be called " convective waves " and these can give rise to contact surfaces. But as these waves propagate the dispersion and damping change completely their nature. From [2.15], [2.17] and [2.18] we can write

$$\alpha_{10}^2, \ \alpha_{20}^2 = \frac{c^2}{2(3a_{10}^2/c + 6\alpha)}\left[\frac{a_{10}^2}{c} + + \frac{a_{T0}^2}{c^2}\left\{\frac{a_{10}^4}{3(\gamma-1)\alpha c^2} + 6\gamma\alpha + 6\frac{a_{10}^2}{c}\right\}\right.$$

$$\left.\pm\left\{\frac{a_{10}^4}{c^2} + \frac{a_{T0}^2}{c^2}\left(\frac{2}{3(\gamma-1)\alpha}\cdot\frac{a_{10}^6}{c^3} + 12\alpha(\gamma-2)\frac{a_{10}^2}{c}\right)\right\}^{1/2}\right] \qquad [2.23]$$

where terms of order $(a_{T0}/c)^4$ are neglected.

When radiation pressure is comparable to gas pressure, [2.11] shows that $a_{10}^2/c = 0\left[12(\gamma-1)\alpha\right]$ and it is possible to expand

$$\left\{\frac{a_{10}^4}{c^2} + \frac{a_{T0}^2}{c^2}\left(\frac{2}{3(\gamma-1)\alpha}\cdot\frac{a_{10}^6}{c^3} + 12\alpha(\gamma-2)\frac{a_{10}^2}{c}\right)\right\}^{1/2}$$

in ascending powers of a_{T0}^2/c^2 provided the speed of light in the medium is large compared to the isothermal sound speed a_{T0}. Retaining only the terms up to first power of a_{T0}^2/c^2 one obtains from [2.23]

$$\alpha_{10}^2 = \frac{c^2}{3}\cdot\frac{a_{10}^2/c + (a_{T0}^2/c^2)\{[a_{10}^4/3(\gamma-1)c^2\alpha] + 3a_1^2/c + 6\alpha(\gamma-1)\}}{a_{10}^2/c + 2\alpha} \qquad [2.24]$$

and

$$\alpha_{20}^2 = a_{T0}^2. \qquad [2.25]$$

In [2.24] we can further neglect the second term in the numerator on the right hand side to get

$$\alpha_{10}^2 = \frac{c^2}{3}\cdot\frac{a_{10}^2/c}{a_{10}^2/c + 2\alpha} = \frac{c^2}{3}\cdot\frac{6(\gamma-1)(1-\beta_0)}{6(\gamma-1)(1-\beta_0)+\beta_0}. \qquad [2.26]$$

Therefore, when $p_{R0} = 0\,(p_{G0})$ and $a_{T0} << c^2$, one of the speeds of propagation of fourth order waves is isothermal sound speed, as in the case of vanishing radiation pressure, *i.e.* $p_{R0} << p_{G0}$. The equation [2.9] can be written as

$$\frac{3}{c^2}\left(\frac{\partial^2}{\partial t^2} - \frac{c^2}{3}\cdot\frac{\partial^2}{\partial x^2}\right)\left(\frac{\partial^2}{\partial t^2} - a_{50}^2\frac{\partial^2}{\partial x^2}\right)\frac{\partial\phi}{\partial t}$$

$$+\frac{1}{c}\left(\frac{3a_{10}^2}{c} + 6\alpha\right)\left(\frac{\partial^2}{\partial t^2} - \alpha_{10}^2\frac{\partial^2}{\partial x^2}\right)\left(\frac{\partial^2}{\partial t^2} - a_{T0}^2\frac{\partial^2}{\partial x^2}\right)\phi$$

$$+3\left(\alpha^2 + \frac{a_{10}^2\alpha}{c}\right)\left(\frac{\partial^2}{\partial t^2} - a_{S0}^2\frac{\partial^2}{\partial x^2}\right)\frac{\partial\phi}{\partial t} = 0 \qquad [2.27]$$

where α_{10} is given by [2.26].

With L as a characteristic length in the flow field we define the non-dimensional quantities

$$\mathrm{x} = \frac{x}{L}, \quad \mathrm{t} = \frac{a_{T0}\,t}{L}, \quad \mathrm{a}_{10}^2 = \frac{a_{10}^2}{a_{T0}\alpha}, \quad \bar{\phi} = \frac{\alpha\phi}{a_{T0}}$$

so that equation [2.27] becomes

$$\left(\frac{1}{\alpha L}\right)^2\left[\left(\frac{3a_{T0}^2}{c^2}\cdot\frac{\partial^2}{\partial t^2}-\frac{\partial^2}{\partial x^2}\right)\left(\frac{\partial^2}{\partial t^2}-\frac{a_{S0}^2}{a_{T0}^2}\cdot\frac{\partial^2}{\partial x^2}\right)\frac{\partial\bar{\phi}}{\partial t}\right]$$

$$+\left(\frac{1}{\alpha L}\right)\left[\left(\frac{3\,a_{10}^2 a_{T0}^2}{c^2}+6\frac{a_{T0}}{c}\right)\left(\frac{\partial^2}{\partial t^2}-\frac{\alpha_{10}^2}{a_{T0}^2}\cdot\frac{\partial^2}{\partial x^2}\right)\left(\frac{\partial^2}{\partial t^2}-\frac{\partial^2}{\partial x^2}\right)\bar{\phi}\right]$$

$$+\left[3\left(1+\frac{a_{T0}\,a_{10}^2}{c}\right)\left(\frac{\partial^2}{\partial t^2}-\frac{a_{S0}^2}{a_{T0}^2}\cdot\frac{\partial^2}{\partial x^2}\right)\frac{\partial\bar{\phi}}{\partial t}\right]=0. \qquad [2.28]$$

It has been shown in paper I, that when p_G and p_R are of the same order of magnitude the neglect of the term $(1/c)(\partial I/\partial t)$ in the radiative transfer equation leads to solutions significantly different from those obtained by retaining this term.

3. INTERACTION OF WAVES OF DIFFERENT ORDERS

As pointed out in paper I fifth, fourth and third order waves dominate in a signalling problem at various times. Therefore. a consideration of interaction of waves of different orders is important. The phrase "interaction of fifth and fourth order waves" will be used for the modifications in the fifth order waves due to the presence of the fourth order operator and vice-versa. In general such a division of a differential equation (2.27) into the three groups does not imply that any wave motion can also be divided into three groups (such a division of waves is however possible for a signalling problem), but this is just a mathematical approach to the basic understanding of the waves and as it is evident from the investigations of Whitham, it helps in approximating the full differential equations by lowest order terms. We shall closely follow Whitham's approach in this investigation.

(*a*) *Interaction of fifth and fourth order waves.* When the third order terms are neglected, the equation (2.27) becomes

$$\frac{3}{c^2}\left(\frac{\partial^2}{\partial t^2}-\frac{c^2}{3}\frac{\partial^2}{\partial x^2}\right)\left(\frac{\partial^2}{\partial t^2}-a_{S0}^2\frac{\partial^2}{\partial x^2}\right)\frac{\partial\phi}{\partial t}$$

$$+3\left(\frac{a_{10}^2}{c^2}+\frac{2\alpha}{c}\right)\left(\frac{\partial^2}{\partial t^2}-\alpha_{10}^2\frac{\partial^2}{\partial x^2}\right)\left(\frac{\partial^2}{\partial t^2}-\alpha_{T0}^2\frac{\partial^2}{\partial x^2}\right)\phi=0. \qquad [3.1]$$

The behaviour of the various wave motions can be found using the principle that in a wave motion with velocity v the derivatives $\partial/\partial t$ and $-v(\partial/\partial x)$ of any quantity are approximately equal. Accordingly, for the fifth order radiation induced waves we can substitute $\partial/\partial t = -(c/\sqrt{3})\,\partial/\partial x$ in equation [3.1] except for the terms forming the factor $\partial/\partial t + (c/\sqrt{3})\,\partial/\partial x$. Thus, in the neighbourhood of the curve $dx/(dt) = c/\sqrt{3}$,

$$\frac{\partial^4}{\partial x^4}\left(\frac{\partial \phi}{\partial t} + \frac{c}{\sqrt{3}}\frac{\partial \phi}{\partial x}\right) + \alpha c \frac{\partial^4 \phi}{\partial x^4} = 0$$

or, omitting the factor $\partial^4/\partial x^4$ from the operator,

$$\frac{\partial \phi}{\partial t} + \frac{c}{\sqrt{3}}\frac{\partial \phi}{\partial x} + \alpha c \phi = 0, \qquad [3.2]$$

with the general solution

$$\phi = f_1[x - (c/\sqrt{3})\, t]\, e^{-\sqrt{3}\alpha x}, \qquad [3.3]$$

where f_1 is an arbitrary function of its argument. Thus this wave is exponentially damped, the exponential damping distance being

$$d_{5,\, c/\sqrt{3}} = 1/(\sqrt{3}\cdot\alpha). \qquad [3.4]$$

Similarly, for the fifth order modified gas dynamic waves we substitute $(\partial/\partial t) = -a_{50}(\partial/\partial x)$ in [3.1] and obtain

$$\frac{\partial \phi}{\partial t} + a_{50}\frac{\partial \phi}{\partial x} + \frac{a_{10}^2}{2}\,\frac{a_{50}^2 - a_{T0}^2}{a_{50}^2}\,\phi = 0. \qquad [3.5]$$

The exponential damping distance, in this case, is

$$d_{5,\, a_{50}} = \frac{2a_{50}^3}{a_{10}^2\,(a_{50}^2 - a_{T0}^2)}. \qquad [3.6]$$

When $p_{R0} = 0\,(p_{G0})$, $a_{10}^2/c = 0\,(\alpha)$ and hence $d_{5,\, a_{50}} = 0\,(a_{50}/\alpha c)$. [3.7]

From [3.4] and [3.7] it is evident that $d_{5,\, c/\sqrt{3}} = 1/\sqrt{3}$ (mean free path of of radiation) and $d_{5,\, a_{50}} << d_{5,\, c/\sqrt{3}}$ so that the fifth order modified gas dynamic waves are damped very rapidly.

It is interesting to put $\partial/\partial t = 0$ in [3.1] except in the factor $\partial/\partial t$ of the fifth order operator, in order to investigate the "convective waves" corresponding to the characteristic $dx/dt = 0$. In this case one obtains

$$\frac{\partial \phi}{\partial t} + \frac{a_{T0}^2\, a_{10}^2}{a_{50}^2}\,\phi = 0 \qquad [3.8]$$

with general solutions

$$\phi = f_2(x) \text{ Exp.} \left[-\frac{a_{T0}^2 a_{10}^2}{a_{50}^2} t \right] \qquad [3.9]$$

where $f_2(x)$ is an arbitrary function of x. Therefore, due to radiation, the discontinuity in contact surfaces is exponentially damped, with "decay time"

$$\tau_{5,0} = \frac{a_{50}^2}{a_{T0}^2 a_{10}^2}. \qquad [3.10]$$

These results are in agreement with the results of paper I, where a solution for a signalling problem is given. For a signalling problem, in which there is a uniform region at rest for $t < o$ and a disturbance is created at some point of it at $t = 0$, the convective waves will be absent. But in an initial value problem with variable initial density distribution, these waves must be present and they will be rapidly damped, because $\tau_{5,0}$ is small.

For the fourth order radiation induced waves, [3.1] may be approximated by using $\partial/\partial t = -\alpha_{10} (\partial/\partial x)$ in all terms except the factor $\partial/\partial t - \alpha_{10} (\partial/\partial x)$ in the fourth order operator. This gives, neglecting a_{T0}^2/c^2 in comparison to unity,

$$\frac{\partial \phi}{\partial t} + \alpha_{10} \frac{\partial \phi}{\partial x} = \frac{c}{3} \frac{\alpha}{(a_{10}^2/c + 2\alpha)^2} \frac{\partial^2 \phi}{\partial x^2} \qquad [3.11]$$

which represents diffusion of waves with diffusion coefficient

$$k_{4,\alpha_{10}} = \frac{c}{3} \frac{\alpha}{(a_{10}^2/c + 2\alpha)^2}. \qquad [3.12]$$

The numerical value of $k_{4,\alpha_{10}}$ is of order of c/α.

The fourth order modified gas dynamic waves are also found to be governed by a diffusion equation of form [3.11] with diffusion coefficient

$$k_{4,a_{T0}} = \frac{a_{50}^2 - a_{T0}^2}{2a_{10}^2}. \qquad [3.13]$$

To get an idea of the magnitude of $k_{4,a_{T0}}$ we take a typical astrophysical situation with $T_0 = 10^6 \,{}^\circ k$, and $p_{R0} = o(p_{G0})$. In this case we find

$$\frac{a_{50}^2 - a_{T0}^2}{c} = 0(10^3), \; \frac{a_{10}^2}{c} = 0(\alpha)$$

even though $a_{T0}^2/c^2 = 0(10^{-7})$. We also notice that

$$k_{4,\alpha 10} >> k_{4,a_{T0}}.$$

(*b*) *Interaction of fourth and third order waves.* When the fifth order operator is omitted, the differential equation [2.27] takes the form

$$\frac{3}{c}\left(\frac{a_{10}^2}{c}+2\alpha\right)\left(\frac{\partial^2}{\partial t^2}-a_{10}^2\frac{\partial^2}{\partial x^2}\right)\left(\frac{\partial^2}{\partial t^2}-a_{T0}^2\frac{\partial^2}{\partial x^2}\right)\phi$$

$$+3\left(\alpha^2+\frac{a_{10}^2\alpha}{c}\right)\left(\frac{\partial^2}{\partial t^2}-a_{S0}^2\frac{\partial^2}{\partial x^2}\right)\frac{\partial\phi}{\partial t}=0\,. \qquad [3.14]$$

As in the previous case, the fourth order waves will be exponentially damped and the third order waves will diffuse. We can easily show that the damping distances of the fourth order waves and the diffusion coefficients of the third order waves are

$$d_{4,\,a_{10}}=\frac{2}{\sqrt{3}}\,\frac{\sqrt{(a_{10}^2/c)}\,\sqrt{(a_{10}^2/c+2\alpha)}}{(a_{10}^2\alpha/c+\alpha^2)}\,, \qquad [3.15]$$

$$d_{4,\,a_{T0}}=\frac{2}{3}\,\frac{a_{T0}\,a_{10}^2}{\alpha\,(a_{10}^2/c+\alpha)\,(a_{S0}^2-a_{T0}^2)}\,, \qquad [3.16]$$

$$k_{3,\,a_{S0}}=\frac{a_{10}^2\,(a_{S0}^2-a_{T0}^2)}{6\alpha\,a_{S0}^2\,(a_{10}^2/c+\alpha)}\,, \qquad [3.17]$$

and

$$k_{3,\,0}=\frac{a_{10}^2\,a_{T0}^2}{3\alpha\,a_{S0}^2\,(a_{10}^2/c+\alpha)}\,, \qquad [3.18]$$

In the derivation of these expressions, terms containing a_{T0}^2/c^2 are neglected in comparison with terms of order unity.

Under astrophysical conditions with $p_{R0}=0\,(p_{G0})$ we notice that

$$d_{4,\,a_{10}}=0\,(1/\alpha)\,, \qquad [3.19]$$

$$d_{4,\,a_{T0}}=0\,(c/\alpha\, a_{T0})\,, \qquad [3.20]$$

$$k_{3,\,a_{S0}}=0\,(c/\alpha)\,, \qquad [3.21]$$

$$k_{3,\,0}=0\,(c/\alpha)\,. \qquad [3.22]$$

(*c*) *Interaction of fifth and third order waves.* The neglect of fourth order operator, reduces equation [2.27] to

$$\frac{3}{c^2}\left(\frac{\partial^2}{\partial t^2}-\frac{c^2}{3}\frac{\partial^2}{\partial x^2}\right)\left(\frac{\partial^2}{\partial t^2}-a_{S0}^2\frac{\partial^2}{\partial x^2}\right)\frac{\partial\phi}{\partial t}$$

$$+3\left(\alpha^2+\frac{a_{10}^2\alpha}{c}\right)\left(\frac{\partial^2}{\partial t^2}-a_{S0}^2\frac{\partial^2}{\partial x^2}\right)\frac{\partial\phi}{\partial t}=0\,, \qquad [3.23]$$

The operator $\partial/\partial t$ is a common factor of the left hand side of [3.23] and hence the third order operator does not affect the fifth order convective waves and *vice versa*. These waves, in the absence of fourth order operator, move with the fluid particles without attenuation and dispersion.

For any other fifth order wave, the equation [3.23] can be approximated by

$$\left(\frac{\partial}{\partial t} + m\frac{\partial}{\partial x}\right)\frac{\partial \phi}{\partial t} + n\,\phi = 0$$

where $m\ (\neq o)$ is a fifth order characteristic speed and n is another non-zero constant. In this case we substitute $\phi = e^{i(\omega t - kx)}$ and obtain

$$k/\omega = 1/m - (n/m)\ 1/\omega^2 \cdot \qquad [3.24]$$

Thus the wave number k is a real function of ω and there is no attenuation or amplification of fifth order waves due to the third order operator. Again [3.24] gives

$$\frac{d}{d\omega}\left(\frac{k}{\omega}\right) = \frac{2n}{m}\frac{1}{\omega^3} \qquad [3.25]$$

and since the fifth order waves are high frequency waves, this means that there is insigificant dispersion in these waves.

Approximating [3.23] in the neighbourhood of $dx/dt = a_{S0}$ by using $\partial/\partial x = -(1/a_{S0})\,\partial/\partial t$ except in the factor $\partial/\partial t + (a_{S0})\,\partial/\partial x$ and substituting $\phi = e^{i(\omega t - kx)}$ in the result we obtain

$$k/\omega = 1/a_{S0} + A\,\omega^2\ , \qquad [3.26]$$

where A is a non-zero constant. Here again we find that k is a real function of ω so that there is no attenuation and amplification of the third order waves. Also

$$d/d\omega\,(k/\omega) = 2\,A\,\omega \qquad [3.27]$$

and since the third order waves are low frequency waves [3 27] means that there is no significant dispersion in these waves due to the fifth order operator.

It is easy to trace the terms in equation [2.2] – [2 8] which give rise to operators of different orders in [2 9]. The continuity and momentum equations are taken as they are in all the three cases and the difference arises on account of the occurrance of different terms in the energy and radiative transfer equations in these operators. We have marked the combinations in [2 8], [2.5] and [2.6] by V, IV and III according as they appear in fifth order or fourth order or third order operators. Since we shall discuss the third order waves in detail in the following we collect here the terms in the energy and radiative transfer equations determing these. These are

$$\left(\frac{\partial}{\partial t}+u\frac{\partial}{\partial x}\right)(p_G+p_R)-\frac{\gamma p_G+4(\gamma-1)p_R}{\rho}\left(\frac{\partial}{\partial t}+u\frac{\partial}{\partial x}\right)\rho$$

$$+(3\gamma-4)\left(\frac{\partial}{\partial t}+u\frac{\partial}{\partial x}\right)p_R=0. \qquad [3.28]$$

$$0=4\pi\alpha\,(\partial B/\partial x)+3\alpha^2 F \qquad [3.29]$$

and

$$c\,(\partial p_R/\partial x)+\alpha F=0. \qquad [3.30]$$

We shall call these equations along with the continuity and momentum equations as set III. Now we ask the following question "Is it possible to approximate the set of full equations [2·2] – [2·8] by the set III and if so under what circumstances?" The answer to the first part is in affirmative. To answer to the second part we have to examine the non-dimensional equation [2·28]. This equation shows that the third order operator becomes more and more dominant with decreasing values of $1/\alpha L$ which is the ratio of the mean free path of radiation to a characteristic length in the flow field. In equation [2·28], the fifth and fourth order terms contain square and first power of $1/\alpha L$ respectively as factors. Thus we have two different situations when $1/\alpha L$ is small. (i) $1/\alpha L$ is small so that the fifth order terms can be neglected but not so small that the fourth order terms can also be neglected. Such a flow may be called "Rosseland flow" since in this case $\partial F/\partial x$ is retained in [2·8] while equations [2·5] and [2·6] reduce to

$$0=4\pi\alpha\frac{\partial B}{\partial x}+\frac{6\alpha}{c}\frac{\partial F}{\partial t}+3\alpha^2 F \qquad [3.31]$$

and

$$\frac{1}{c}\frac{\partial F}{\partial t}+c\frac{\partial p_R}{\partial x}+\alpha F=0 \qquad [3.32]$$

giving Rosseland diffusion approximation to radiative transfer equation when the terms containing $1/c$ are neglected. (ii) $1/\alpha L$ is sufficiently small so that both the fourth and fifth order terms can be neglected. We define such a flow to be "flow in large". Thus the set III can be used when we are interested in changes in flow and physical parameters over distances very large compared to the mean free path of radiation in the medium. When a characteristic length, say the distances between two points P and Q, of the flow field is much larger than the mean free path of radiation, and flux from P is almost absorbed before reaching Q and the motion of the medium takes place without any heat exchange between points whose mutual distance are much larger than the mean free path of radiation. This is not the only example where a set of differential equations is approximated by retaining only those terms which lead to the lowest order operator. In magnetohydrodynamics, the assumption of infinite conductivity is made to achieve such a goal. In aerodynamics, the assumptions of zero viscosity and zero heat conductivity remove the higher order terms. It is surprising to note that in all these three cases the approximate equations represent isentropic flow. Whatever may be the opacity of a medium, it should be regarded as one with low opacity if the changes in flow and physical parameters are to be considered across distances over which a ray of radiation is not significantly absorbed; otherwise the medium should be regarded as optically thick. While considering the structure of a shock wave, the interest lies in the changes across the shock region and we cannot presuppose the width of the shock region to be much larger than the mean free path of radiation in order to apply Rosseland diffusion approximation. It is worthy of notice that Masani's[21] work on the propagation of shock waves is based on the equations in set III. He considers the propagation of shock waves in hot and massive stars defined by Boury's[18] models where α is of the order of unity in C G.S. system and L can be taken to be radius of the star. Actually for temperatures of the order of 10^6 °K, the set III can be used when $1/\alpha L \leqslant 10^{-5}$. Thus the present analysis, while pointing out its limitations, gives a theoretical support to Masani's work.

The discussion of the interaction of waves in this section has been done in somewhat arbitrary fashion. While discussing the effect of fifth and fourth order operators on the third order waves one should proceed in a more logical sequence starting with a solution of the lowest order equation and building up the effect of the higher order terms as perturbations. Here we shall take the signalling problem, discussed in reference 2, and find out asymptotic solution for large αL. This will clearly show the diffusive effect of higher order terms in the "flow in large".

We wish to solve the equation [2·28] for $x > 0$, $t > 0$ with the following initial and boundary conditions:

$$\text{At}\quad t=0\ ,\ \bar{\phi}=\bar{\phi}_t=\bar{\phi}_{tt}=\bar{\phi}_{ttt}=\bar{\phi}_{tttt}=0 \ \text{ for } x>0 \tag{3.33}$$

$$\left.\begin{aligned} \text{At}\quad x=0\ ,\ (\partial\bar{\phi}/\partial x) &= (\alpha L)\bar{\beta} && \text{for } t\geqslant 0\\ &=0 && \text{for } t<0\\ (\partial\bar{\phi}/\partial t) &= -(\alpha L)\bar{\epsilon} && \text{for } t\geqslant 0\\ &=0 && \text{for } t<0 \end{aligned}\right\} \tag{3.34}$$

$\beta = a_{T0}\bar{\beta}$ and $p_0\,\epsilon = p_0\,a_{T0}^2\,\bar{\epsilon}$ are the velocity and total pressure respectively imposed on $x=0$ for $t \geqslant 0$ in an otherwise undisturbed medium. $\bar{\phi}$, being a potential function, is taken to be continuous at $x=0$, $t=0$.

Using the Laplace trasform

$$\bar{\bar{\phi}} = \int_0^\infty \bar{\phi}\, e^{-pt}\, dt \tag{3.35}$$

the solution is found to be

$$\bar{\phi} = \frac{1}{2\pi i}\int_{\Gamma} \bar{\bar{\phi}}\, e^{pt}\, dp \qquad [3.36]$$

where Γ is the path such that $Re\,p$ = constant and to the right of the all singularities and

$$\bar{\bar{\phi}} = A_1 e^{\bar{\gamma}_1 x} + A_2 e^{\bar{\gamma}_2 x}, \qquad [3.37]$$

$$\bar{\gamma}_1, \bar{\gamma}_2 = -\left[\frac{\bar{\delta}_2 \pm \sqrt{\bar{\delta}_2^2 - 4 b_2^2 p^3 \bar{\delta}_1}}{2\bar{\delta}_1}\right]^{1/2}, \qquad [3.38]$$

$$A_1 = \frac{p\bar{\beta} + \bar{\gamma}_2\bar{\epsilon}}{p^2(\bar{\gamma}_1 - \bar{\gamma}_2)}, \quad A_2 = \frac{p\bar{\beta} + \bar{\gamma}_1\bar{\epsilon}}{p^2(\bar{\gamma}_2 - \bar{\gamma}_1)}, \qquad [3.39]$$

$$\bar{\delta}_1 = \frac{1}{(\alpha L)^2} a_{s0}^2\, p + \frac{1}{(\alpha L)} a_{10}^2, \qquad [3.40]$$

$$\bar{\delta}_2 = \frac{1}{(\alpha L)^2}\left(1 + \frac{3 a_{s0}^2}{c^2}\right) p^3 + \frac{1}{(\alpha L)}(\bar{\alpha}_{10}^2 + 1)\left(\frac{3 a_{10}^2}{c} + 6\right)\frac{1}{c} p^2$$

$$+ 3(1 + a_{10}^2/c)\, a_{s0}^2\, p \qquad [3.41]$$

$$b_2^2 = \frac{1}{(\alpha L)^2}\frac{3}{c^2} p^2 + \frac{1}{(\alpha L)}\left(\frac{3 a_{10}^2}{c} + 6\right)\frac{1}{c} p + 3\left(1 + \frac{a_{10}^2}{c}\right) \qquad [3.42]$$

$$\text{and } \bar{\alpha} = \frac{\alpha_{10}}{a_{T0}}, \quad a_{s0} = \frac{a_{s0}}{a_{T0}}, \quad c = \frac{c}{a_{T0}}. \qquad [3.43]$$

Assuming $1/\alpha L$ to be small one can appropriate $\bar{\gamma}_1, \bar{\gamma}_2, A_1, A_2$ by expanding these functions in ascending powers of $1/\alpha L$. It is found that

$$\bar{\gamma}_1 = -\xi_1\sqrt{\alpha L}\, p^{1/2} - (\xi_2/\sqrt{\alpha L})\, p^{3/2} + \cdots \qquad [3.44]$$

$$\bar{\gamma}_2 = -\frac{1}{a_{s0}} p + \frac{\xi_3 a_{s0}}{2(\alpha L)} p^2 + \cdots, \qquad [3.45]$$

$$A_1 = -\frac{\bar{\beta} - \bar{\epsilon}/a_{s0} + [\bar{\epsilon}\, a_{s0}\xi_3/2(\alpha L)]\, p + \cdots}{\xi_1\sqrt{\alpha L}\{1 - 1/(\xi_1 a_{s0}\sqrt{\alpha L})p^{1/2} + \cdots\}}\frac{1}{p^{3/2}}, \qquad [3.46]$$

$$A_2 = -\frac{\bar{\epsilon} - \{\bar{\beta}/(\xi_1\sqrt{\alpha L})\}p^{1/2} + \cdots}{1 - \{1/(\xi_1 a_{s0}\sqrt{\alpha L})\}\, p^{1/2} + \cdots}\frac{1}{p^2} \qquad [3.47]$$

where the constants ξ_1, ξ_2, ξ_3 etc., are very complicated functions of $\bar{a}_{10}$, a_{S0}, c and a_{10}. But neglecting terms or order $1/c^2$ with comparison with terms of order unity, it is found that

$$\xi_1 = (a_{S0}/a_{10})[3 + 3\,(a_{10}^2/c)]^{1/2},$$

$$\xi_2 = (a_{S0}^2 - 1)\, a_{10}/\{2a_{S0}^3 (3 + 3a_{10}^2/c)^{1/2}\} \qquad [3.48]$$

and

$$\xi_3 = \frac{a_{10}^2\,(a_{S0}^2 - 1)}{a_{S0}^6\,(3 + 3\,a_{10}^2/c)}.$$

One can write [3.36] as $\bar{\phi} = \bar{\phi}_1 + \bar{\phi}_2$ [3.49]

where

$$\bar{\phi}_1 = \frac{1}{2\pi i}\int_\Gamma A_1\, e^{\bar{\gamma}_1 x + pt}\, dp \qquad [3.50]$$

and

$$\bar{\phi}_2 = \frac{1}{2\pi i}\int_\Gamma A_2\, e^{\bar{\gamma}_2 x + pt}\, dp \qquad [3.51]$$

The characteristics of the third order operator are $dx/dt = 0$ and $dx/dt = \pm a_{S0}$. The first corresponds to the convective waves, while last two to isentropic waves. In the present signalling problem convective waves are absent due to uniform initial state but there is diffusion from the disturbance at the wall at $x = 0$ and this diffusion is represented by $\bar{\phi}_1$. In this case it is sufficient to retain only the most dominant terms in $\bar{\gamma}_1$ and A_1 so that

$$\bar{\phi}_1 = -\frac{1}{2\pi i}\,\frac{\bar{\beta} - \bar{\epsilon}/a_{S0}}{\xi_1\sqrt{\alpha L}}\int_\Gamma \frac{1}{p^{3/2}}\, e^{(pt - \xi_1\sqrt{\alpha L}\, x\, p^{1/2})}\, dp$$

and this leads to

$$\bar{\phi}_1 = -(\bar{\beta} - \bar{\epsilon}/a_{S0})\left[\frac{2\sqrt{\alpha L}\, t^{1/2}\, a_{10}}{\sqrt{\pi(3 + 3a_{10}^2/c)}\, a_{S0}}\exp\left\{-\frac{(3 + 3\,a_{10}^2/c)\, a_{S0}^2}{4\, a_{10}^2}\,(\alpha L)\,\frac{x^2}{t}\right\}\right.$$

$$\left. -\,(\alpha L)\, x\, erfc\left\{\frac{\sqrt{3 + 3(a_{10}^2/c)}\; a_{S0}}{2\, a_{10}}\sqrt{\alpha L}\,\frac{x}{\sqrt{t}}\right\}\right] \qquad [3.52]$$

where

$$erfc\{y\} = \frac{2}{\sqrt{\pi}}\int_y^\infty e^{-\tau^2}\, d\tau.$$

This clearly shows the diffusive effect of the fourth and fifth order terms on the third order convective wave. But it is important to realise that $\bar{\phi}_1$ is not significant for a signalling problem since its effect is confined to a region close to the wall at $x = 0$. The third order isentropic wave is represented by $\bar{\phi}_2$ and this is the most important term. We can write

$$\bar{\phi}_2 = -\frac{1}{2\pi i}\int_{\Gamma} \frac{\bar{\epsilon} - (\bar{\beta}/\xi_1\sqrt{\alpha L})\, p^{1/2}}{1 - 1/(\xi_1 a_{S0}\sqrt{\alpha L})p^{1/2}}\, \frac{1}{p^2}\, e^{t g(p)}\, dp \qquad [3.53]$$

where

$$g(p) = \left(1 - \frac{x}{a_{S0}\, t}\right) p + \frac{\xi_3\, a_{S0}}{2(\alpha L)}\, \frac{x}{t}\, p^2 . \qquad [3.54]$$

The method of steepest descent used by Lick[4], can be employed here also to evaluate $\bar{\phi}_2$ when $|x - a_{S0}\, t|$ is small. It is found that

$$\bar{\phi}_2 = -\bar{\epsilon}\,(t - x/a_{S0})\,(\alpha L)\, H(t - x/a_{S0})$$

$$+ \frac{\bar{\epsilon} \quad \bar{\beta}\,\{a_{S0}^2 |x - a_{S0}\, t| / [(a_{S0}^2 - 1)x]\}^{1/2}}{1 - \{|x - a_{S0}\, t| / [(a_{S0}^2 - 1)\, x]\}^{1/2}} \left[(\alpha L)\, \frac{|x - a_{S0}\, t|}{2\, a_{S0}}\; erfc \left\{ \frac{c_1 |x - a_{S0} t| \sqrt{\alpha L}}{x^{1/2}} \right\} \right.$$

$$\left. - c_2 (\alpha L x)^{1/2} \exp\left\{ - \frac{c_1^2 (x - a_{S0} t)^2 (\alpha L)}{x} \right\} \right] \qquad [3.55]$$

where

$$H(t - x/a_{S0}) = 0 \text{ when } t < x/a_{S0}$$

$$= 1 \text{ when } t > x/a_{S0},$$

$$C_1 = \left[\frac{a_{S0}^2}{2(a_{S0}^2 - 1)} \right]^{1/2} \frac{(3 + 3\, a_{10}^2/c)^{1/2}\, a_{S0}}{a_{10}}$$

and

$$C_2 = \left[\frac{a_{S0}^2 - 1}{2\pi\, a_{S0}^3} \right]^{1/2} \frac{a_{10}}{a_{S0}\,(3 + 3\, a_{10}^2\, c)^{1/2}} .$$

This solution is valid for small values of $|x - a_{S0} t|$, *i.e.* near the wave front $x = a_{S0} t$ and under present approximation

$$\frac{\bar{\epsilon} - \bar{\beta}\sqrt{\{a_{S0}^2\, |x - a_{S0} t| \,/\, [(a_{S0}^2 - 1)\, x]\}}}{1 - \sqrt{\{|x - a_{S0} t| \,/\, [(a_{S0}^2 - 1)x]\}}}$$

should be taken to be $\bar{\epsilon}$ and in fact this has already been done in the first term of $\bar{\phi}_2$. However, we have retained this term in order to show that out of the two boundary values $\bar{\epsilon}$ and $\bar{\beta}$ the effect of $\bar{\beta}$ is small in comparison with that of $\bar{\epsilon}$ on the "flow in large." This solution represents a wave moving with third order characteristic speed a_{S0} and diffusive effects of higher order terms in it.

4. VARIATION OF THE FIRST DERIVATIVES ALONG CHARACTERISTIC LINES

Non-linear wave propagation :

A general property of the characteristics of a system of differential equations is that the variations along these curves in the nth order partial derivatives of the dependent variables can be completely determined if all the lower order derivatives are given on them (see Courant and Hilbert[20], page 618). In this section, the variations of the first order partial derivatives are determined along the characteristic

$$dx/dt = c/\sqrt{3} \qquad [4.1]$$

and

$$dx/dt = a_5 + u. \qquad [4.2]$$

It is to be noted that in RGD [4.1], [4.2] and $(dx/dt) = u$ are the only curves across which a discontinuity in certain derivatives of dependent variables can exist.

Consider discontinuities in the first derivatives of the flow quantities propagating along the characteristics [4.1]. As [4.1] is one of the outermost characteristics, the flow ahead of it remains undisturbed and it is convenient to take the flow ahead to be of constant state given by $T = T_0$, $p_G = p_{G0}$, $p_R = p_{R0} \equiv (4\sigma/3c)\, T_0^4$, $u = 0$, $F = 0$ and $\rho = \rho_0$.

The equation of the characteristic can be written as $t - [\sqrt{3}/c]\, x = 0$, and the flow quantities behind it can be expanded as

$$\begin{aligned} p_G &= p_{G0} + p_{G1}(x)\,\tau + \cdots , \\ p_R &= p_{R0} + p_{R1}(x)\,\tau + \cdots , \\ \rho &= \rho_0 + \rho_1(x)\,\tau + \cdots , \\ u &= 0 + u_1(x)\,\tau + \cdots , \end{aligned} \qquad [4.3]$$

and

$$F = 0 + F_1(x)\,\tau + \cdots$$

where

$$\tau = t - [\sqrt{3}/c]\, x. \qquad [4.4]$$

Then $p_{G1}(x)$, $p_{R1}(x)$ etc., give the magnitudes of the discontinuities in t-derivatives. Substituting [4.3] in [2.1]–[2.3], [2.5]–[2.6] and [2.8] and equating the various powers of τ it is found that

$$\frac{dF_1(x)}{dx} = -\left[\sqrt{3}\cdot\alpha - \frac{\sqrt{3}\cdot a_{10}^2\,(a_{50}^2 - a_{T0}^2)}{2\,(\gamma-1)\,c^3\,(1 - 3\,a_{50}^2/c^2)}\right] F_1(x), \qquad [4.5]$$

$$p_{G1}(x) = \frac{\sqrt{3}\cdot a_{50}^2}{c^3\,(1 - 3\,a_{50}^2/c^2)}\, F_1(x), \qquad [4.6]$$

$$u_1(x) = \frac{F}{c^2 \rho_0 (1 - 3\, a_{50}^2/c^2)} F_1(x) \qquad [4.7]$$

$$\rho_1(x) = \frac{\sqrt{3}}{c^3 (1 - 3\, a_{50}^2/c^2)} F_1(x), \qquad [4.8]$$

$$T_1(x) = \frac{\sqrt{3}\cdot(a_{50}^2 - a_{T0}^2)}{R\, \rho_0\, c^3 (1 - 3\, a_{50}^2/c^2)} F_1(x) \qquad [4.9]$$

and
$$p_{R1}(x) = [1/c\sqrt{3}]\, F_1(x). \qquad [4.10]$$

From [4.6] and [4.10]

$$p_{G1} = \frac{3\, a_{50}^2}{c^2 (1 - 3\, a_{50}^2/c^2)} p_{R1}(x). \qquad [4.11]$$

In the right hand expression of [4.5], the second term in the square brackets can be neglected in comparison with the first. This equation shows that the non-linearity of the equations of motion does not contribute anything to the radiation induced waves and whatever may be the initial value of F_1, it ultimately $\to 0$ as $t \to \infty$. Thus the discontinuities in the first derivatives of the flow quantities in radiation induced waves are exponentially damped and formation of a front, carrying discontinuities in the flow quantities themselves, is not possible from a continuous flow. From the expressions [4.6]—[4.11], it follows that the quantities P_{G1}, u_1, ρ_1 and T_1 are small compared with F_1 and p_{R1} as it should be in radiation induced waves.

Now consider discontinuities in the first derivatives of the flow quantities propagating along characteristic [4.2]. In general, the flow ahead of this curve will be disturbed by radiation induced waves. But we have just seen that for radiation induced waves u_1, p_{G1}, ρ_1, T_1 are small compared to p_{R1} and F_1 and hence the main effects can be seen for the special case in which $u = 0$ and p_G, ρ, T are constant ahead of [4.2]. Now the equations [2.5], [2.3] and [2.8] give

$$\left(\frac{\partial^2}{\partial x^2} - \frac{3}{c^2}\frac{\partial^2}{\partial t^2} - \frac{6\alpha}{c}\frac{\partial}{\partial t} - 3\alpha^2\right) F = 0, \qquad [4.12]$$

$$\partial p_R/\partial x = 0 \qquad [4.13]$$

and
$$3(\gamma - 1)\,\partial p_R/\partial t + (\gamma - 1)(\partial F/\partial x) = 0. \qquad [4.14]$$

Elimination of p_R between [2.6], [4.13] and [4.14] gives

$$\partial^2 F/\partial x^2 = 0 \qquad [4.15]$$

and $$(1/c)(\partial F/\partial t) + \alpha F = 0 \cdot \qquad [4.16]$$

The flux F satisfies an over-determined system of two partial differential equation namely [4.12] and [4.15]. Equation [4.16] is not independent of [4.12] and [4.15]. Now let us consider a particular solution $F \equiv 0$ of [4.15] and [4.16). In this case, from [4.14] $(\partial p_R/\partial t) = 0$ and from [4.13] p_R = constant. Thus the assumption made just now leads to a particular solution $F = 0$ and p_R = constant ahead of [4.2]. Now the flow quantities behind characteristic [4.2], which now may be written as $t - x/a_{50} = 0$, can be expanded in the form [4.3] where the relation [4.4] is to be replaced by

$$\tau = t - x/a_{50} \cdot$$

Substituting in [2.1]—[2.3], [2.5]—[2.6] and [2.8] and equating various powers of τ it can be shown that

$$du_1(x)/dx = -\frac{a_{10}^2}{2a_{50}}\,\frac{a_{50}^2 - a_{T0}^2}{a_{50}^2}\,u_1(x) + \frac{\gamma+1}{2a_{50}^2}\,u_1^2(x), \qquad [4.17]$$

$$\rho_1(x) = (\rho_0/a_{50})\,u_1(x), \qquad [4.18]$$

$$p_{G1}(x) = \rho_0\,a_{50}\,u_1(x) \qquad [4.19]$$

and $$F_1(x) = p_{R1}(x) \equiv 0, \qquad [4.20]$$

where a_{10}^2/c^2 is neglected in comparison with terms of order unity. The equation [4.20] implies that whatever may be the nature of discontinuities in the time derivatives of u, p_G, ρ the time derivatives of F and p_R are always continuous. The first term in the right hand side of [4.17] corresponds to small amplitude waves governed by linear equation. This term represents the exponential damping of u_1 and the damping distance agrees with that given by [3.6]. The second term results from the non-linearity of equations. The solution of [4.17] is

$$u_1 = \frac{1}{B/A + (1/U_1 - B/A)e^{Ax}}, \qquad [4.21]$$

where $$A = \frac{a_0^2}{2a_{50}}\,\frac{a_{50}^2 - a_{T0}^2}{a_{50}^2} > 0, \qquad [4.22]$$

$$B = \frac{\gamma+1}{2a_{50}^2}$$

and U_1 is the value of u_1 at some point, say $x = 0$. Since we are moving along a characteristic $dx/dt = a_{50} > 0$, $x > 0$ in [4.21].

When $U_1 < 0$, $1/U_1 - B/A < -B/A$ but $|1/U_1 - B/A| > B/A$ so that u_1 remains negative for all finite values of $x > 0$ and monotonically increases from U_1 to 0 as x varies from 0 to ∞.

When $0 < U_1 < A/B$, $u_1(x)$ monotonically decreases from U_1 to 0 as x increases from 0 to ∞.

When $U_1 > A/B$, $1/U_1 - B/A < 0$ and u_1 monotonically increases from U_1 to infinity as x increases from 0 to X, given by

$$X = (1/A)\, ln\, \{U_1/(U_1 - A/B)\} \cdot \qquad [4.23]$$

Now $u_1 < 0$ corresponds to an expansion wave, while $u_1 > 0$ to a compression wave. Therefore, as in ordinary gas-dynamics, expansion wave never leads to the breakdown of continuity of the flow. But for a compression wave the situation is completely different. In ordinary gas-dynamics a compression wave always leads to the formation of a shock front, whereas in *RGD* a compression wave leads to the formation of a shock front if and only if

$$U_1 > \frac{A}{B} = \frac{a_{10}^2\,(a_{50}^2 - a_{T0}^2)}{(\gamma + 1)\, a_{50}} \cdot \qquad [4.24]$$

The numerical value of the expression on the right hand side when $T_0 = 10^6$ and $p_{R0} = 0\,(p_{G0})$ will be of the order of $10^{16}\,\alpha$. This is an important result as it tells us that a shock front can be formed only if the initial disturbance producing the compression wave, is sufficiently strong. This also explains why Zel'dovich[12] and Heaslet and Baldwin[15] find a discontinuity in p_G, ρ, T, u in the structure of only strong shock waves. Again, if a shock front is formed, the discontinuity will be only in p_G, ρ, u, T and not in p_R and F, as shown by equations [4.17]—[4.20]. This result is also in agreement with the results of the authors cited above. The above result also gives a theoretical support to the basic assumptions made in a previous paper[3] on the structure of a shock wave with radiation.

5. Simple Waves and Shock Waves in *RGD*

At the end of § 3 the conditions, under which the equations of motion in *RGD* can be approximated by the set III, have been stated. In this approximation, as it can be seen from equations [3.29], [3.30] and [2.7], radiation pressure and radiation energy density are replaced by their values in thermodynamic equilibrium

$$p_R = (4\sigma/3c)\, T^4, \quad E_R = (4\sigma/c)\, T^4 \cdot \qquad [5.1]$$

The equation of energy does not contain the flux term as if the motion is isentropic in large. Using [5.1] we can write the equations [2.2], [2.3] and [3.28] as

$$\frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial x} + \rho\frac{\partial u}{\partial x} = 0\,, \qquad [5.2]$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{dx} + \frac{1}{\rho}\frac{\partial p}{\partial x} = 0 \qquad [5.3]$$

and

$$\left(\frac{\partial}{\partial t} + u\frac{\partial}{\partial x}\right)p - a_S^2\left(\frac{\partial}{\partial t} + u\frac{\partial}{\partial x}\right)\rho = 0 \qquad [5.4]$$

where

$$p = p_G + p_R\cdot \qquad [5.5]$$

a_S is given by [2.14] in terms of β and β is related to the density ρ and total pressure p by

$$\frac{(1-\beta)\,\rho^4}{\beta^4\,p^3} = \frac{4\sigma}{3cR^4}\,. \qquad [5.6]$$

The equations [5.2]—[5.4] can be replaced by an equivalent system of three characteristic equations

$$dp + a_S\,\rho\,du = 0 \quad \text{on } dx/dt = u + a_S\,, \qquad [5.7]$$

$$dp - a_S\,\rho\,du = 0 \quad \text{on } dx/dt = u - a_S \qquad [5.8]$$

and

$$dp - a_S^2\,d\rho = 0 \quad \text{on } dx/dt = u. \qquad [5.9]$$

For the discussion of forward facing simple wave running into a region of constant state (see Courant and Friedrich[17]), the two relations [5.8] and [5.9] are taken to be valid throughout the flow field, so that

$$p - p_0 = \int_{\rho_0}^{\rho} a_S^2\,d\rho \qquad [5.10]$$

and

$$u = \int_{\rho_0}^{\rho} a_S\,d\rho/\rho. \qquad [5.11]$$

Now the expression [2.14] is written as

$$a_S^2 = \Gamma\,(p/\rho) \qquad [5.12]$$

where

$$\Gamma = \frac{(4-3\gamma)\,\beta^2 - 12\,(\gamma-1)\,\beta + 16\,(\gamma-1)}{\{1-12\,(\gamma-1)\}\,\beta + 12\,(\gamma-1)} \qquad [5.13]$$

involves β, which is a complicated function of p and ρ as shown by [5.6]. In ordinary gas dynamics $\Gamma = \gamma$, which is a constant and equations [5.10] and [5.11] can be at once integrated. However, [5.10] with [5.6], [5.12] and [5.13] expresses p in terms of ρ and [5.11] expresses u in terms of ρ. On characteristic $dx/dt = u + a_S$ all the three equations [5.7] – [5.9] are valid and hence p, ρ, u are constant along this characteristic. Thus the velocity of propagation of the wave is $u + a_S$. From [5.6], [5.10] – [5.13] it is found that

$$\frac{2a_S \rho^2}{p} \frac{d}{d\rho} (u + a_S) = \phi(\beta) + \psi(\beta) \qquad [5.14]$$

where $\phi(\beta)$ and $\psi(\beta)$ are functions of β only, given by

$$\phi(\beta) = \Gamma(1 + \Gamma) \qquad [5.15]$$

and

$$\psi(\beta) = \frac{(4-3\gamma)(4-3\Gamma)\beta(1-\beta)[\{1-12(\gamma-1)\}\beta^2 + 24(\gamma-1)\beta - 16(\gamma-1)]}{(4-3\beta)[\{1-12(\gamma-1)\}\beta + 12(\gamma-1)]^2}. \qquad [5.16]$$

It can be shown that for $0 \leq \beta \leq 1$ and $1 \leq \gamma \leq 2$, $\phi(\beta) + \psi(\beta)$ is always positive, so that in this case

$$(d/d\rho)(u + a_S) > 0$$

and the propagation speed $u + a_S$ is a monotonic increasing function of ρ and hence of p. Therefore the wave region of higher density and pressure moves with higher velocity and a compression simple wave ultimately ends in a discontinuous flow. The appearance of discontinuity in the third order simple wave ultimately ends in a discontinuous flow. The appearance of discontinuity in the third order simple wave means that the approximate equations are not valid now and the neglected higher order terms, introducing diffusion in these waves, are crucial. However, as Whitham pointed out in the case of bores, the higher order terms need not be included explicitly. The solution by third order terms can be saved by introducing discontinuities, satisfying "Rankine-Hugoniote conditions" for the set III. Such discontinuities, still called shock waves, will be different from shock waves in *RGD* in which the radiation flux, radiation pressure and radiation energy density will be continuous. The Rankine-Hugoniot conditions for the third order terms are

$$\rho_1 u_1 = \rho_2 u_2 = m \text{ (say)} \qquad [5.17]$$

$$p_{G1} + p_{R1} + \rho_1 u_1^2 = p_{G2} + p_{R2} + \rho_2 u_2^2 \qquad [5.18]$$

and

$$E_{G1} + E_{R1}/\rho_1 + \tfrac{1}{2} u_1^2 + (u_1/m)(p_{G1} + p_{R1})$$
$$= E_{G2} + E_{R2}/\rho_2 + \tfrac{1}{2} u_2^2 + (u_2/m)(p_{G2} + p_{R2}) \quad [5.19]$$

where p_R and E_R are given by [5.1], suffixes 1 and 2 refer to quantities on the two sides of the shock and u_1, u_2 are velocities relative to the shock front. Thus the Rankine-Hugoniot conditions [5.17]—[5.19], first derived by Sachs[6], apply only to the reduced set III *i.e.*, for flow in large. Finding structure of a shock in *RGD* means joining the parameters on the two sides '1' and '2' (supposed to be at infinity on both sides) by solutions of the full equations of *RGD* and the structure may contain a discontinuous front satisfying the Ranking-Hugoniot conditions

$$\rho_3 u_3 = \rho_4 u_4 = m\,, \quad [5.20]$$

$$p_{G3} + \rho_3 u_3^2 = p_{G4} + \rho_4 u_4^2 \quad [5.21]$$

and

$$E_{G3} + \frac{1}{2} u_3^2 + \frac{u_3 p_{G3}}{m} + \frac{1}{\rho_3} (\mathbf{E}_R + \mathbf{p}_R)$$

$$= E_{G4} + \frac{1}{2} u_4^2 + \frac{u_4 p_{G4}}{m} + \frac{1}{\rho_4} (\mathbf{E}_R + \mathbf{p}_R) \quad [5.22]$$

for full differential equations in *RGD*, where suffixes 3 and 4 refer to states on the two sides of the shock and $\mathbf{E}_R$, $\mathbf{p}_R$ are the non-equilibrium values of E_R, p_R on the both sides of shock. It is important to note that in both sets [5.17] – [5.19] and [5.20] [5.22] the radiation flux does not give any contribution.

Now we shall come back to the discussion of simple waves. The function $\psi(\beta)$ in [5.14] is due to the dependence of Γ on β and hence on ρ. The numerical values of $\phi(\beta)$ and $\psi(\beta)$ are given in Table I for $\gamma = \frac{5}{3}$ and this table shows that $\psi(\beta)$ can be neglected in comparison with $\phi(\beta)$. Therefore for $\gamma = \frac{5}{3}$, Γ can be taken to be constant and the discussion of simple waves in *RGD* becomes exactly the same as that for a polytropic gas given by Courant and Friedrich[17].

6. FORMATION OF SHOCK WAVES IN SPHERICAL, CYLINDRICAL AND PLANE MOTION

The propagation of spherical shock waves in stellar envelopes has been studied extensively with a view to study the phenomena of ejection of mass from the stars. The general investigation of formation of shock waves

TABLE I

β	$\phi(\beta)$	$\psi(\beta)$
0.0	3.111	0 00000
0.1	3.173	– 0.00021
0.2	3.238	– 0.00086
0.3	3.307	– 0.00199
0.4	3.379	– 0.00368
0.5	3.459	– 0.00606
0.6	3.549	– 0.00941
0.7	3 656	– 0.01429
0.8	3.7)4	– 0.02173
0.9	4.005	– 0.03194
1.00	4.444	– 0 00000

in plane, cylindrical and spherical motion in *RGD* will be considered here. For one dimensional motion, this problem has already been considered in §4 with full equations of *RGD* and it has been found that even for plane motion, shock wave is formed only if the initial disturbance is very strong. Therefore, it is assumed here that the changes in flow quantities over small distances (*e.g*, distances comparable to mean free path of radiation) are not important and our interest lies in " flow in large ". This assumption can be made for discussing waves in hot and massive stars such as defined by Boury's[18] models. This problem has also been investigated by Pack[12] in the absence of radiation but here we shall include radiation terms and show that his results can be immediately obtained by a very simple alternative method. The equation of continuity is

$$\frac{\partial \rho}{\partial t}+u\frac{\partial \rho}{\partial x}+\rho\frac{\partial u}{\partial x}+\frac{\alpha \rho u}{x}=0, \qquad [6.1]$$

where $\alpha=0$, 1, 2 for plane, cylindrical and spherical motion and x represents radial distance when $\alpha=1$, 2. The equations of momentum and energy are [5 3] and [5.4] with relations [5.6] and [5.12] in physical variables. From these equations we can obtain

$$\left\{\frac{\partial}{\partial t}+(u+a_S)\frac{\partial}{\partial x}\right\}p+\rho a_S\left\{\frac{\partial}{\partial t}+(u+a_S)\frac{\partial}{\partial x}\right\}u+\frac{\alpha\rho a_S^2 u}{x}=0 \qquad [6.2]$$

and

$$\left\{\frac{\partial}{\partial t}+(u-a_S)\frac{\partial}{\partial x}\right\}p-\rho a_S\left\{\frac{\partial}{\partial t}+(u-a_S)\frac{\partial}{\partial x}\right\}u+\frac{\alpha\rho a_S^2 u}{x}=0\cdot \qquad [6.3]$$

Let us consider a forward facing wave (diverging wave for $\alpha = 1, 2$) running into a uniform state given by $p = p_0$, $\rho = \rho_0$ and $u = 0$. Then, since the front of the disturbance must be bounded by $dx/dt = u + a_S = a_{S0}$, there exist discontinuities in the first derivatives of the flow quantities across the characteristic $x - a_{S0} t = 0$ with a proper choice of the origin of t. We can expand the flow quantities behind this characteristic as

$$p = p_0 + p_1(x)\,\tau + \cdots,$$

$$\rho = \rho_0 + \rho_1(x)\,\tau + \cdots, \qquad [6.4]$$

$$u = 0 + u_1(x)\,\tau + \cdots$$

with $\tau = t - x/a_{S0}$.

Substituting [6.4] in [5.3], [5.6], [5.12], [6.1] and [6.2] and equating different powers of τ it is found that

$$\frac{du_1}{dx} + \frac{\alpha}{2x} u_1 = \frac{D}{a_{s0}^2} u_1^2, \qquad [6.5]$$

$$\rho_1 = \rho_0/a_{s0}\, u_1 \qquad [6.6]$$

and

$$p_1 = \rho_0\, a_{s0}\, u_1 \qquad [6.7]$$

where $D = (1/2\Gamma_0)\,[\phi(\beta_0) + \psi(\beta_0)]$

and the suffix 0 represents the values of quantities in the uniform state. The equations [6.5]—[6.7] are the same as those obtained by Pack.

If U_1 be the value of u_1 when $x = x_0$, the solution of [6.5] is found to be

$$\frac{1}{u_1} = \left(\frac{x}{x_0}\right)^{\alpha/2} \left\{\frac{1}{U_1} + \frac{2D}{(2-\alpha)\,a_{s0}^2} x_0\right\} - \frac{2D}{(2-\alpha)\,a_{s0}^2} x, \text{ for } \alpha = 0, 1 \qquad [6.8]$$

and

$$\frac{1}{u_1} = x\left[\frac{1}{U_1 x_0} - \frac{D}{a_{s0}^2} \log\frac{x}{x_0}\right], \text{ for } \alpha = 2. \qquad [6.9]$$

Same results [6.8] and [6.9] are obtained for a backward facing wave (convergent wave for $\alpha = 1, 2$) by proceeding exactly in the same way except with $\tau = t + x/a_{s0}$ and substituting [6.4] in [6.3] instead of [6.2].

u_1 becomes infinite *i.e.*, a shock wave is formed at $x = y$ given by

$$y^{1-\alpha/2} = x_0^{-\alpha/2}\, \frac{1/U_1 + [2D/\{(2-\alpha)\,a_{s0}^2\}]\,x_0}{2D/(2-\alpha)\,a_{s0}^2}, \text{ for } \alpha = 0, 1 \qquad [6.10]$$

and

$$y = x_0 \,\text{Exp.}\,[a_{s0}^2/(D\,U_1 x_0)], \text{ for } \alpha = 2. \qquad [6.11]$$

U_1 represents the time rate of change of u at the wave front when it is at $x = x_0$ and for a forward facing compression wave $U_1 > 0$ and for a backward facing compression wave $U_1 < 0$.

(*A*) *Effect of change in volume due to cylindrical and spherical motion* :

The quantity $a_{s0}^2 / |U_1| D$ has the dimension of length and we define the non-dimensional quantities ξ and ξ_0 by

$$\xi = \frac{|U_1| D}{a_{s0}^2} y \quad , \quad \xi_0 = \frac{|U_1| D}{a_{s0}^2} x_0 . \qquad [6.12]$$

[6.10] and [6.11] give

$$\xi^{1-\alpha/2} = \xi_0^{-\alpha/2} \left[\xi_0 + \frac{2-\alpha}{2} \frac{|U_1|}{U_1} \right] \quad , \quad \text{for } \alpha = 0, 1 \qquad [6.13]$$

and

$$\xi = \xi_0 \text{ Exp.} \left(\frac{|U_1|}{U_1} \frac{1}{\xi_0} \right) \quad , \quad \text{for } \alpha = 2 . \qquad [6.14]$$

The graphs of ξ versus ξ_0 are shown in Fig. 1. Curves in set I represent forward facing wave with $U_1 > 0$ and these in set II backward facing wave with $U_1 < 0$. The curves, for $\alpha = 0$, in both sets are straight lines showing that breakdown of continuity occurs after the propagation of wave through a constant distance $|\xi - \xi_0| = 1$. The curves for $\alpha = 1$ and $\alpha = 2$, in both sets, asymptotically tend to that for $\alpha = 0$ as $\xi_0 \to \infty$. For a forward facing wave the distance $\xi - \xi_0$, travelled before the shock is formed, is greatest for spherical motion and least for the plane motion and in cases $\alpha = 1, 2$; $\xi - \xi_0 \to \infty$ as $\xi_0 \to 0$. Thus a forward compression wave, created near origin, ends into a shock wave at a very large but finite distance in cylindrical and spherical motion. For $U_1 < 0$ and $\alpha = 1$, [6 13] shows that for $\xi_0 = \frac{1}{2}$, $u_1 \to \infty$ at $\xi = 0$; for $\xi_0 > \frac{1}{2}$, $u_1 \to \infty$ at points for which $\xi > 0$ and there is no positive value of ξ where $u_1 \to \infty$ for $\xi_0 < \frac{1}{2}$ In this case the right hand side of [6.8] vanishes for $x = 0$. Thus in set II $\xi - \xi_0$ curve, for $\alpha = 1$, is a part of ξ_0 axis for $0 < \xi_0 \leq \frac{1}{2}$. Thus in cylindrical and spherical motion converging compression wave ends in discontinuity either at origin or before reaching it. Putting all the results together we find that a compression wave always ends in a shock wave of the set III.

(*B*) *Effect of variation of U_1 on y*:

The non-dimensional quantities η and V_1 are introduced by

$$\eta = \frac{y}{x_0} \text{ and } V_1 = \frac{U_1 D x_0}{a_{s0}^2} . \qquad (6\ 15]$$

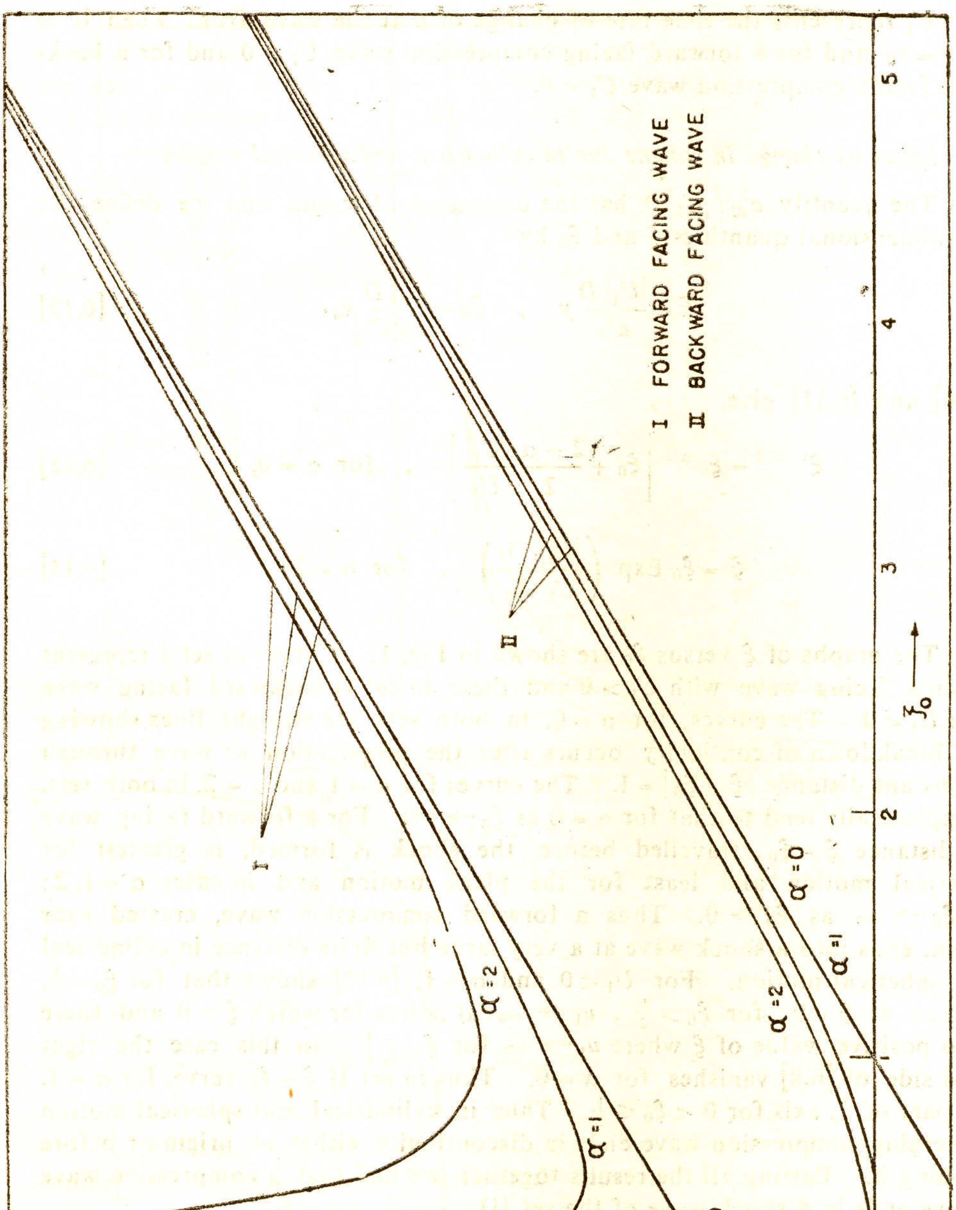
I FORWARD FACING WAVE
II BACKWARD FACING WAVE
I
II
α = 0
α = 1
α = 2
α = 0
α = 1
α = 2
ξ
ξ₀
0
1
2
3
4
5
6
0
1
2
3
4
5

FIG. 1

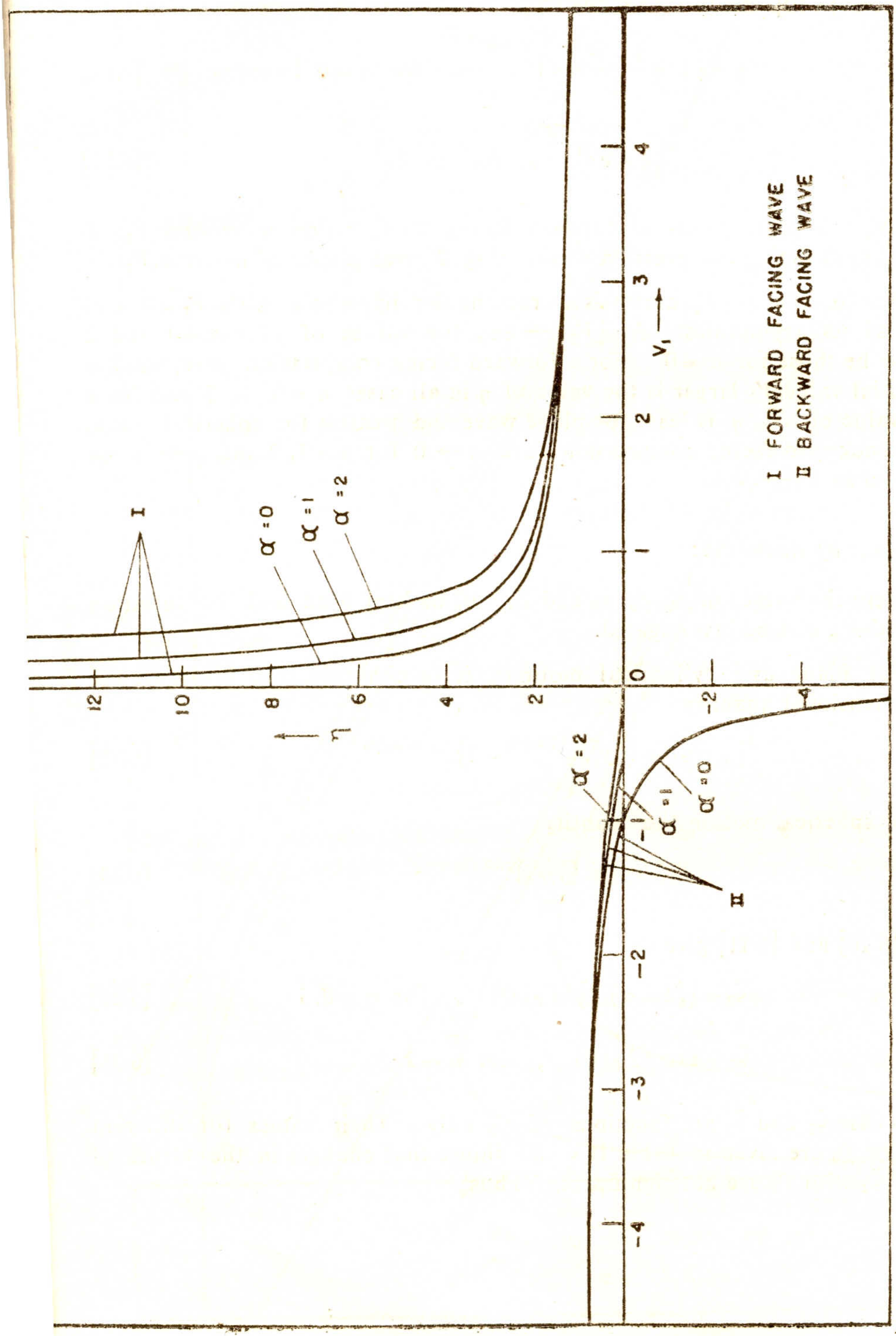

FIG. 2
η VERSUS V_1

Equations [6.10] and [6.11] reduce to

$$\eta = \left(1 + \frac{2-\alpha}{2}\frac{1}{V_1}\right)^{2/2-\alpha} \quad , \quad \text{for } \alpha = 0, 1 \qquad [6.16]$$

and

$$\eta = e^{1/V_1} \quad , \quad \text{for } \alpha = 2. \qquad [6.17]$$

Here $V_1 > 0$ corresponds to forward facing compression wave and $V_1 < 0$ to backward facing compression wave. Fig. 2 gives graphs of η versus V_1.

For $\alpha = 0$, $\eta - V_1$ curve is a rectangular hyperbola with $V_1 = 0$ and $\eta = 1$ as its asymptotes. As $|V_1| \to \infty$, the values of η for $\alpha = 1$ and 2 tend to be those for $\alpha = 0$. For a forward facing compression wave, smaller the initial value V_1 larger is the value of η in all cases $\alpha = 0, 1, 2$ and for a fixed value of V_1, η is least for plane wave and greatest for spherical wave. For a backward facing compression wave $\eta \to 0$ for $\alpha = 1, 2$ and $\eta \to -\infty$ for $\alpha = 0$ as $V_1 \to -0$.

(*C*) *Effect of Radiation*:

Here the values of p_0, ρ_0, x_0 and U_1 will be kept fixed and the variation of y with β_0 will be investigated.

For plane and cylindrical motions, it is convenient to introduce the non-dimensional quantity

$$\zeta_1 = \{(y/x_0)^{1-\alpha/2} - 1\}^{U_1 x_0 \rho_0/p_0} \qquad [6.18]$$

and for spherical motion the quantity

$$\zeta_2 = (y/x_0)^{\rho_0 V_1 x_0/p_0}. \qquad [6.19]$$

Then [6.10] and [6.11] give us

$$\zeta_1 = [(2-\alpha)/2]\,(\Gamma_0/D) \quad , \quad \text{for } \alpha = 0, 1 \qquad [6.20]$$

and

$$\zeta_2 = \exp.\,(\Gamma_0/D) \quad , \quad \text{for } \alpha = 2\cdot \qquad [6.21]$$

In this case ζ_1 and ζ_2 are functions of β_0 only. Their values for different values of β_0 are given in Table II which shows that changes in the values of ζ_1 and ζ_2 with β_0 are not significant. Thus

TABLE II

β_0	ζ_1		ζ_2
	$\alpha = 0$	$\alpha = 1$	$\alpha = 2$
0.0	1.143	0.5714	3.136
0.1	1.149	0.5746	3.155
0.2	1.156	0.5778	3 176
0.3	1.162	0.5812	3.198
0.4	1.169	0.5849	3.221
0.5	1.178	0.5888	3.247
0.6	1.187	0.5933	3.276
0.7	1.197	0.5985	3.310
0.8	1.210	0 6052	3.355
0.9	1.229	0.6147	3.419
1.0	1.250	0.6250	3.490

we come to an important conclusion that if we are interested in the variations of flow quantities over large distances, such that the equations with third order terms are sufficient approximations to full equations in RGD, the radiation does not appreciably affect the formation of a shock wave.

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