HOUSTON JOURNAL OF MATHEMATICS, Volume 17, No. 1, 1991

SOME REMARKS ON THE DIRICHLET PROBLEM WITH CRITICAL GROWTH FOR THE n-LAPLACIAN

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1. Introduction. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary. Let $f(t) = h(t)e^{b|t|^{n/n-1}}$ be a function of critical growth (as in **[1]**, see also definition (2.1)) and consider the following problem

$$(1.1) \qquad \begin{gathered} -\Delta_n u = f(u)u^{n-2} \text{ in } \Omega \\ u \in W_o^{1,n}(\Omega) \end{gathered}$$

where $\Delta_n u =$ div $(|\nabla u|^{n-2}\nabla u)$ is the n-Laplacian. Now the natural questions one can ask are : under what conditions we can obtain

a) existence of a positive solution?
b) existence of multiple solutions?

In the case (a), if we assume that $\Omega = B(R)$, the ball of radius R, then by standard shooting argument it follows that (1.1) admit a positive radial solution if R is sufficiently large. In **[1]**, this result has been extended to an arbitrary Ω by a variational method. More precisely, for $u \in W_o^{1,n}(\Omega)$, define

$$(1.2) \qquad \|u\|^n = \int_\Omega |\nabla u|^n dx$$

$$(1.3) \qquad J(u) = \frac{1}{n}\|u\|^n - \int_\Omega F(u)dx$$

$$(1.4) \qquad \partial B(\Omega, f) = \{u \in W_o^{1,n}(\Omega) \setminus \{0\}; \|u\|^n = \int_\Omega f(u)u^{n-1}dx\}$$

$$(1.5) \qquad \frac{a(\Omega, f)^n}{n} = \inf\{J(u); u \in \partial B(\Omega, f)\}$$

$$(1.6) \qquad \lambda_1(\Omega) = \inf\{\|u\|^n; \int_\Omega |u|^n dx = 1\}$$

then under the conditions $f'(0) < \lambda_1(\Omega), \lim_{t\to\infty} h(t)t^{n-1} = \infty$, it has been shown that infimum is achieved in (1.5) and a minimizer is a positive solution of (1.1).

In this paper, we study the continuity of $a(\Omega, f)$ with respect to f and prove the following

THEOREM 1.1. *Let $f(t) = h(t)e^{\delta|t|^{n/n-1}}$ be a function of critical growth such that*

$$\overline{\lim_{t\to\infty}}\, h(t)t^{n-1} = \infty \tag{1.7}$$

For $\lambda > 0$, denote $a_\lambda = a(\Omega, \lambda f)$. Then $\lambda \to a_\lambda$ is continuous for $\lambda \in (0, \lambda_f(\Omega)/f'(0))$.

In the case (b), again if we assume that $\Omega = B(R)$, and $f'(0) = 0$, then by the shooting argument we can show that for a given integer $k \geq 0$, there exist a $R_o = R_o(k, f)$ such that for all $R \geq R_o$, (1.1) admit at least k-pair of non trivial solutions.

In this paper we exhibit the above phenomena by using the variational principle when Ω is an arbitrary domain. More precisely, we have the following.

THEOREM 1.2. *Let $f(t) = h(t)e^{\delta|t|^{n/n-1}}$ be a function of critical growth such that $f'(0) = 0$, $\underline{\lim}_{t\to\infty} h(t)t^{n-1} h(t)t^{n-1} > 0$. Let $k \geq 0$ be an integer. Then if Ω is of type (k, f) (see definition (2.2)) then (1.1) admit k-pair of nontrivial solutions.*

2. Preliminaries. In this section we recall some definitions and known results from [**1**].

Definition 2.1. Let $h : \mathbb{R} \to \mathbb{R}$ be a C^1-function and $b > 0$. Let $f(t) = h(t)e^{b|t|^{n/n-1}}$. We say that f is a function of critical growth if the following holds:

There exist constants $M > 0$, $\delta \in [0, 1)$ such that for every $\epsilon > 0$ and for every $t > 0$

(H_1) $f(0) = 0, f(t) > 0, f(t)t^{n-1} = f(-t)(-t)^{n-1}$

(H_2) $f'(t) > \frac{f(t)}{t}$ where $f'(t) = \frac{df}{dt}(t)$

(H_3) $F(t) \leq M(1 + f(t)t^{n-2+\delta})$ where $F(t) = \int_o^t f(s)s^{n-2}ds$ is the primitive of f.

(H_4) $\lim_{t\to\infty} h(t)e^{-\epsilon|t|^{n/n-1}} = 0, \lim_{t\to\infty} h(t)e^{\epsilon|t|^{n/n-1}} = \infty.$

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary and $f(t) = h(t)e^{b|t|^{n/n-1}}$ be a function of critical growth. For $x \in \Omega$, let $d(x, \partial\Omega)$ to denote the distance from x to $\partial\Omega$ and

$$\frac{1}{\eta^n} = \frac{1}{n}\left(\frac{be}{n-1}\right)^{n-1} \inf\left\{h(t)t^{n-1}; t \geq \left(\frac{n-1}{b}\right)^{n-1}\right\} \tag{2.1}$$

$$\omega_n = \mathrm{Vol}(S^{n-1}) \tag{2.2}$$

$$\alpha_n = n\omega_n^{1/n-1} \tag{2.3}$$

then we have

Definition 2.2. Let $k \geq 0$ be an integer. We say that Ω is of type $(k,\ f)$ if there exist k distinct points $x_1, \ldots x_k$ in Ω, $(R_1 \ldots R_k)$ and $(l_1 \ldots l_k)$ of positive numbers such that

(i) $R_i < d(x_i, \partial\Omega), B(x_i, R_i) \cap B(x_j, R_j) = \phi$ for $i \neq j$

(ii) $\log l_i + \left(\frac{\eta}{l_i}\right)^{n/n-1} < \log R_i$

(iii) $k \max_{1 \leq i \leq k} \left(\frac{n-1}{\log R_i/l_i}\right)^{n-1} \leq \frac{\alpha_n^{n-1}}{\omega_n}$

Let us recall now some known results which are needed for the proof of the theorems. Let f and Ω be as above. Then

THEOREM 2.1.
1) $J : W_o^{1,n}(\Omega) \to \mathbb{R}$ satisfy the Palais-Smale condition on the interval $\left(-\infty, \frac{1}{n}\left[\frac{\alpha_n}{b}\right]^{n-1}\right)$.
2) Let $f'(0) < \lambda_1(\Omega)$ and $\overline{\lim}_{t\to\infty} h(t)t^{n-1} = \infty$. Then there exists $u \geq 0$ in $W_o^{1,n}(\Omega)$ satisfying (1.1) and

$$J(u) = \frac{a(\Omega,\ f)^n}{n} < \frac{1}{n}\left[\frac{\alpha_n}{b}\right]^{n-1}$$

*For the proof, see [**1**].*

LEMMA 2.1.
a) Let $f'(0) < \lambda_1(\Omega)$. Then for $u \in W_o^{1,n}(\Omega) \setminus \{0\}$, there exist a unique $\gamma > 0$ such that $\gamma u \in \partial B(\Omega,\ f)$ where

$$\partial B(\Omega,\ f) = \{u \in W_o^{1,n}(\Omega) \setminus \{0\}; \|u\|^n = \int_\Omega f(u)u^{n-1}dx\} \tag{2.4}$$

Further if $\|u\|^n \leq \int_\Omega f(u)u^{n-1}dx$, *then* $\gamma \leq 1$.

b) Let $\{u_m\}$, $\{v_m\}$ be sequences in $W_o^{1,n}(\Omega)$ converging weakly to u and v respectively. Then

(i) If $\overline{\lim}_{m\to\infty} \|u_m\|^n < \left(\frac{\alpha_n}{b}\right)^{n-1}$, then for any integer $l \geq 0$,

$$\lim_{m\to\infty} \int_\Omega \frac{f(u_m)}{u_m} v_m^l dx = \int_\Omega \frac{f(u)}{u} v^l dx. \tag{2.5}$$

(ii) If $\sup_m \int_\Omega f(u_m)u_m^{n-1}dx < \infty$, then

$$\lim_{m\to\infty} \int_\Omega F(u_m)dx = \int_\Omega F(u)dx \tag{2.6}$$

$$\lim_{m\to\infty} \int_\Omega |f(u_m)u_m^{n-2}|dx = \int_\Omega |f(u)u^{n-2}|dx \tag{2.7}$$

c) Let $I(u) = \int_\Omega [f(u)u^{n-1} - nF(u)]dx$. Then there exist a constant $M_1 > 0$ such that for $u \in W_o^{1,n}(\Omega)$,

$$\int_\Omega f(u)u^{n-1}dx \leq M_1(1 + I(u)) \tag{2.9}$$

For the proof of (a) see step (2) in Lemma (3.4) and for (b) and (c), see (3), (4), (5) in Lemma (3.1) of [1].

3. Proof of the Theorems.

Let f and Ω be as in section 2. For $\lambda > 0$ and $u \in W_o^{1,n}(\Omega)$, let

$$J_\lambda(u) = \frac{1}{n}\|u\|^2 - \lambda \int_\Omega F(u)dx \tag{3.1}$$

$$I(u) = \int_\Omega [f(u)u^{n-1} - nF(u)]dx \tag{3.2}$$

$$B_\lambda = \{u \in W_o^{1,n}(\Omega) \setminus \{0\}; \|u\|^n \leq \lambda \int_\Omega f(u)u^{n-1}dx\} \tag{3.3}$$

$$\partial B_\lambda = \{u \in B_\lambda; \|u\|^n = \lambda \int_\Omega f(u)u^{n-1}dx\} \tag{3.4}$$

$$b_\lambda = \frac{a(\Omega, \lambda f)^n}{\lambda}. \tag{3.5}$$

PROOF OF THE THEOREM 1.1:

From (3.1), (3.2) and (3.4) we have for $u \in \partial B_\lambda$, $J(u) = \frac{\lambda}{n} I(u)$ and from (a) of lemma (2.1) it follows easily that for $\lambda < \lambda_1(\Omega)/f'(0)$

$$\begin{aligned} b_\lambda &= \frac{a(\Omega, \lambda f)^n}{\lambda} = \frac{n}{\lambda} \inf \{J(u); u \in \partial B_\lambda\} \\ &= \frac{n}{\lambda} \inf \{\frac{\lambda}{n} I(u); u \in \partial B)\lambda\} \\ &= \inf \{I(u); u \in B_\lambda\} \end{aligned} \tag{3.6}$$

For $\lambda_1 \leq \lambda_2$, $B_{\lambda_1} \subset B_{\lambda_2}$ and hence from (3.6) $b_{\lambda_2} \leq b_{\lambda_1}$. Therefore b_λ is a non increasing function of λ.

Let $\{\lambda_k, u_k\}_{k \geq 0} \in (0, \lambda_1(\Omega)/f'(0) x \partial B_{\lambda_k}$ such that

$$\lim_{k \to \infty} \lambda_k = \lambda_o, b_{\lambda_k} = I(u_k) \tag{3.7}$$

Note that for given λ_k, then from theorem (2.1) u_k exist satisfying (3.7)

Step 1. For a subsequence of $\{\lambda_k, u_k\}$ we will prove that

$$\overline{\lim_{k \to \infty}} b_{\lambda_k} \leq b_{\lambda_o}. \tag{3.8}$$

From (a) of Lemma (2.1) we can choose $\gamma_k > 0$ such that $v_k = \gamma_k u_o \in \partial B_{\lambda_k}$. We claim that $\{\gamma_k\}$ is bounded. Suppose not, then by Fatou's lemma, we have

$$\begin{aligned} \infty = \lambda_o \int_\Omega \underline{\lim}_{k \to \infty} \frac{f(\gamma_k u_o)}{\gamma_k} u_o^{n-1} dx &\leq \underline{\lim}_{k \to \infty} \lambda_k \int_\Omega \frac{f(\gamma_k u_o)}{\gamma_k} u_o^{n-1} dx \\ &= \lim_{k \to \infty} \|u_o\|^n \end{aligned}$$

which is a contradiction. Hence $\{\gamma_k\}$ is bounded. Let a subsequence $\gamma_k \to \gamma_o$. Since $\gamma_k u_o \to \gamma_o u_o$ in $W_o^{1,n}(\Omega)$, hence we have

$$\begin{aligned} \lambda_o \int_\Omega f(u_o) u_o^{n-1} dx = \|u_o\|^n \\ &= \lim_{k \to \infty} \lambda_k \int_\Omega \frac{f(\gamma_k u_o)}{\gamma_k} u_o^{n-1} dx \\ &= \lambda_o \int_\Omega \frac{f(\gamma_o u_o)}{\gamma_o} u_o^{n-1} dx \end{aligned}$$

This implies that $\gamma_o = 1$. Now $v_k \in \partial B_{\lambda_k}$ and $v_k \to u_o$ in $W_o^{1,n}(\Omega)$ implies that

$$\begin{aligned}\overline{\lim_{k\to\infty}}\, b_{\lambda_k} &\le \overline{\lim_{k\to\infty}}\, I(v_k)\\ &= \overline{\lim_{k\to\infty}}\,(I(v_k) - I(v_o)) + I(u_o)\\ &= I(u_o) = b_{\lambda_o}.\end{aligned}$$

This proves step (1).

Step 2. For a subsequence of $\{\lambda_k, u_k\}$ we will prove that

$$b_{\lambda_o} \le \underline{\lim_{k\to\infty}}\, b_{\lambda_k}. \tag{3.9}$$

Since $I(u_k) = b_{\lambda_k}$, therefore from theorem (2.1) $0 \le I(u_k) < \frac{1}{\lambda_k} \left(\frac{\alpha_n}{b}\right)^{n-1}$. Hence from (c) of lemma (2.1) $\{\int_\Omega f(u_k)u_k^{n-1}dx\}$ is bounded. Since $u_k \in \partial B_{\lambda_k}$ implies that $\{\|u_k\|\}$ is bounded. Let for a subsequence

$$u_k \to u \text{ weakly in } W_o^{1,n}(\Omega) \text{ and almost every } x \in \Omega. \tag{3.10}$$

and one of the following holds

(i) $\lambda_k \le \lambda_o$
(ii) $\lambda_k \ge \lambda_o$

In the case of (i) and by the monotonicity of b_λ, we have

$$b_{\lambda_o} \le \underline{\lim_{k\to\infty}}\, b_{\lambda_k}$$

In the case of (ii) we have the following.

Claim 1. $u \in \partial B_{\lambda_o}(\Omega)$.

Suppose $u \equiv 0$, then from (2.6), we have

$$\lim_{k\to\infty} \int_\Omega F(u_k)dx = 0 \tag{3.11}$$

then by monotonicity of b_λ, (3.11) and from theorem (2.1), we have

$$\begin{aligned}\frac{1}{\lambda_o} \overline{\lim_{k\to\infty}}\, \|u_k\|^n &= \overline{\lim_{k\to\infty}}\, \{I(u_k) + n\int_\Omega F(u_k)dx\}\\ &= \overline{\lim_{k\to\infty}}\, b_{\lambda_k}\\ &\le b_{\lambda_o} < \frac{1}{\lambda_o}\left(\frac{\alpha_n}{b}\right)^{n-1}\end{aligned}$$

this implies that $\overline{\lim}_{k\to\infty} \|u_k\|^n < \left(\frac{\alpha_n}{b}\right)^{n-1}$. Hence from (2.5), we have

$$\lim_{k\to\infty} \int_\Omega f(u_k)u_k^{n-1}dx = 0 \tag{3.12}$$

Let $\lambda_+ = \sup\{\lambda_k\}$. Then by monotonicity of b_λ, (3.11) and (3.12) we have

$$0 < b_{\lambda_+} \leq \underline{\lim}_{k\to\infty} b_{\lambda_k} = \underline{\lim}_{k\to\infty} I(u_k) = 0$$

which is a contradiction. Hence $u \not\equiv 0$.

Let $\varphi \in C_o^\infty(\Omega)$, then from (2.7) and by dominated convergence theorem, we have

$$\lim_{k\to\infty} \int_\Omega f(u_k)u_k^{n-2}\varphi dx = \int_\Omega f(u)u^{n-2}\varphi dx \tag{3.13}$$

since u_k satisfy

$$\int_\Omega |\nabla u_k|^{n-2}\nabla u_k \cdot \nabla\varphi dx = \lambda_k \int_\Omega f(u_k)u_k^{n-2}dx$$

From (3.13) and letting $k \to \infty$ in the above equation, we obtain for all $\varphi \in C_o^\infty(\Omega)$

$$\int_\Omega |\nabla u|^{n-2}\nabla u \cdot \nabla\varphi dx = \lambda_o \int_\Omega f(u)u^{n-2}\varphi dx$$

this implies that $u \in \partial B_{\lambda_o}$ and this proves the claim

Now from (a) of lemma (2.1), we can choose $\gamma_k > 0$ such that $v_k = \gamma_k u_k \in \partial B_{\lambda_o}$.

Claim 2. ($\{\gamma_k\}$ bounded and $\overline{\lim}_{k\to\infty} \gamma_k \leq 1$.

Suppose $\{\gamma_k\}$ is unbounded. Since $u \not\equiv 0$, then by Fatou's lemma

$$\infty = \lambda_o \int_\Omega \underline{\lim}_{k\to\infty} \frac{f(\gamma_k u_k)}{\gamma_k} u_k^{n-1}dx \leq \underline{\lim}_{k\to\infty} \|u_k\|^n < \infty$$

which is a contradiction. Hence $\{\gamma_k\}$ is bounded.

Suppose $\overline{\lim}\,\gamma_k > 1$, then for a subsequence, we can choose an $\epsilon > 0$ such that

$$1 + \epsilon \leq \gamma_k \text{ and } \gamma_k \to \gamma_o \tag{3.14}$$

Since $\|u_k\|$ is bounded and $v_k = \gamma_k u_k \in \partial B_{\lambda_o}(\Omega)$, we have from (3.14)

$$\sup_k \int_\Omega f((1+\epsilon)u_k)u_k^{n-1}dx < \infty.$$

This implies that

$$\lim_{k\to\infty} \int_\Omega f(u_k)u_k^{n-1}dx = \int_\Omega f(u)u^{n-1}dx$$

Hence we have from (3.15)

$$\begin{aligned}
\|u\|^n &\leq \underline{\lim}_{k\to\infty} \|u_k\|^n \\
&= \underline{\lim}_{k\to\infty} \lambda_k \int_\Omega f(u_k)u_k^{n-1}dx \\
&= \lambda_o \int_\Omega f(u)u^{n-1}dx \\
&= \|u\|^n
\end{aligned}$$

This shows that $u_k \to u$ in $W_o^{1,n}(\Omega)$. Therefore we have

$$\begin{aligned}
\lambda_o \int_\Omega f(u)u^{n-1}dx &= \|u\|^n \\
&= \lim_{k\to\infty} \|u_k\|^n \\
&= \lambda_o \lim_{k\to\infty} \int_\Omega \frac{f(\gamma_k u_k)}{\gamma_k} u_k^{n-1}dx \\
&= \gamma_o \int_\Omega \frac{f(\gamma_o u)}{\gamma_o} u^{n-1}dx
\end{aligned}$$

This implies that $\gamma_o = 1$ contradicting (3.14). This proves the claim (2).

Let for a subsequence, $\gamma_k \to \gamma_o$, $v_k \to \gamma_o u$ weakly in $W_o^{1,n}(\Omega)$ and almost every $x \in \Omega$. Since $v_k \in \partial B_{\lambda_o}$ and $\{\|v_k\|\}$ is bounded and hence from (2.6), claim (1),

$$\lim_{k\to\infty}\left\{\int_\Omega F(u_k)dx - \int_\Omega F(v_k)dx\right\} \le 0$$

Therefore

$$\begin{aligned} b_{\lambda_o} &\le \underline{\lim}_{k\to\infty} I(v_k) \\ &= \underline{\lim}_{k\to\infty} \{I(v_k) - I(u_k) + I(u_k)\} \\ &= \underline{\lim}_{k\to\infty} \left\{\left(\frac{\gamma_k^n}{\lambda_o} - \frac{1}{\lambda_k}\right)\|u_k\|^n + \int_\Omega (F(u_k) - F(v_k))dx + b_{\lambda_k}\right\} \\ &\le \underline{\lim}_{k\to\infty} b_{\lambda_k}. \end{aligned}$$

and this proves the step (2).

Now from step (1) and (2) it follows that $\lambda \to b_\lambda$ is continuous and hence from the definition of $b_\lambda, \lambda \to a_\lambda$ is continuous. This proves the theorem.

In order to prove the Theorem (1.2), we need few lemmas.

Let $0 < l < R$, then define the Moser function [3] $m_{l,R}$ by

$$m_{l,R}(x) = \frac{1}{\omega_n^{1/n}}\begin{cases} (\log R/l)^{1-1/n} & o \le |x| \le l \\ \frac{\log R/|x|}{(\log R/l)^{1/n}} & 1 \le |x| \le R \\ 0 & |x| > R \end{cases} \tag{3.15}$$

Clearly $m_{l,R} \in W^{1,n}(\mathbb{R}^n)$ with support contained in $B(0,R)$ with $\|m_{l,R}\| = 1$.

For $x_o \in \Omega, 0 < l < R < d(x_o, \partial\Omega)$, let $W_{l,R}(x) = m_{l,R}(x - x_o)$. Then $\|W_{l,R}\| = 1, W_{l,R} \in W_o^{1,n}(\Omega)$ with support contained in $B(x_o, R)$.

LEMMA 3.1. *Let η be as in (2.1) and $x_o \in \Omega$. Assume that there exist $0 < l < R < d(x_o, \partial\Omega)$ such that*

$$\log l + \left(\frac{\eta}{l}\right)^{n/n-1} < \log R \tag{3.16}$$

then

$$\max_{t\in\mathbb{R}} J(tW_{l,R}) < \frac{\omega_n}{n}\left(\frac{n-1}{n\log R/l}\right)^{n-1}. \tag{3.17}$$

PROOF: Let $t_o > 0$ such that $J(t_oW_{l,R}) = \max_{t\in\mathbb{R}} J(tW_{l,R})$ and

$$\tau = t_oW_{l,R}(l) = \frac{t_o}{\omega_n^{1/n}}(\log R/l)^{(n-1)/n} \text{ then}$$

$$\begin{aligned} t_o^n &= t_o^{n-1}\int_\Omega f(t_oW_{l,R})W_{l,R}^{n-1}dx \\ &\geq t_o^{n-1}\int_{B(x_o,l)} f(t_oW_{l,R})W_{l,R}^{n-1}dx \\ &= \frac{\omega_n l^n}{n}h(\tau)\tau^{n-1}e^{b\tau^{n/n-1}} \end{aligned}$$

that is

$$\frac{\tau^n}{(\log R/l)^{n-1}} \geq \frac{l^n}{n}h(\tau)\tau^{n-1}e^{b\tau^{n/n-1}} \tag{3.18}$$

we claim that

$$\tau < \left(\frac{n-1}{b}\right)^{(n-1)/n}. \tag{3.19}$$

Suppose (3.19) is not true, then we have for

$$\begin{aligned} \tau &\geq \left(\frac{n-1}{b}\right)^{(n-1)/n} \\ \frac{e^{b\tau^{n/n-1}}}{\tau^n} &\geq \left(\frac{be}{n-1}\right)^{n-1}. \end{aligned} \tag{3.20}$$

Hence from (3.20) and (3.18) we have

$$\begin{aligned} 1 &\geq l^n(\log R/l)^{n-1}\left(\frac{be}{n-1}\right)^{n-1} \inf\left\{h(s)s^{n-1}; s \geq \left(\frac{n-1}{b}\right)^{n-1}\right\} \\ &= \frac{l^n}{\eta^n}(\log R/l)^{n-1}. \end{aligned}$$

This implies that $\log R \leq \log l + \left(\frac{\eta}{l}\right)^{n/n-1}$, contradicting (3.17). This proves (3.19). Now we have from (3.19)

$$\begin{aligned}\max_{t\in\mathbb{R}} J(tW_{l,R}) &= J(t_o W_{l,R})\\ &\leq \frac{t_o^n}{n}\\ &< \frac{\omega_n}{n}\left(\frac{n-1}{\text{blog } R/l}\right)^{n-1}\end{aligned}$$

This concludes the lemma.

As an immediate consequence of this lemma, we have the following.

COROLLARY 3.1. *Let $k \geq 0$ be an integer and Ω is of type (k, f) (see definition 2.2). Let $W_i(x) = m_{l_i,R_i}(x - x_i)$, then*

$$\max_{(t_1\ldots t_k)\in\mathbb{R}^k} J(t_1W_1 + \cdots + t_kW_k) < \frac{1}{n}\left(\frac{\alpha_n}{b}\right)^{n-1}. \tag{3.21}$$

PROOF: Since W_i has disjoint support and from lemma (3.1)

$$\begin{aligned}\max_{(t_1\ldots t_k)\in\mathbb{R}^k} J(t_1W_1 + \cdots + t_kW_k) &= \sum_{i=1}^{k} \max_{t\in\mathbb{R}} J(tW_i)\\ &< \frac{\omega_n}{n}\sum_{i=1}^{k}\left(\frac{n-1}{\text{blog } R_i/l_i}\right)^{n-1}\\ &\leq \frac{k\omega_n}{n}\max_{1\leq i\leq k}\left(\frac{n-1}{\text{blog } R_i/l_i}\right)^{n-1}\\ &\leq \frac{1}{n}\left(\frac{\alpha_n}{b}\right)^{n-1}.\end{aligned}$$

This proves the corollary.

LEMMA 3.2. *Let $f'(0) = 0$, then there exist a $0 < \mathcal{L}^n < \left(\frac{\alpha_n}{b}\right)^{n-1}$ and a $\delta > 0$ such that*

$$J(u) \geq \delta \text{ for } \|u\| = \mathcal{L}. \tag{3.22}$$

PROOF: Suppose (3.22) is not true, then for every $\mathcal{L}$ in $\left(0, \left(\frac{\alpha_n}{b}\right)^{(n-1)/n}\right)$, there exist a sequence $\{u_{k,\mathcal{L}}\}$ in $W_o^{1,n}(\Omega)$ with $\|u_{k,\mathcal{L}}\| = \mathcal{L}$ such that

$$(3.23) \qquad \begin{cases} u_{k,\mathcal{L}} \to u_{\mathcal{L}} \text{ weakly in } W_o^{1,n}(\Omega) \text{ and} \\ \text{almost every } x \in \Omega \\ \lim_{k\to\infty} J(u_{k,\mathcal{L}}) = 0 \end{cases}$$

From (b) of lemma (2.1), and (3.23) we have

$$(3.24) \qquad \begin{aligned} \lim_{k\to\infty} \int_\Omega f(u_{k,\mathcal{L}}) u_{k,\mathcal{L}}^{n-1} dx &= \int_\Omega f(u_{\mathcal{L}}) u_{\mathcal{L}}^{n-1} dx \\ \lim_{k\to\infty} \int_\Omega F(u_{k,\mathcal{L}}) dx &= \int_\Omega F(u_{\mathcal{L}}) dx \end{aligned}$$

Hence from (3.23) and (3.24), we have

$$(3.25) \qquad \begin{aligned} \mathcal{L}^n = \|u_{k,\mathcal{L}}\|^n &= \lim_{k\to\infty} \left\{ nJ(u_{k,\mathcal{L}} + n \int_\Omega F(u_{k,\mathcal{L}}) dx \right\} \\ &= n \int_\Omega F(u_{\mathcal{L}}) dx \end{aligned}$$

This implies that $u_{\mathcal{L}} \not\equiv 0$. Next we claim that for each $\mathcal{L}$,

$$(3.26) \qquad \mathcal{L}^n \leq \int_\Omega f(u_{\mathcal{L}}) u_{\mathcal{L}}^{n-1} dx$$

Suppose not, then from (3.25), we have

$$0 < \int_\Omega [f(u_{\mathcal{L}}) u_{\mathcal{L}}^{n-1} - nF(u_{\mathcal{L}})] dx \leq 0$$

which is a contradiction. Hence (3.26) holds. Let $v_{\mathcal{L}} = \frac{u_{\mathcal{L}}}{\|u_{\mathcal{L}}\|}$ and $v_{\mathcal{L}} \to v_o$ weakly in $W_o^{1,n}(\Omega)$. Since $\|u_{\mathcal{L}}\| \leq \mathcal{L}$, hence from (3.26) and (b) of Lemma (2.1) we have

$$\begin{aligned} 1 &\leq \lim_{\mathcal{L}\to 0} \int_\Omega \frac{f(u_{\mathcal{L}})}{u_{\mathcal{L}}} v_{\mathcal{L}}^n dx \\ &= \int_\Omega f'(0) v_o^n dx = 0 \end{aligned}$$

which is a contradiction. This proves the lemma.

In order to complete the proof of the theorem (1.2), let us recall an abstract theorem of Bartalo-Benci'-Fortunato [2] (See also Rabinowitz [4]).

THEOREM 3.1. *Let E be Banach space and $I \in C^1(E, \mathbb{R})$ be even with $I(0) = 0$. Let I satisfies the following.*

1) There exist a positive constant β such that I satisfies Palais-Smale condition on $(0, \beta)$

2) There exist two closed subspaces V_1 and V_2 of E and positive constants $\mathcal{L}, \delta, \beta'$ with $\delta < \beta' < \beta$ such that

$$I(u) \leq \beta' \text{ for all } u \in V_1$$
$$I(u) \geq \delta \text{ for all } u \in V_2 \text{ with } \|u\| = \mathcal{L}$$
$$\dim V_1 < \infty, \ \operatorname{codim} V_2 < \infty.$$

Then there exist at least $(\dim V_1 - \operatorname{Codim} V_2)$ pair of critical points of I with values in $[\delta, \beta']$.

PROOF OF THE THEOREM 1.2: Let $W_i(x) = m_{l_i, R_i}(x - x_i)$. Taking $E = W_o^{1,n}(\Omega), I = J, \beta = \frac{1}{n}\left(\frac{\alpha_n}{b}\right)^{n-1}, V_2 = E, V_1 =$ Span $\{W_1, \ldots W_k\}$ in the theorem (3.1), it follows that the hypothesis of the theorem (3.1) is satisfied by making use of (1) of theorem (2.1), corollary (3.1) and lemma (3.2). This implies that J have k pair of critical points which implies the theorem.

Acknowledgement. I would like to thank Dr. Yadava for many helpful discussions.

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