Comparative study of various quasiprobability distributions in different models of correlated-emission lasers

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We make a comparative study of various quasiprobability distributions in phase-sensitive quantum-optical systems. Starting from a general, linear master equation for the field, which emerges in different models of correlated-emission lasers, we derive the Fokker-Planck equations in the Glauber-Sudarshan P, the antinormal ordering Q, the Wigner W, the complex P, and the positive P representations and find the steady-state solutions for the five distributions. Simple relations between the complex and positive P functions are discovered for the first time. Various moments calculated by using these distributions are found to be identical, as expected. An application of these distributions to the two-photon correlated-emission laser shows that the intracavity field can be near-perfectly squeezed in the phase quadrature and the maximum quadrature squeezing is reached when the mean laser amplitude vanishes.

I. INTRODUCTION

Recently, several mechanisms for the correlated-emission laser (CEL) have been considered.[1-6] The correlated emission is based on using atoms prepared in a coherent superposition of the states between which the laser emission takes place. The initial atomic coherence can lead to the reduction in either phase or amplitude noise. It can even lead to the squeezing[7] in one of the quadratures of the field. The microscopic theories of the (single-mode) CEL show that the dynamical equation for the density matrix for the field mode a can be written in the form[1]

$$\dot{\rho}=\Lambda_1(a^\dagger\rho a-\rho aa^\dagger)+\Lambda_2(a\rho a^\dagger-a^\dagger a\rho)+\Lambda_3(\rho a^{\dagger 2}-a^\dagger\rho a^\dagger)+\Lambda_4(a^{\dagger 2}\rho-a^\dagger\rho a^\dagger)+f[a^\dagger,\rho]+\text{H.c.}, \tag{1.1}$$

with $\text{Re}\Lambda_1>0$, $\text{Re}\Lambda_2>0$. Note that the mean amplitude satisfies the equation

$$\langle\dot{a}\rangle=(\Lambda_1-\Lambda_2)\langle a\rangle-(\Lambda_3-\Lambda_4)\langle a^\dagger\rangle+f\ . \tag{1.2}$$

Clearly the usual net gain is $\Lambda_1-\Lambda_2$. The effects of cavity loss are contained in Λ_2. The parameters Λ_3,Λ_4, and f depend on the input coherences of the active atoms and the corresponding terms are phase sensitive. The Λ_3 and Λ_4 terms are important because they lead to correlated emission and phase locking and thus give rise to quantum noise quenching and even squeezing. The term f acts as a "driving force" and it can also lead to phase locking. The explicit form of the parameters Λ depend on the specific model of the correlated-emission laser. It may be noted that Eq. (1.1) has a fairly general structure and, in fact, other models of the CEL presently under investigation also lead to dynamical equations of the form (1.1). In view of the widespread applicability of Eq. (1.1) to various models of the CEL, it is important to explore the general results that follow from Eq. (1.1). This is the main theme of this paper. One useful way to study the consequence of Eq. (1.1) is to transform Eq. (1.1) into equations for the c-number distributions associated with ρ. The differential equations for the distribution functions can be solved by standard methods.

The organization of this paper is as follows. In Sec. II we derive the Fokker-Planck equations for the Glauber-Sudarshan P distribution,[8,9] the Q distribution,[10-12] and the Wigner distribution W.[13-15] In Sec. III we solve these Fokker-Planck equations in the steady state and show that the variances in the two quadratures of the field as well as in the photon number found from the Glauber-Sudarshan P, the Q, and the Wigner W distributions are the same. In Sec. IV we study the generalized P distributions.[16-18] We first obtain the Fokker-Planck equations in the complex P and positive P distributions and give their steady-state solutions. Then we use the two generalized P distributions to calculate variances in the amplitude and phase quadratures of field as well as in the photon number, which are the same as those obtained in Sec. III, as expected. A connection between the two generalized P distributions is pointed out. In Sec. V, the general results are employed to study explicitly the two-photon CEL. Finally, we discuss and summarize our results in Sec. VI.

II. FOKKER-PLANCK EQUATIONS FOR TWO-DIMENSIONAL DISTRIBUTIONS

Using the standard transformations, the master equation (1.1) can be transformed into a differential equation for a corresponding c-number distribution. In this section we use the Glauber-Sudarshan P distribution, the antinormal ordering Q distribution, and the Wigner distribution W (symmetric ordering). These distributions are related to the density matrix ρ in the following way ($|\alpha\rangle$

denotes the coherent state with amplitude α):

$$\rho=\int P(\alpha,\alpha^*)|\alpha\rangle\langle\alpha|d^2\alpha \ , \tag{2.1}$$

$$Q(\alpha,\alpha^*)=\pi^{-1}\langle\alpha|\rho|\alpha\rangle \ , \tag{2.2}$$

$$W(\alpha,\alpha^*)=\pi^{-2}\int d^2\beta\,\mathrm{Tr}\{\exp[-\beta(\alpha^*-a^\dagger)+\beta^*(\alpha-a)]\rho\} \ . \tag{2.3}$$

Using the transformation rules listed in Table I, we find the Fokker-Planck equations for the distributions P, Q, and W,

$$\frac{\partial\Phi(\alpha,\alpha^*)}{\partial t}=\left[-\frac{\partial}{\partial\alpha}d_\alpha-\frac{\partial}{\partial\alpha^*}d_\alpha^*+2\frac{\partial^2}{\partial\alpha\,\partial\alpha^*}D_{\alpha^*\alpha}+\frac{\partial^2}{\partial\alpha^2}D_{\alpha\alpha}+\frac{\partial^2}{\partial\alpha^{*2}}D_{\alpha\alpha}^*\right]\Phi(\alpha,\alpha^*) \ . \tag{2.4}$$

Here Φ can be P, Q, or W. The drift coefficient d_α depends on the parameters Λ and f, whereas the diffusion coefficients $D_{\alpha^*\alpha}$ and $D_{\alpha\alpha}$ depend on the parameters Λ only. While the drift coefficients are the same in the three distributions P, Q, and W, the diffusion coefficients are different in the three distributions. These coefficients are listed in Table II.

The Fokker-Planck equation (2.4) can also be written in terms of the quadratures of the field

$$\alpha=x+iy \ . \tag{2.5}$$

This corresponds to writing the field annihilation operator a as

$$a=a_1+ia_2 \ , \tag{2.6}$$

where the Hermitian operators a_1 and a_2 are the two quadatures. Equation (2.4) in terms of the variables x and y reads as

$$\frac{\partial\Phi(x,y)}{\partial t}=\left[-\frac{\partial}{\partial x}d_x-\frac{\partial}{\partial y}d_y+\frac{\partial^2}{\partial x^2}D_{xx}+\frac{\partial^2}{\partial y^2}D_{yy}+\frac{\partial^2}{\partial x\,\partial y}2D_{xy}\right]\Phi(x,y) \ . \tag{2.7}$$

TABLE II. The drift and diffusion coefficients in terms of the complex variables α and α^* [see Eq. (2.4)] in the Glauber-Sudarshan P, the Q, and the Wigner W representations.

Φ	P	Q	W
d_α		$(\Lambda_1-\Lambda_2)\alpha+(\Lambda_4-\Lambda_3)\alpha^*+f$	
$D_{\alpha^*\alpha}$	$\mathrm{Re}\Lambda_1$	$\mathrm{Re}\Lambda_2$	$\frac{1}{2}\mathrm{Re}(\Lambda_1+\Lambda_2)$
$D_{\alpha\alpha}$	Λ_4	Λ_3	$\frac{1}{2}(\Lambda_3+\Lambda_4)$

The new drift and diffusion coefficients are listed in Table III. Note that the diffusion matrix

$$\underline{D}=\begin{bmatrix} D_{xx} & D_{xy} \\ D_{xy} & D_{yy}\end{bmatrix}$$

for the P distribution need not be positive definite.

Finally we also give the form of the Fokker-Planck equation in polar coordinates

$$\alpha=re^{i\phi} \ , \tag{2.8}$$

which is useful in studying the phase and amplitude fluctuations,

$$\frac{\partial\Phi(r,\phi)}{\partial t}=\left[-\frac{1}{r}\frac{\partial}{\partial r}rd_r-\frac{\partial}{\partial\phi}d_\phi+\frac{1}{r}\frac{\partial^2}{\partial r^2}rD_{rr}+\frac{\partial^2}{\partial\phi^2}D_{\phi\phi}+\frac{2}{r}\frac{\partial^2}{\partial r\partial\phi}rD_{r\phi}\right]\Phi(r,\phi) \ , \tag{2.9}$$

where Φ is normalized according to

$$\int\int\Phi(r,\phi)r\,dr\,d\phi=1 \ . \tag{2.10}$$

The diffusion coefficients in the P, Q, and W representations are tabulated in Table IV, whereas the drift coefficients can be simply written as

$$d_r=\mathrm{Re}(d_\alpha e^{-i\phi})+rD_{\phi\phi} \ , \tag{2.11a}$$

$$d_\phi=\frac{1}{r}\mathrm{Im}(d_\alpha e^{-i\phi})-2D_{r\phi}/r \ . \tag{2.11b}$$

It is interesting to note that the usual gain Λ_1 (loss Λ_2)

TABLE I. The rules for transforming a master equation into Fokker-Planck equations in the (Glauber-Sudarshan) P, the generalized P, the Q, and the Wigner W representations.

Φ	P	Generalized P	Q	W
$a\rho$	αP	αP	$\left[\alpha+\frac{\partial}{\partial\alpha^*}\right]Q$	$\left[\alpha+\frac{1}{2}\frac{\partial}{\partial\alpha^*}\right]W$
$a^\dagger\rho$	$\left[\alpha^*-\frac{\partial}{\partial\alpha}\right]P$	$\left[\beta-\frac{\partial}{\partial\alpha}\right]P$	α^*Q	$\left[\alpha^*-\frac{1}{2}\frac{\partial}{\partial\alpha}\right]W$
ρa	$\left[\alpha-\frac{\partial}{\partial\alpha^*}\right]P$	$\left[\alpha-\frac{\partial}{\partial\beta}\right]P$	αQ	$\left[\alpha-\frac{1}{2}\frac{\partial}{\partial\alpha^*}\right]W$
$\rho a^\dagger$	α^*P	βP	$\left[\alpha^*+\frac{\partial}{\partial\alpha}\right]Q$	$\left[\alpha^*+\frac{1}{2}\frac{\partial}{\partial\alpha}\right]W$

TABLE III. The drift and diffusion coefficients in terms of the quadrature variables x and y [see Eq. (2.7)] in the P, Q, and W representations. The drift coefficients are the same for all P, Q, and W.

Φ	P	Q	W
d_x	$x\,\mathrm{Re}(\Lambda_1-\Lambda_2-\Lambda_3+\Lambda_4)-y\,\mathrm{Im}(\Lambda_1-\Lambda_2+\Lambda_3-\Lambda_4)+\mathrm{Re}f$		
d_y	$x\,\mathrm{Im}(\Lambda_1-\Lambda_2-\Lambda_3+\Lambda_4)+y\,\mathrm{Re}(\Lambda_1-\Lambda_2+\Lambda_3-\Lambda_4)+\mathrm{Im}f$		
D_{xx}	$\frac{1}{2}\mathrm{Re}(\Lambda_1+\Lambda_4)$	$\frac{1}{2}\mathrm{Re}(\Lambda_2+\Lambda_3)$	$\frac{1}{4}\mathrm{Re}(\Lambda_1+\Lambda_2+\Lambda_3+\Lambda_4)$
D_{yy}	$\frac{1}{2}\mathrm{Re}(\Lambda_1-\Lambda_4)$	$\frac{1}{2}\mathrm{Re}(\Lambda_2-\Lambda_3)$	$\frac{1}{4}\mathrm{Re}(\Lambda_1+\Lambda_2-\Lambda_3-\Lambda_4)$
D_{xy}	$\frac{1}{2}\mathrm{Im}\Lambda_4$	$\frac{1}{2}\mathrm{Im}\Lambda_3$	$\frac{1}{2}\mathrm{Im}(\Lambda_3+\Lambda_4)$

terms in the master equation (1.1) contribute to the diffusion coefficients in the amplitude r and phase ϕ of the field if one uses the P (Q) representation. In the Wigner representation the diffusion coefficients depend on the combination of both. In addition, the diffusion coefficients in all three representations depend on the phase-sensitive Λ_3 (Λ_4) terms in the master equation (1.1).

III. VARIANCES IN THE TWO QUADRATURES OF THE FIELD AND THE NATURE OF THE DISTRIBUTION FUNCTIONS

Using the Fokker-Planck equation (2.7), the variances in the two field quadratures a_1 and a_2 can be calculated. The equations of motion for the variances can be written in the form

$$\begin{aligned}
\frac{d}{dt}\langle(\delta x)^2\rangle &= 2A_{xx}\langle(\delta x)^2\rangle+2A_{xy}\langle\delta x\delta y\rangle+2D_{xx}\ ,\\
\frac{d}{dt}\langle(\delta y)^2\rangle &= 2A_{yx}\langle\delta x\delta y\rangle+2A_{yy}\langle(\delta y)^2\rangle+2D_{yy}\ ,\\
\frac{d}{dt}\langle\delta x\delta y\rangle &= (A_{xx}+A_{yy})\langle\delta x\delta y\rangle\\
&\quad + A_{yx}\langle(\delta x)^2\rangle+A_{xy}\langle(\delta y)^2\rangle+2D_{xy}\ ,
\end{aligned}\tag{3.1}$$

where $A_{ij}=\partial d_i/\partial j$ $(i,j=x,y)$ is defined by

$$\begin{aligned}
A_{xx} &= \mathrm{Re}(\Lambda_1-\Lambda_2-\Lambda_3+\Lambda_4)\ ,\\
A_{xy} &= -\mathrm{Im}(\Lambda_1-\Lambda_2+\Lambda_3-\Lambda_4)\ ,\\
A_{yx} &= \mathrm{Im}(\Lambda_1-\Lambda_2-\Lambda_3+\Lambda_4)\ ,\\
A_{yy} &= \mathrm{Re}(\Lambda_1-\Lambda_2+\Lambda_3-\Lambda_4)\ .
\end{aligned}\tag{3.2}$$

The interpretation of variances in terms of the operators is given in Table V. It is easy to show that

$$\begin{aligned}
&D_{jj}^P-\tfrac{1}{4}A_{jj}=D_{jj}^Q+\tfrac{1}{4}A_{jj}=D_{jj}^W,\quad j=x,y\\
&D_{xy}^P-\tfrac{1}{4}(A_{xy}+A_{yx})=D_{xy}^Q+\tfrac{1}{4}(A_{xy}+A_{yz})=D_{xy}^W\ .
\end{aligned}\tag{3.3}$$

The relations given in Table V and Eqs. (3.3) are sufficient to prove that the equations of motion for the quadrature variances $\langle(\Delta a_1)^2\rangle$ and $\langle(\Delta a_2)^2\rangle$ and covariance $\langle(\Delta a_1)(\Delta a_2)\rangle$ are identical irrespective of the choice of the representation, as expected. The relations (3.3) are quite interesting. For example, if A_{xx} is negative, then the diffusion coefficient D_{xx}^P in the P representation is smallest followed by those in the Wigner and Q representations. It should be remembered that Λ_1 (Λ_2) correspond to phase-insensitive gain (loss), and Λ_3 and Λ_4 correspond to phase-sensitive loss and gain. The sign of A_{xx} depends on the strength of the phase-sensitive terms compared to phase-insensitive ones.

The Fokker-Planck equation (2.7) has the drift coefficients which are linear in x and y and the diffusion coefficients which are constants, and hence its solution can be written down immediately provided that the diffusion matrix $\underline{D}$ is positive definite. If the system under consideration exhibits interesting nonclassical features, then the P distribution does not exist. However, the Q and W distributions exist. The Q and W distributions will be Gaussian if initially they are Gaussian.

In the following discussion, we choose f to be real, which is always possible by the replacement $a^\dagger\rightarrow a^\dagger e^{-i\arg f}$. We further assume that all Λ's are real (i.e., $A_{xy}=A_{yx}=D_{xy}=0$), so that a_1 and a_2 defined by Eq. (2.6) are the amplitude and phase quadrature operators, respectively. In order for the system to approach a steady state, the condition $\Lambda_1-\Lambda_2<-|\Lambda_3-\Lambda_4|$ must be met. Thus the steady-state solutions are

TABLE IV. The diffusion coefficients in terms of amplitude and phase variables r and ϕ [see Eq. (2.9)] in the P, Q, and W representations.

Φ	P[a]	Q[b]	W
D_{rr}	$\frac{1}{2}[\mathrm{Re}\Lambda_1+\lvert\Lambda_4\rvert\cos(\theta_4-2\phi)]$	$\frac{1}{2}[\mathrm{Re}\Lambda_2+\lvert\Lambda_3\rvert\cos(\theta_3-2\phi)]$	$\frac{1}{2}(D_{rr}^P+D_{rr}^Q)$
$D_{\phi\phi}$	$[\mathrm{Re}\Lambda_1-\lvert\Lambda_4\rvert\cos(\theta_4-2\phi)]/2r^2$	$[\mathrm{Re}\Lambda_2-\lvert\Lambda_3\rvert\cos(\theta_3-2\phi)]/2r^2$	$\frac{1}{2}(D_{\phi\phi}^P+D_{\phi\phi}^Q)$
$D_{r\phi}$	$(2r)^{-1}\lvert\Lambda_4\rvert\sin(\theta_4-2\phi)$	$(2r)^{-1}\lvert\Lambda_3\rvert\sin(\theta_3-2\phi)$	$\frac{1}{2}(D_{r\phi}^P+D_{r\phi}^Q)$

[a] $\Gamma_4=|\Gamma_4|e^{i\theta_4}$.
[b] $\Gamma_3=|\Gamma_3|e^{i\theta_3}$.

TABLE V. The relations among the variances and covariance of the field quadratures with those of the Glauber-Sudarshan P, the Q, and the Wigner W distributions.

Φ	P	Q	W
$\langle(\Delta a_1)^2\rangle$	$\langle(\delta x)^2\rangle+\frac{1}{4}$	$\langle(\delta x)^2\rangle-\frac{1}{4}$	$\langle(\delta x)^2\rangle$
$\langle(\Delta a_2)^2\rangle$	$\langle(\delta y)^2\rangle+\frac{1}{4}$	$\langle(\delta y)^2\rangle-\frac{1}{4}$	$\langle(\delta y)^2\rangle$
$\langle(\Delta a_1)(\Delta a_2)\rangle$	$\langle\delta x\delta y\rangle+\frac{i}{4}$	$\langle\delta x\delta y\rangle+\frac{i}{4}$	$\langle\delta x\delta y\rangle+\frac{i}{4}$

$$\Phi=\frac{1}{2\pi\sqrt{\sigma_1\sigma_2}}\exp\left[-\frac{(x-x_0)^2}{2\sigma_1}-\frac{y^2}{2\sigma_2}\right], \quad (3.4)$$

where

$$x_0=\frac{f}{\Lambda_2-\Lambda_1+\Lambda_3-\Lambda_4}, \quad (3.5)$$

and σ_1 and σ_2 for different representations are listed in Table VI. The following relations are noticed:

$$\sigma_j^W=\tfrac{1}{2}(\sigma_j^P+\sigma_j^Q) . \quad (3.6)$$

The quantity x_0 has the physical meaning of mean-field amplitude in the steady state, $x_0=\langle a\rangle=\langle a^\dagger\rangle$, independent of representations. This can be proved by either using Eq. (3.4) or directly solving Eq. (1.2) and its complex conjugate. The σ's represent the widths (or, say, variances) of the Gaussian distributions (3.4),

$$\langle(\delta x)^2\rangle=\sigma_1 , \quad \langle(\delta y)^2\rangle=\sigma_2 . \quad (3.7)$$

The distribution functions (3.4) can be used to calculate the quadrature variances by direct integrations. It follows from Table V and Eqs. (3.7) and (3.6) that

$$\langle(\Delta a_j)^2\rangle=\sigma_j^P+\tfrac{1}{4}=\sigma_j^Q-\tfrac{1}{4}=\sigma_j^W \quad (3.8a)$$

$$=\frac{\Lambda_1+\Lambda_2\pm(\Lambda_3+\Lambda_4)}{4[\Lambda_2-\Lambda_1\pm(\Lambda_3-\Lambda_4)]}, \quad (3.8b)$$

where the plus signs "+" are for $j=1$ and the minus signs "−" for $j=2$. Note that different representations give the same results, as expected. The relations among the widths of the distributions P, Q, and W can be found from Eqs. (3.8a) as $\sigma_j^Q>\sigma_j^W>\sigma_j^P$ $(j=1,2)$. The Heisenberg uncertainty principle $\langle(\Delta a_1)^2\rangle\langle(\Delta a_2)^2\rangle\geq\frac{1}{16}$ gives rise to the restrictions $4\sigma_1^Q\sigma_2^Q\geq\sigma_1^Q+\sigma_2^Q$ and $\sigma_1^W\sigma_2^W\geq\frac{1}{16}$ for the Q and W distributions, respectively. Squeezing occurs whenever $\sigma_j^P<0$ ($j=1$ or 2), i.e., $\Lambda_1<|\Lambda_4|$, and the P distribution in (3.4) diverges in such cases and is thus unacceptable.

The distributions (3.4) are also useful in the calculation of the photon-number statistics.[19] In particular, the mean photon number $\langle\hat{n}\rangle=\langle a^\dagger a\rangle$ and the Mandel parameter[20]

$$Q_M=\frac{\langle(\Delta\hat{n})^2\rangle-\langle\hat{n}\rangle}{\langle\hat{n}\rangle}=\langle\hat{n}\rangle[g^{(2)}(0)-1] \quad (3.9)$$

can be calculated easily. Here $\langle(\Delta\hat{n})^2\rangle$ is the photon-number variance and $g^{(2)}(0)$ the normalized second-order correlation function. The Mandel parameter Q_M has the property of being negative if the width of the photon-number distribution is narrower than Poissonian. Its lower bound is -1, corresponding to a pure number (Fock) state.

For the Q function in (3.4) we find

$$\begin{aligned}\langle\hat{n}\rangle&=\langle aa^\dagger\rangle-1=\langle x^2+y^2\rangle_Q-1\\&=x_0^2+\sigma_1^Q+\sigma_2^Q-1\\&=x_0^2+\sum_{j=1}^{2}\langle:(\Delta a_j)^2:\rangle ,\end{aligned} \quad (3.10a)$$

$$\begin{aligned}Q_M&=\frac{\langle aaa^\dagger a^\dagger\rangle-\langle aa^\dagger\rangle^2-\langle aa^\dagger\rangle}{\langle\hat{n}\rangle}-1\\&=\frac{\langle(x^2+y^2)^2\rangle_Q-\langle x^2+y^2\rangle_Q^2-\langle x^2+y^2\rangle_Q}{\langle\hat{n}\rangle}-1\\&=\frac{4x_0^2(\sigma_1^Q-\frac{1}{2})+2\sum_{j=1}^{2}(\sigma_j^Q-\frac{1}{2})^2}{x_0^2+\sigma_1^Q+\sigma_2^Q-1}\\&=\frac{4x_0^2\langle:(\Delta a_1)^2:\rangle+2\sum_{j=1}^{2}\langle:(\Delta a_j)^2:\rangle^2}{x_0^2+\sum_{j=1}^{2}\langle:(\Delta a_j)^2:\rangle} .\end{aligned} \quad (3.10b)$$

TABLE VI. The widths (variances) of the P, Q, W, and P_+ distributions in the steady state.

	σ_1	σ_2
P (no squeezing)	$\dfrac{\Lambda_1+\Lambda_4}{2(\Lambda_2-\Lambda_1+\Lambda_3-\Lambda_4)}$	$\dfrac{\Lambda_1-\Lambda_4}{2(\Lambda_2-\Lambda_1-\Lambda_3+\Lambda_4)}$
Q	$\dfrac{\Lambda_2+\Lambda_3}{2(\Lambda_2-\Lambda_1+\Lambda_3-\Lambda_4)}$	$\dfrac{\Lambda_2-\Lambda_3}{2(\Lambda_2-\Lambda_1-\Lambda_3+\Lambda_4)}$
W	$\dfrac{\Lambda_1+\Lambda_2+\Lambda_3+\Lambda_4}{4(\Lambda_2-\Lambda_1+\Lambda_3-\Lambda_4)}$	$\dfrac{\Lambda_1+\Lambda_2-\Lambda_3-\Lambda_4}{4(\Lambda_2-\Lambda_1-\Lambda_3+\Lambda_4)}$
P_+ (phase squeezing)	$\dfrac{\Lambda_1+\Lambda_4}{\Lambda_2-\Lambda_1+\Lambda_3-\Lambda_4}$	$\dfrac{\lvert\Lambda_1-\Lambda_4\rvert}{\Lambda_2-\Lambda_1-\Lambda_3+\Lambda_4}$
P_+ (amplitude squeezing)	$\dfrac{\lvert\Lambda_1+\Lambda_4\rvert}{\Lambda_2-\Lambda_1+\Lambda_3-\Lambda_4}$	$\dfrac{\Lambda_1-\Lambda_4}{\Lambda_2-\Lambda_1-\Lambda_3+\Lambda_4}$

Here : : denotes normal ordering of the field operators a and $a^\dagger$, $\langle :(\Delta a_j)^2: \rangle = \langle (\Delta a_j)^2 \rangle - \frac{1}{4}$, and use has been made of Eq. (3.8a). Note that the last expression in Eq. (3.10b) is representation independent. For a large amplitude $x_0^2 \gg \sigma_1^Q, \sigma_2^Q$, it follows from Eqs. (3.10) and (3.8a) that

$$\langle \hat{n} \rangle \simeq x_0^2 \ , \tag{3.11a}$$

$$Q_M \simeq 4(\sigma_1^Q - \tfrac{1}{2}) = 4\langle :(\Delta a_1)^2: \rangle \ . \tag{3.11b}$$

Consequently, sub-Poissonian statistics is equivalent to amplitude squeezing in this limit.[21] For arbitrary x_0^2, it is easy to see from Eq. (3.10b) that (i) sub-Poissonian statistics ($Q_M < 0$) always means the amplitude squeezing ($\sigma_1^Q < \frac{1}{2}$) but (ii) amplitude squeezing ($\sigma_1^Q < \frac{1}{2}$) may not imply sub-Poissonian statistics ($Q_M < 0$), especially for small σ_1^Q (which requires large σ_2^Q via the Heisenberg uncertainty principle) or small x_0^2.[7]

For the Wigner function W in (3.4), using the relations[14]

$$\langle \tfrac{1}{2}(a^\dagger a + a a^\dagger) \rangle = \langle x^2 + y^2 \rangle_W \ , \tag{3.12a}$$

$$\tfrac{1}{6}\langle a^{\dagger 2} a^2 + a^\dagger a a^\dagger a + a^\dagger a^2 a^\dagger + a a^{\dagger 2} a + a a^\dagger a a^\dagger + a^2 a^{\dagger 2} \rangle = \langle (x^2+y^2)^2 \rangle_W \ , \tag{3.12b}$$

and $[a, a^\dagger] = 1$, one obtains[15]

$$\langle \hat{n} \rangle = \langle x^2 + y^2 \rangle_W - \tfrac{1}{2} = x_0^2 + \sigma_1^W + \sigma_2^W - \tfrac{1}{2} = x_0^2 + \sum_{j=1}^{2} \langle :(\Delta a_j)^2: \rangle \ , \tag{3.13a}$$

$$Q_M = \frac{\langle (x^2+y^2)^2 \rangle_W - \langle x^2+y^2 \rangle_W^2 - \langle \hat{n} \rangle^2 - \langle \hat{n} \rangle}{\langle \hat{n} \rangle} = \frac{4x_0^2(\sigma_1^W - \frac{1}{4}) + 2\sum_{j=1}^{2}(\sigma_j^W - \frac{1}{4})^2}{x_0^2 + \sigma_1^W + \sigma_2^W - \frac{1}{2}} = \frac{4x_0^2 \langle :(\Delta a_1)^2: \rangle + 2\sum_{j=1}^{2} \langle :(\Delta a_j)^2: \rangle^2}{x_0^2 + \sum_{j=1}^{2} \langle :(\Delta a_j)^2: \rangle} \ , \tag{3.13b}$$

which are the same as those obtained from the Q function. When $x_0 \gg 1, \sigma_1^W, \sigma_2^W$, Eqs. (3.13) reduce to

$$\langle \hat{n} \rangle \simeq x_0^2, \tag{3.14a}$$

$$Q_M \approx 4(\sigma_1^W - \tfrac{1}{4}) = 4\langle :(\Delta a_1)^2: \rangle \ , \tag{3.14b}$$

which are identical to Eqs. (3.11). A comparison of Eq. (3.13b) with Eq. (3.10b) indicates that $\sigma_j^Q - \frac{1}{4} = \sigma_j^W$ ($j = 1,2$), in agreement with Eq. (3.8a).

IV. EQUATIONS AND SOLUTIONS FOR THE GENERALIZED *P* DISTRIBUTIONS

When the diffusion matrix of the Fokker-Planck equation (2.4) in the Glauber-Sudarshan P representation is not semipositive definite (i.e., $\Lambda_1^2 < \Lambda_4^2$), which indicates the appearance of squeezed states of light, the Glauber-Sudarshan P distribution does not exist as a well-behaved function (c.f. Sec. III). In this situation, the generalized P distributions are well-behaved, normal ordering functions.

We next discuss the generalized P distributions for the master equation (1.1). The generalized P distributions exist for the present problem and are very useful for systems exhibiting nonclassical features. The generalized P distributions $P(\alpha,\beta)$ are nondiagonal expansions of the density operator in terms of coherent states

$$\rho = \int_D \int d\mu(\alpha,\beta) P(\alpha,\beta) \frac{|\alpha\rangle\langle\beta^*|}{\langle\beta^*|\alpha\rangle} \ , \tag{4.1}$$

where $\alpha = \alpha_x + i\alpha_y$, $\beta = \beta_x + i\beta_y$, $d\mu(\alpha,\beta)$ is an integration measure, and D is the domain of integrations. Two choices can be made regarding the integration measure $d\mu(\alpha,\beta)$. The first one is $d\mu(\alpha,\beta) = d\alpha\, d\beta$, where α and β are to be integrated on some contours C and C', respectively. This leads to what is called the complex P representation, $P_c(\alpha,\beta)$. The second is $d\mu(\alpha,\beta) = d^2\alpha\, d^2\beta$, giving rise to the positive P representation, $P_+(\{\alpha\})$. Here $\{\alpha\}$ represents α_x, β_x, α_y, and β_y collectively.

The transformation rules for converting a master equation into the corresponding Fokker-Planck equation in the generalized P representation have been given in Table I. For the master equation (1.1), the corresponding Fokker-Planck equation for the complex $P_c(\alpha,\beta)$ function is Eq. (2.4) with replacement $\alpha^* \to \beta$, and the Fokker-Planck equation for the positive $P_+(\{\alpha\})$ function can be obtained by rewriting that for the complex $P_c(\alpha,\beta)$ function.[16–18] For simplicity, we assume in the following discussion that f and all Λ's are real, as in Sec. III.

A. Complex *P* representation

The Fokker-Planck equation in the complex $P_c(\alpha,\beta)$ representation for the master equation (1.1) is

$$\frac{\partial P_c(\alpha,\beta)}{\partial t} = \left[-\frac{\partial}{\partial\alpha} d_\alpha - \frac{\partial}{\partial\beta} d_\beta + 2\frac{\partial^2}{\partial\alpha\partial\beta} D_{\beta\alpha} + \frac{\partial^2}{\partial\alpha^2} D_{\alpha\alpha} + \frac{\partial^2}{\partial\beta^2} D_{\beta\beta} \right] P_c(\alpha,\beta) \ , \tag{4.2}$$

with the following drift and diffusion coefficients:

$$d_\alpha = (\Lambda_1 - \Lambda_2)\alpha + (\Lambda_4 - \Lambda_3)\beta + f \ , \tag{4.3a}$$

$$d_\beta = (\Lambda_1 - \Lambda_2)\beta + (\Lambda_4 - \Lambda_3)\alpha + f, \tag{4.3b}$$

$$D_{\beta\alpha} = \Lambda_1, \quad D_{\alpha\alpha} = D_{\beta\beta} = \Lambda_4 \ . \tag{4.4}$$

The steady-state solution of Eq. (4.2) is (when $\Lambda_1 - \Lambda_2 < -|\Lambda_3 - \Lambda_4|$)

$$P_c(\alpha,\beta) = N \exp\{(\Lambda_4^2 - \Lambda_1^2)^{-1} \times [(\Lambda_1\Lambda_2 - \Lambda_1^2 + \Lambda_4^2 - \Lambda_3\Lambda_4)\alpha\beta - \tfrac{1}{2}(\Lambda_2\Lambda_4 - \Lambda_1\Lambda_3)(\alpha^2 + \beta^2) + f(\Lambda_4 - \Lambda_1)(\alpha + \beta)]\} \ , \tag{4.5a}$$

which reduces to

$$P_c(\alpha,\beta)=N_0\exp\left\{\frac{\Lambda_2-\Lambda_1}{\Lambda_4^2-\Lambda_1^2}[\Lambda_1\alpha\beta-\tfrac{1}{2}\Lambda_4(\alpha^2+\beta^2)-x_0(\Lambda_1-\Lambda_4)(\alpha+\beta)]\right\}, \tag{4.5b}$$

if $\Lambda_3=\Lambda_4$. Here N and N_0 are normalization constants which depend on the integration contours C and C'. Notice that when there is no squeezing (i.e., $\Lambda_1^2>\Lambda_4^2$), one can set $\beta=\alpha^*$ in Eqs. (4.5). Thus Eq. (4.5a) reduces to the Glauber-Sudarshan P distribution in Eq. (3.4) and the integration in this case is over the whole α plane. The integration contours C and C' are to be chosen for the phase- and amplitude-squeezing cases separately. When the system exhibits phase squeezing ($\Lambda_1<\Lambda_4$), the integration contours are C, $\alpha_y=0$; and C', $\beta_y=0$. On the other hand, when the amplitude squeezing occurs ($\Lambda_1+\Lambda_4<0$), the integration contours are C, $\alpha_x=x_0$; and C', $\beta_x=x_0$. It is possible to calculate various operator moments by using the complex $P_c(\alpha,\beta)$ functions (4.5). We shall return to this point in Sec. IV C after investigating the positive P function.

B. Positive *P* representation

The Fokker-Planck equation in the positive $P_+(\{\alpha\})$ representation for the master equation (1.1) is

$$\frac{\partial P_+(\{\alpha\})}{\partial t}=\left\{-\frac{\partial}{\partial\alpha_x}d_x^\alpha-\frac{\partial}{\partial\beta_x}d_x^\beta-\frac{\partial}{\partial\alpha_y}d_y^\alpha-\frac{\partial}{\partial\beta_y}d_y^\beta\right\}P_+(\{\alpha\})+\text{diffusion terms}, \tag{4.6}$$

with drift coefficients

$$\begin{aligned}d_x^\alpha&=(\Lambda_1-\Lambda_2)\alpha_x+(\Lambda_4-\Lambda_3)\beta_x+f,\\ d_x^\beta&=(\Lambda_1-\Lambda_2)\beta_x+(\Lambda_4-\Lambda_3)\alpha_x+f,\\ d_y^\alpha&=(\Lambda_1-\Lambda_2)\alpha_y+(\Lambda_4-\Lambda_3)\beta_y,\\ d_y^\beta&=(\Lambda_1-\Lambda_2)\beta_y+(\Lambda_4-\Lambda_3)\alpha_y.\end{aligned} \tag{4.7}$$

The diffusion terms are different for the phase- and amplitude-squeezing cases. We give their derivation in the Appendix. We discuss the two situations separately in the following.

Case 1. Phase squeezing ($\Lambda_4>\Lambda_1>0$). The Fokker-Planck equation for the positive P distribution can be written as (see the Appendix)

$$\frac{\partial P_+(\{\alpha\})}{\partial t}=\left\{-\frac{\partial}{\partial\alpha_x}d_x^\alpha-\frac{\partial}{\partial\beta_x}d_x^\beta-\frac{\partial}{\partial\alpha_y}d_y^\alpha-\frac{\partial}{\partial\beta_y}d_y^\beta+\Lambda_4\frac{\partial^2}{\partial\alpha_x^2}+\Lambda_4\frac{\partial^2}{\partial\beta_x^2}+2\Lambda_1\frac{\partial^2}{\partial\alpha_x\partial\beta_x}\right\}\times P_+(\{\alpha\}). \tag{4.8}$$

Notice that the diffusion matrix $\underline{D}_{4\times4}$ is now semipositive definite. In order to solve Eq. (4.8) we make the transformation

$$\mathcal{E}_1=(\alpha_x+\beta_x)/\sqrt{2},\quad \mathcal{E}_2=(\beta_x-\alpha_x)/\sqrt{2}; \tag{4.9}$$

then Eq. (4.8) reduces to

$$\dot P_+=\left\{-\frac{\partial}{\partial\mathcal{E}_1}[(\Lambda_1-\Lambda_2-\Lambda_3+\Lambda_4)\mathcal{E}_1+\sqrt{2}f]-\frac{\partial}{\partial\mathcal{E}_2}(\Lambda_1-\Lambda_2+\Lambda_3-\Lambda_4)\mathcal{E}_2-\frac{\partial}{\partial\alpha_y}d_y^\alpha-\frac{\partial}{\partial\beta_y}d_y^\beta+(\Lambda_1+\Lambda_4)\frac{\partial^2}{\partial\mathcal{E}_1^2}+|\Lambda_1-\Lambda_4|\frac{\partial^2}{\partial\mathcal{E}_2^3}\right\}P_+. \tag{4.10}$$

The steady-state solution has the form (when $\Lambda_1-\Lambda_2<-|\Lambda_3-\Lambda_4|$)

$$\begin{aligned}P_+&=\frac{1}{2\pi\sqrt{\sigma_{\text{ph},1}\sigma_{\text{ph},2}}}\exp\left\{-\frac{(\mathcal{E}_1-\sqrt{2}x_0)^2}{2\sigma_{\text{ph},1}}-\frac{\mathcal{E}_2^2}{2\sigma_{\text{ph},2}}\right\}\delta(\alpha_y)\delta(\beta_y)\\ &=\frac{1}{2\pi\sqrt{\sigma_{\text{ph},1}\sigma_{\text{ph},2}}}\exp\left\{-\frac{(\alpha_x+\beta_x-2x_0)^2}{4\sigma_{\text{ph},1}}-\frac{(\beta_x-\alpha_x)^2}{4\sigma_{\text{ph},2}}\right\}\delta(\alpha_y)\delta(\beta_y),\end{aligned} \tag{4.11}$$

where $\sigma_{\text{ph},1}$ and $\sigma_{\text{ph},2}$ have been listed in Table VI (ph means phase squeezing). Here x_0 still has the physical meaning of mean-field amplitude, $\langle a\rangle=\langle\alpha_x\rangle=x_0=\langle a^\dagger\rangle$. It is straightforward to obtain from the Heisenberg uncertainty principle that $\sigma_{\text{ph},1}\geq\sigma_{\text{ph},2}/(1-2\sigma_{\text{ph},2})>\sigma_{\text{ph},2}$. A pictorial representation of the quadratic form in Eq. (4.11) is shown in Fig. 1(a). A comparison of σ_j's in Table VI shows that

$$\begin{aligned}\sigma_{\text{ph},1}&=2\sigma_1^P,\\ \sigma_{\text{ph},2}&=-2\sigma_2^P.\end{aligned} \tag{4.12}$$

The quadrature variances can be calculated by using (4.11) and performing direct integrations

$$\begin{aligned}\langle(\Delta a_j)^2\rangle&=\tfrac{1}{4}+\langle:(\Delta a_j)^2:\rangle\\ &=\tfrac{1}{4}-\tfrac{1}{2}(-1)^j\langle(\delta\mathcal{E}_j)^2\rangle_{P_+}\\ &=\tfrac{1}{4}-\tfrac{1}{2}(-1)^j\sigma_{\text{ph},j}\\ &=\tfrac{1}{4}+\sigma_j^P,\quad j=1,2.\end{aligned} \tag{4.13}$$

Here use has been made of Eqs. (4.12). Equations (4.13) give rise to the same results as those in Eqs. (3.8). The

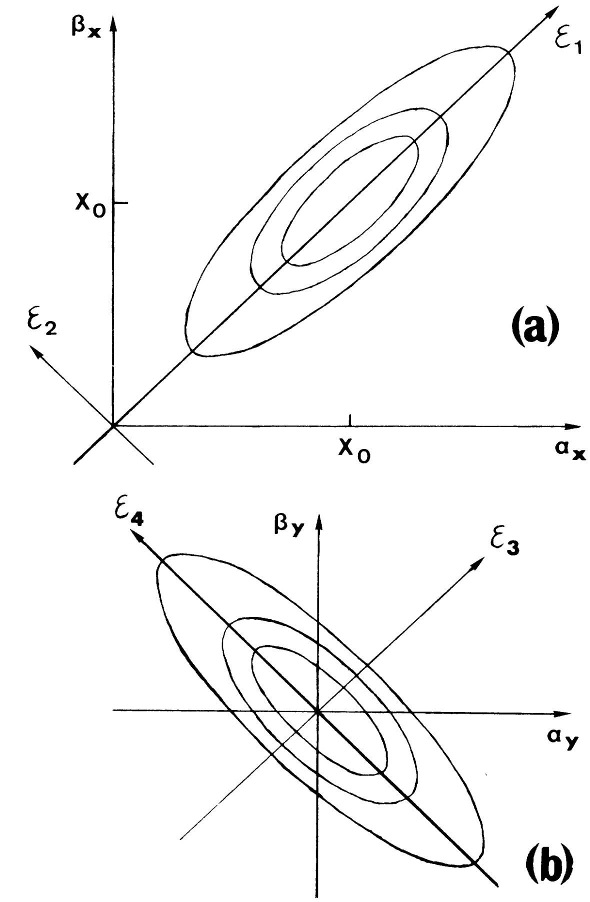

FIG. 1. Illustration of the positions and shapes of two-dimensional Gaussians in the positive P representation for squeezed states obeying $\langle(\Delta a_1)^2\rangle\langle(\Delta a_2)^2\rangle=\frac{1}{8}$ (i.e., twice the quantum limit). (a) Phase squeezing: $(2\sigma_{\mathrm{ph},1})^{-1}(\mathcal{E}_1-\sqrt{2}x_0)^2+(2\sigma_{\mathrm{ph},2})^{-1}\mathcal{E}_2^2=1$ [see Eq. (4.11)]; (b) amplitude squeezing: $(2\sigma_{\mathrm{am},1})^{-1}\mathcal{E}_3^2+(2\sigma_{\mathrm{am},2})^{-1}\mathcal{E}_4^2=1$ [see Eq. (4.19)]. In both cases three lines represent, from inside to outside, 20%, 50%, and 80% squeezing, respectively.

mean photon number and the Mandel parameter Q_M can also be calculated as

$$\begin{aligned}\langle\hat{n}\rangle=\langle a^\dagger a\rangle=\langle\beta\alpha\rangle_{P_+}&=\langle\beta_x\alpha_x\rangle_{P_+}\\&=\tfrac{1}{2}\langle\mathcal{E}_1^2-\mathcal{E}_2^2\rangle_{P_+}\\&=\tfrac{1}{2}(2x_0^2+\sigma_{\mathrm{ph},1}-\sigma_{\mathrm{ph},2})\\&=x_0^2+\sigma_1^P+\sigma_2^P\\&=x_0^2+\sum_{j=1}^{2}\langle:(\Delta a_j)^2:\rangle\ ,\end{aligned}\tag{4.14}$$

$$\begin{aligned}Q_M&=\frac{\langle a^\dagger a^\dagger aa\rangle-\langle a^\dagger a\rangle^2}{\langle\hat{n}\rangle}\\&=\frac{2x_0^2\sigma_{\mathrm{ph},1}+\frac{1}{2}(\sigma_{\mathrm{ph},1}^2+\sigma_{\mathrm{ph},2}^2)}{\langle\hat{n}\rangle}\\&=\frac{4x_0^2\sigma_1^P+2(\sigma_1^P)^2+2(\sigma_2^P)^2}{x_0^2+\sigma_1^P+\sigma_2^P}\\&=\frac{4x_0^2\langle:(\Delta a_1)^2:\rangle+2\sum_{j=1}^{2}\langle:(\Delta a_j)^2:\rangle^2}{x_0^2+\sum_{j=1}^{2}\langle:(\Delta a_j)^2:\rangle}\ ,\end{aligned}\tag{4.15}$$

which also agrees with the previous results obtained from other functions.

Case 2. Amplitude squeezing ($\Lambda_4<-\Lambda_1<0$). The Fokker-Planck equation for the positive P_+ distribution is (see the Appendix)

$$\frac{\partial P_+(\{\alpha\})}{\partial t}=\left[-\frac{\partial}{\partial\alpha_x}d_x^\alpha-\frac{\partial}{\partial\beta_x}d_x^\beta-\frac{\partial}{\partial\alpha_y}d_y^\alpha-\frac{\partial}{\partial\beta_y}d_y^\beta-\Lambda_4\frac{\partial^2}{\partial\alpha_y^2}-\Lambda_4\frac{\partial^2}{\partial\beta_y^2}-2\Lambda_1\frac{\partial^2}{\partial\alpha_y\partial\beta_y}\right]\times P_+(\{\alpha\})\ .\tag{4.16}$$

The diffusion matrix $\underline{D}_{4\times4}$ is again semipositive definite. We now use a different transformation to solve Eq. (4.16). Let us define $\mathcal{E}_3$ and $\mathcal{E}_4$ by

$$\mathcal{E}_3=(\alpha_y+\beta_y)/\sqrt{2},\quad\mathcal{E}_4=(\beta_y-\alpha_y)/\sqrt{2}\ ,\tag{4.17}$$

then Eq. (4.16) becomes

$$\dot{P}_+=\left[-\frac{\partial}{\partial\alpha_x}d_x^\alpha-\frac{\partial}{\partial\beta_x}d_x^\beta-\frac{\partial}{\partial\mathcal{E}_3}(\Lambda_1-\Lambda_2-\Lambda_3+\Lambda_4)\mathcal{E}_3-\frac{\partial}{\partial\mathcal{E}_4}(\Lambda_1-\Lambda_2+\Lambda_3-\Lambda_4)\mathcal{E}_4+|\Lambda_1+\Lambda_4|\frac{\partial^2}{\partial\mathcal{E}_3^2}+(\Lambda_1-\Lambda_4)\frac{\partial^2}{\partial\mathcal{E}_4^2}\right]P_+\ .\tag{4.18}$$

The steady-state solution is (if $\Lambda_1-\Lambda_2<-|\Lambda_3-\Lambda_4|$)

$$\begin{aligned}P_+&=\frac{1}{2\pi\sqrt{\sigma_{\mathrm{am},1}\sigma_{\mathrm{am},2}}}\exp\left[-\frac{\mathcal{E}_3^2}{2\sigma_{\mathrm{am},1}}-\frac{\mathcal{E}_4^2}{2\sigma_{\mathrm{am},2}}\right]\delta(\alpha_x-x_0)\delta(\beta_x-x_0)\\&=\frac{1}{2\pi(\sigma_{\mathrm{am},1}\sigma_{\mathrm{am},2})^{1/2}}\exp\left[-\frac{(\alpha_y+\beta_y)^2}{4\sigma_{\mathrm{am},1}}-\frac{(\beta_y-\alpha_y)^2}{4\sigma_{\mathrm{am},2}}\right]\delta(\alpha_x-x_0)\delta(\beta_x-x_0)\ ,\end{aligned}\tag{4.19}$$

where $\sigma_{\mathrm{am},1}$ and $\sigma_{\mathrm{am},2}$ have been given in Table VI (am represents amplitude squeezing). A comparison of σ_j's in Table VI gives rise to

$$\sigma_{\mathrm{am},1}=-2\sigma_1^P\ ,\qquad \sigma_{\mathrm{am},2}=2\sigma_2^P\ . \tag{4.20}$$

Once again we have $\langle a\rangle=\langle a^\dagger\rangle=x_0$. One can find from the Heisenberg uncertainty principle that $\sigma_{\mathrm{am},2}\geq\sigma_{\mathrm{am},1}/(1-2\sigma_{\mathrm{am},1})>\sigma_{\mathrm{am},1}$. A pictorial illustration of the quadratic form in Eq. (4.19) is given in Fig. 1(b).

The calculation of quadrature variances from the distribution (4.19) is straightforward:

$$\begin{aligned}\langle(\Delta a_j)^2\rangle&=\tfrac{1}{4}+\langle:(\Delta a_j)^2:\rangle\\&=\tfrac{1}{4}+\tfrac{1}{2}(-1)^j\langle\mathcal{E}_{j+2}^2\rangle_{P_+}\\&=\tfrac{1}{4}+\tfrac{1}{2}(-1)^j\sigma_{\mathrm{am},j}=\tfrac{1}{4}+\sigma_j^P,\quad j=1,2\ .\end{aligned} \tag{4.21}$$

In the last step above, Eqs. (4.20) have been used. One sees that the same results as those given by Eqs. (3.8) are obtained. The average photon number and the Mandal parameter Q_M can also be found:

$$\begin{aligned}\langle\hat n\rangle=\langle\beta\alpha\rangle&=x_0^2-\langle\beta_y\alpha_y\rangle\\&=x_0^2-\tfrac{1}{2}\langle\mathcal{E}_3^2-\mathcal{E}_4^2\rangle\\&=x_0^2-\tfrac{1}{2}(\sigma_{\mathrm{am},1}-\sigma_{\mathrm{am},2})=x_0^2+\sigma_1^P+\sigma_2^P\ ,\end{aligned} \tag{4.22}$$

$$\begin{aligned}Q_M&=\frac{\langle a^\dagger a^\dagger aa\rangle-\langle a^\dagger a\rangle^2}{\langle\hat n\rangle}\\&=\frac{2x_0^2|\sigma_{\mathrm{am},1}|+\frac{1}{2}(\sigma_{\mathrm{am},1}^2+\sigma_{\mathrm{am},2}^2)}{\langle\hat n\rangle}\\&=\frac{4x_0^2\sigma_1^P+2(\sigma_1^P)^2+2(\sigma_2^P)^2}{x_0^2+\sigma_1^P+\sigma_2^P},\end{aligned} \tag{4.23}$$

which are the same as Eqs. (4.14) and (4.15).

C. Connections among three *P* representations

So far we have discussed three P representations: the Glauber-Sudarshan P representation $P(\alpha)$, the complex P representation $P_c(\alpha,\beta)$, and the positive P representation $P_+(\{\alpha\})$. The Glauber-Sudarshan P representation is a good, simple representation for the classical state, whereas the generalized P representations are useful, genuine representations for nonclassical states. It is interesting to note the connections among the three P representations. (i) When there is no squeezing, one can obtain the Glauber-Sudarshan P function from the complex P function by setting $\beta=\alpha^*$ in it and discarding the integration contours C and C',

$$P_c(\alpha,\alpha^*)=P(\alpha)\ . \tag{4.24}$$

An example of this relation is two distributions presented in Eqs. (4.5a) and (3.4). (ii) When squeezing exists, the positive P function may be obtained from the complex P function by substituting the integration contours C and C' into it and multiplying it by two δ functions. For example, one finds from Eqs. (4.5a), (4.11), and (4.19) that for phase squeezing,

$$P_+(\alpha_x,\beta_x,\alpha_y,\beta_y)=P_c(\alpha,\beta)|_{\alpha_y=\beta_y=0}\delta(\alpha_y)\delta(\beta_y)\ , \tag{4.25a}$$

and for the amplitude squeezing

$$P_+(\alpha_x,\beta_x,\alpha_y,\beta_y)=P_c(\alpha,\beta)|_{\alpha_x=\beta_x=x_0}\delta(\alpha_x-x_0)\delta(\beta_x-x_0)\ . \tag{4.25b}$$

To our knowledge, simple relations (4.25) connecting the complex and positive P distributions are found for the first time.

With relations (4.25) it is easy to see that, for calculating $\langle a\rangle$, $\langle\hat n\rangle$, $\langle(\Delta a_j)^2\rangle$, and Q_M by using the complex P function (4.5a), one obtains the same results as those by using the positive P functions (4.11) and (4.19).

V. TWO-PHOTON CORRELATED-EMISSION LASER

The results of the early sections are applicable to a wide variety of optical systems exhibiting phase locking, such as various types of (single-mode) correlated-emission lasers, degenerate parametric amplifiers, systems involving down conversion, etc. We illustrate some of our results by applying the formalism of the preceding sections to the two-photon correlated-emission laser[1] (two-photon CEL). The two-photon CEL consists of coherently pumped, cascade three-level atoms interacting with a single mode of the radiation field. We consider the situation where the jth atom is injected into the laser cavity at time t_j with initial populations ρ_{aa}, ρ_{bb}, and ρ_{cc} and initial coherences $\rho_{ab}^j(t_j)=[\rho_{ba}^j(t_j)]^*=\bar\rho_{ab}e^{-i\nu t_j}$, $\rho_{bc}^j(t_j)=[\rho_{cb}^j(t_j)]^*=\bar\rho_{bc}e^{-i\nu t_j}$, and $\rho_{ac}^j(t_j)=[\rho_{ca}^j(t_j)]^*=\bar\rho_{ac}e^{-i2\nu t_j}$, where a, b, and c denote the top, middle, and bottom levels, respectively, ν is the laser frequency, and $\bar\rho_{ab}$, $\bar\rho_{bc}$, and $\bar\rho_{ac}$ are the same for all atoms. We assume that the cavity-mode frequency is on resonance with both $a-b$ and $b-c$ transitions so that $\nu=\omega_a-\omega_b=\omega_b-\omega_c$ (see Fig. 2). Here $\hbar\omega_l$ $(l=a,b,c)$ is the energy of the atomic level l. The master equation for the reduced field density operator can be obtained, which is of the form of Eq. (1.1) with the coefficients Λ's and f given below:[22]

$$\begin{aligned}f&=-is(\bar\rho_{ab}+\bar\rho_{bc})\ ,\\\Lambda_1&=\tfrac{1}{2}\alpha_0(\rho_{aa}+\rho_{bb}-\tfrac{1}{2}|\bar\rho_{ab}+\bar\rho_{bc}|^2)\ ,\\\Lambda_2&=\tfrac{1}{2}\alpha_0(\rho_{bb}+\rho_{cc}-\tfrac{1}{2}|\bar\rho_{ab}+\bar\rho_{bc}|^2)+\tfrac{1}{2}\gamma\ ,\\\Lambda_3&=\Lambda_4=-\tfrac{1}{2}\alpha_0[\bar\rho_{ac}-\tfrac{1}{2}(\bar\rho_{ab}+\bar\rho_{bc})^2]\ .\end{aligned} \tag{5.1}$$

Here $\alpha_0=2r_ag^2/\Gamma^2$ is the linear gain coefficient, γ is the cavity loss rate, $s=r_ag/\Gamma$, r_a is the atomic injection rate, g is the atom-field coupling constant (for simplicity taken to be the same for the $a-b$ and $b-c$ transitions), and Γ is the atomic decay rate (same for all levels).

In the following we restrict our discussion to a situation in which the linear gain $\alpha_0(\rho_{aa}-\rho_{cc})$ is less than the loss γ, i.e., $\Lambda_1<\Lambda_2$. In this case, the laser intensity is maintained by the atomic coherences $\bar{\rho}_{ab}$ and $\bar{\rho}_{bc}$ involving the middle level b, so that such a resonant two-photon CEL is still an active device. For simplicity, we assume that the two initial atomic coherences $\bar{\rho}_{ab}$ and $\bar{\rho}_{bc}$ have the same phase. Let $\theta_{ij}=\arg\bar{\rho}_{ij}$ $(i,j=a,b,c)$, then the choice $\theta_{ab}=\theta_{bc}=\pi/2$ leads to f and all Λ's real (note $\theta_{ab}=\theta_{ab}+\theta_{bc}$), and all previous steady-state distribution functions apply here.

It follows from Table III and Eqs. (5.1) that the diffusion coefficients in the Glauber-Sudarshan P representation are

$$D_{xx}^P=\tfrac{1}{4}\alpha_0[\rho_{aa}+\rho_{bb}+|\bar{\rho}_{ac}|-(|\bar{\rho}_{ab}|+|\bar{\rho}_{bc}|)^2]\ , \tag{5.2a}$$

$$D_{yy}^P=\tfrac{1}{4}\alpha_0(\rho_{aa}+\rho_{bb}-|\bar{\rho}_{ac}|)\ , \tag{5.2b}$$

$$D_{xy}^P=0\ . \tag{5.2c}$$

While D_{xx}^P is always positive, D_{yy}^P becomes negative when $\rho_{aa}+\rho_{bb}<|\bar{\rho}_{ac}|$, indicating squeezing in the phase quadrature a_2. The explicit expressions for the quadrature variances are found by substituting Eqs. (5.1) into Eq. (3.8b),

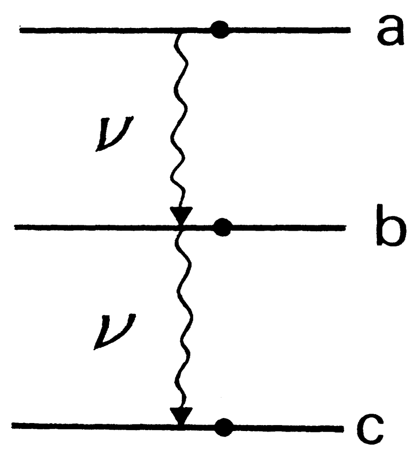

FIG. 2. Energy-level diagram for the two-photon CEL with $\nu=\omega_a-\omega_b=\omega_b-\omega_c$. Atoms are prepared initially in a coherent superposition of levels a, b, and c.

$$\langle(\Delta a_1)^2\rangle=\frac{\rho_{aa}+2\rho_{bb}+\rho_{cc}+2|\bar{\rho}_{ac}|-2(|\bar{\rho}_{ab}|+|\bar{\rho}_{bc}|)^2+\gamma/\alpha_0}{4(\rho_{cc}-\rho_{aa}+\gamma/\alpha_0)}\ , \tag{5.3a}$$

$$\langle(\Delta a_2)^2\rangle=\frac{\rho_{aa}+2\rho_{bb}+\rho_{cc}-2|\bar{\rho}_{ac}|+\gamma/\alpha_0}{4(\rho_{cc}-\rho_{aa}+\gamma/\alpha_0)}\ . \tag{5.3b}$$

The mean amplitude follows from Eqs. (3.5) and (5.1),

$$x_0=\frac{2s(|\bar{\rho}_{ab}|+|\bar{\rho}_{bc}|)}{\gamma+\alpha_0(\rho_{cc}-\rho_{aa})}\ . \tag{5.4}$$

For the initial atomic populations

$$\rho_{aa}=\tfrac{1}{2}[1-\epsilon-(\lambda-1)\gamma/\alpha_0]\ ,$$
$$\rho_{bb}=\lambda\gamma/\alpha_0\ , \tag{5.5}$$
$$\rho_{cc}=\tfrac{1}{2}[1+\epsilon-(\lambda+1)\gamma/\alpha_0]$$

and coherences $|\bar{\rho}_{ij}|=(\rho_{ii}\rho_{jj})^{1/2}$ $(i,j=a,b,c)$ with $\epsilon=[2(2\lambda+1)\gamma/\alpha_0]^{1/2}\ll1$ and $(\gamma/\alpha_0)^{1/2}\ll1$, we find

$$\langle(\Delta a_1)^2\rangle\simeq(2\epsilon)^{-1}\ , \tag{5.6a}$$

$$\langle(\Delta a_2)^2\rangle\simeq\tfrac{1}{4}\epsilon\ll\tfrac{1}{4}\ , \tag{5.6b}$$

$$\langle(\Delta a_1)^2\rangle\langle(\Delta a_2)^2\rangle\simeq\tfrac{1}{8}\ , \tag{5.6c}$$

which indicates nearly 100% intracavity phase squeezing and the product of the variances being twice the quantum limit set by the Heisenberg uncertainty principle. Meanwhile, the mean laser amplitude and mean photon number are

$$\langle a\rangle=x_0\simeq\left[\frac{\lambda}{2\lambda+1}\right]^{1/2}\frac{2s}{\alpha_0}=\left[\frac{\lambda}{2\lambda+1}\right]^{1/2}\frac{\Gamma}{g}\ , \tag{5.7a}$$

$$\langle\hat{n}\rangle\simeq x_0^2+\langle(\Delta a_1)^2\rangle\ . \tag{5.7b}$$

To ensure the linear master equation valid, we restrict $\lambda\ll1$ here.

To achieve larger squeezing one should reduce λ, which, on the other hand, decreases the mean amplitude and mean photon number too. The minimum quadrature noise is reached when $\lambda=0$ (i.e., $\rho_{bb}=\bar{\rho}_{ab}=\bar{\rho}_{bc}=0$),

$$\langle(\Delta a_2)^2\rangle_{\min}=\tfrac{1}{4}(2\gamma/\alpha_0)^{1/2}\ . \tag{5.8}$$

In this case, the mean amplitude vanishes, $\langle a\rangle=x_0=0$, and accompanying distributions are all centered at the origin of respective coordinate systems. The mean photon number in this case is just provided by noises

$$\langle\hat{n}\rangle=\sum_{j=1}^{2}\langle:(\Delta a_j)^2:\rangle\simeq(2\epsilon)^{-1}\ . \tag{5.9}$$

VI. DISCUSSION

Our study so far is based on the linear master equation (1.1). For a nonlinear master equation one can linearize it by setting $a=\hat{A}+\langle a\rangle$, with $\langle a\rangle$ being the mean amplitude, and dropping terms containing three or more operators $\hat{A}$ and $\hat{A}^\dagger$. This procedure will lead to a new field master equation of the type (1.1) with $f=0$ and $a\ (a^\dagger)\to\hat{A}\ (\hat{A}^\dagger)$. Our general results presented in this work are applicable to this linearized master equation.

In summary, starting from the master equation (1.1) we have derived the corresponding Fokker-Planck equations in the Glauber-Sudarshan P, the antinormal ordering Q, the Wigner W, the complex P, and the positive P representations and made a comparative study of these quasiprobability distributions. We have proven that the quadrature variances found from the Fokker-Planck equations in the Glauber-Sudarshan P, the Q, and the Wigner representations are the same. Let $a=a_1+ia_2$, with a_1 and a_2 being the amplitude and phase quadrature operators of the field, respectively, and assuming all Λ's real, we have solved the five Fokker-Planck equations in the steady state. The Glauber-Sudarshan P function (when there is no squeezing), the Q function, and the Wigner functions (3.4) are two-dimensional Gaussian with different widths. When squeezing occurs, the positive P functions (4.11) and (4.19) consist of a two-dimensional Gaussian and two δ functions for both the phase- and amplitude-squeezing cases. Simple relations (4.25) between the complex P functions (4.5a) and the positive P functions (4.11) and (4.19) are found for the first time. The mean amplitude and mean photon number of the field, the variances in the amplitude and phase quadratures, and the Mandel parameter Q_M (i.e., photon-number variance) directly calculated from these distribution functions (when applicable) are identical. Moreover, we have applied our general formalism to the (resonant) two-photon CEL. For the initial atomic conditions (5.5) nearly perfect intracavity squeezing can be achieved in the phase quadrature. We find further, correspondingly, that (i) the product of the amplitude and phase noise is just twice the quantum limit set by the Heisenberg uncertainty principle, and (ii) maximum quadrature squeezing is reached when $\langle a \rangle = 0$.

ACKNOWLEDGMENTS

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APPENDIX: DERIVATION OF THE DIFFUSION COEFFICIENTS IN THE POSITIVE P REPRESENTATION

Since a symmetric matrix can always be factorized into the product of another matrix and its adjoint, we write

$$\begin{bmatrix} D_{\alpha\alpha} & D_{\beta\alpha} \\ D_{\beta\alpha} & D_{\beta\beta} \end{bmatrix} = \begin{bmatrix} B^{11} & B^{12} \\ B^{21} & B^{22} \end{bmatrix} \begin{bmatrix} B^{11} & B^{21} \\ B^{12} & B^{22} \end{bmatrix} = \underline{B}\,\underline{B}^T , \tag{A1}$$

which leads to

$$\begin{aligned} &B^{11}B^{11}+B^{12}B^{12}=D_{\alpha\alpha} , \\ &B^{11}B^{21}+B^{12}B^{22}=D_{\beta\alpha} , \\ &B^{21}B^{21}+B^{22}B^{22}=D_{\beta\beta} . \end{aligned} \tag{A2}$$

We now have four unknown quantities to be determined from three equations. We can let $B^{22}=0$, giving rise to

$$\begin{aligned} &B^{21}=\sqrt{\Lambda_4} , \\ &B^{11}=\Lambda_1/\sqrt{\Lambda_4} , \\ &B^{12}=[(\Lambda_4^2-\Lambda_1^2)/\Lambda_4]^{1/2} , \end{aligned} \tag{A3}$$

where Eqs. (4.4) have been used. By separating $\underline{B}$ into real and imaginary parts $\underline{B}=\underline{B}_x+i\underline{B}_y$, the diffusion matrix in the positive P representation is given by

$$\underline{B}_{4\times4} = \begin{bmatrix} \underline{B}_x\underline{B}_x^T & \underline{B}_x\underline{B}_y^T \\ \underline{B}_y\underline{B}_x^T & \underline{B}_y\underline{B}_y^T \end{bmatrix} . \tag{A4}$$

For phase squeezing, we have $\Lambda_4 > \Lambda_1 > 0$, and thus B^{21}, B^{11}, and B^{12} are real. Consequently, $\underline{B}_y=0$ and

$$\underline{D}_{4\times4} = \begin{bmatrix} \underline{B}_x\underline{B}_x^T & 0 \\ 0 & 0 \end{bmatrix} , \tag{A5}$$

with

$$\underline{B}_x\underline{B}_x^T = \begin{bmatrix} \Lambda_4 & \Lambda_1 \\ \Lambda_1 & \Lambda_4 \end{bmatrix} . \tag{A6}$$

For amplitude squeezing, we have $\Lambda_4 < -\Lambda_1 < 0$ so that B^{11}, B^{12}, and B^{21} are purely imaginary now. Thus $\underline{B}_x=0$ and

$$\underline{D}_{4\times4} = \begin{bmatrix} 0 & 0 \\ 0 & \underline{B}_y\underline{B}_y^T \end{bmatrix} , \tag{A7}$$

with

$$\underline{B}_y\underline{B}_y^T = \begin{bmatrix} |\Lambda_4| & -\Lambda_1 \\ -\Lambda_1 & |\Lambda_4| \end{bmatrix} . \tag{A8}$$

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