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Existence of positive solutions of the semilinear Dirichlet problem with critical growth for the *n*-laplacian

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Existence of Positive Solutions of the Semilinear Dirichlet Problem with Critical Growth for the n-Laplacian

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1. - Introduction

Let Ω be a bounded open set in $\mathbb{R}^n$ with smooth boundary. We are looking for a solution of the following problem:

Let $1 < p \leq n$, find $u \in W_0^{1,p}(\Omega) \backslash \{0\}$ such that

$$
(1.1) \qquad \begin{aligned} \Delta_p u &= f(x,u)|u|^{p-2} \qquad \text{in } \Omega \\ u &\geq 0, \end{aligned}
$$

where $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2} \nabla u)$ is the p-Laplacian and $f : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ is a C^1-function with $f(x,0) = 0$, $f(x,t) \geq 0$ for $t \geq 0$ and of critical growth.

For $p = 2$ and $n \geq 3$, Brézis-Nirenberg [4] have studied the existence and non-existence of solution of (1.1) when f has critical growth of the form $u^{(n+2)/(n-2)} + \lambda u$. A generalization of this result, on the same lines, for the p-Laplacian with $p \leq n$ and $p^2 \leq n$, has been studied by Garcia Azorero-Peral Alonso [7]. When $p = n$, in view of the Trudinger [13] imbedding, a critical growth function $f(x,u)$ behaves like $\exp\left(b|u|^{n/(n-1)}\right)$ for some $b > 0$. In this context, when $p = n = 2$ and Ω is a ball in $\mathbb{R}^2$, existence of a solution of (1.1) has been studied by Adimurthi [1], Atkinson-Peletier [2]. The method used by Atkinson-Peletier is a shooting method and hence cannot be generalized to solve (1.1) in an arbitrary domain. Whereas in Adimurthi [1], (1.1) is solved via variational method which is in the spirit of Brézis-Nirenberg [4] and, based on this method, we prove the following main result in this paper.

Let $f(x,t) = h(x,t)\exp(b|t|^{n/(n-1)})$ be a function of critical growth and $F(x,t)$ be its primitive (see definition (2.1)). For $u \in W_0^{1,n}(\Omega)$, let

$$
(1.2) \qquad J(u) = \frac{1}{n} \int_\Omega |\nabla u|^n \, dx - \int_\Omega F(x,u) \, dx
$$

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$$\lambda_1(u) = \inf\left\{\int_\Omega |\nabla u|^n \,\mathrm{d}x;\ u \in W_0^{1,n}(\Omega),\ \int_\Omega |u|^n \,\mathrm{d}x = 1\right\} \tag{1.3}$$

$$\alpha_n = n\omega_n^{1/(n-1)}, \text{ where } \omega_n = \text{ Volume of } S^{n-1}. \tag{1.4}$$

THEOREM *Let* $f(x,t) = h(x,t)\exp(b|t|^{n/(n-1)})$ *be a function of critical growth on* Ω. *Then*

1) $J : W_0^{1,n}(\Omega) \to \mathbb{R}$ *satisfies the Palais-Smale Condition on the interval* $\left(-\infty,\ \frac{1}{n}\left(\frac{\alpha_n}{b}\right)^{n-1}\right)$;

2) *Let* $f'(x,t) = \frac{\partial}{\partial t} f(x,t)$ *and further assume that*

$$\sup_{x\in\overline{\Omega}} f'(x,0) < \lambda_1(\Omega) \tag{1.5}$$

$$\overline{\lim_{t\to\infty}}\ \inf_{x\in\overline{\Omega}} h(x,t)t^{n-1} = \infty, \tag{1.6}$$

then there exists some $u_0 \in W_0^{1,n}(\Omega)\backslash\{0\}$ *such that*

$$\begin{aligned} \Delta_n u_0 &= f(x,u_0)u_0^{n-2} && \text{in } \Omega \\ u_0 &\geq 0 \\ u_0 &= 0 && \text{on } \partial\Omega. \end{aligned} \tag{1.7}$$

The method adopted to solve (1.7) in Brézis-Nirenberg [4] does not work because of the critical growth is of exponential type. Here we adapt the method of artificial constraint due to Nehari [11]. The main idea of the proof is as follows:

Define

$$\frac{a(\Omega,f)^n}{n} = \inf\left\{J(u);\ \int_\Omega |\nabla u|^n \,\mathrm{d}x = \int_\Omega f(x,u)u^{n-1}\mathrm{d}x,\ u \not\equiv 0\right\}, \tag{1.8}$$

then the minimizer of (1.8) is a solution of (1.7).

It has to be noted that α_n is the best constant appearing in Moser's [10] result about the Trudinger's imbedding of $W_0^{1,n}(\Omega)$. In view of this, one expects that J should satisfy the Palais-Smale Condition on $\left(-\infty,\ \frac{1}{n}\left(\frac{\alpha_n}{b}\right)^{n-1}\right)$. Therefore, in order to get a minimizer of (1.8), the question remains to show that

$$a(\Omega,f)^n < \left(\frac{\alpha_n}{b}\right)^{n-1} \tag{1.9}$$

and this has been achieved by showing the following relation

$$\sup_{\int_\Omega |\nabla w|^n \,dx \le 1} \int_\Omega f(x, a(\Omega, f)w)w^{n-1}\,dx \le a(\Omega, f). \tag{1.10}$$

In the forthcoming paper (jointly with Yadava), we discuss the bifurcation and multiplicity results for (1.7) when $n = 2$.

2. - Preliminaries

Let Ω be a bounded domain with smooth boundary. In view of the Trudinger-Moser [13,10] imbedding, we have the following definition of functions of critical growth.

DEFINITION 2.1. Let $h : \overline{\Omega} \times \mathbb{R} \to \mathbb{R}$ be a C^1-function and $b > 0$. Let $f(x,t) = h(x,t)\exp\left(b|t|^{n/(n-1)}\right)$. We say that f is a function of critical growth on Ω if the following holds:

There exist constants $M > 0$, $\sigma \in [0, 1)$ such that, for every $\epsilon > 0$, and for every $(x, t) \in \overline{\Omega} \times (0, \infty)$,

(H_1) $f(x, 0) = 0$, $f(x, t) > 0$, $f(x, t)t^{n-1} = f(x, -t)(-t)^{n-1}$;

(H_2) $f'(x, t) > \dfrac{f(x, t)}{t}$, where $f'(x, t) = \dfrac{\partial f}{\partial t}(x, t)$;

(H_3) $F(x, t) \le M(1 + f(x, t)t^{n-2+\sigma})$, where

$$F(x, t) = \int_0^t f(x, s)s^{n-2}\,ds$$

is the primitive of f;

(H_4) $\lim\limits_{t\to\infty} \sup\limits_{x\in\overline{\Omega}} h(x, t)\exp\left(-\epsilon t^{n/(n-1)}\right) = 0$,

$$\lim_{t\to\infty} \inf_{x\in\overline{\Omega}} h(x, t)\exp\left(\epsilon t^{n/(n-1)}\right) = \infty.$$

Let $A(\Omega)$ denote the set of all functions of critical growth on Ω.

EXAMPLES. In view of (H_1), it is enough to define f on $\overline{\Omega} \times (0, \infty)$.

1) For $m \ge 1$, $b > 0$, $\beta \ge 0$ and $0 \le \alpha < \dfrac{n}{n-1}$, $f(x, t) = t^m \exp(\beta t^\alpha)\exp\left(bt^{n/(n-1)}\right)$ is in $A(\Omega)$.

2) $f(x, t) = t^2 e^{-t}\exp\left(t^{n/(n-1)}\right)$ is in $A(\Omega)$.

3) Let $f(x, t) = h(x, t)\exp\left(bt^{n/(n-1)}\right)$, satisfying (H_1) and (H_4).

Further assume that $h'(x,t) \geq \frac{h(x,t)}{t}$ for $(x,t) \in \overline{\Omega} \times (0,\infty)$. Then f is in $A(\Omega)$.

For

$$\frac{f'(x,t)}{f(x,t)} = \frac{h'(x,t)}{h(x,t)} + \frac{nb}{n-1} t^{1/(n-1)} > \frac{1}{t}$$

and hence f satisfy (H_2).

Let $\epsilon > 0$, and $\sigma = \frac{1}{n-1}$

$$\begin{aligned} F(x,t) - F(x,\epsilon) &= \frac{n-1}{nb} \int_{\epsilon}^{t} h(x,s) s^{n-1-\sigma} \frac{\mathrm{d}}{\mathrm{d}s} \exp\left(bs^{n/(n-1)}\right) \mathrm{d}s \\ &\leq \frac{n-1}{nb} \left[f(x,t)t^{n-2-\sigma} - f(x,\epsilon)\epsilon^{n-2-\sigma}\right]. \end{aligned}$$

This implies that there exists a constant $M > 0$ such that $F(x,t) \leq M[1 + f(x,t)t^{n-2-\sigma}]$ for $(x,t) \in \overline{\Omega} \times (0,\infty)$. This shows that f satisfy (H_3) and hence $f \in A(\Omega)$.

Let $W_0^{1,n}(\Omega)$ be the usual Sobolev space and $f(x,t) = h(x,t) \exp\left(bt^{n/(n-1)}\right)$ be in $A(\Omega)$. For $u \in W_0^{1,n}(\Omega)$, define

$$\|u\|^n = \int_{\Omega} |\nabla u|^n \, \mathrm{d}x \tag{2.1}$$

$$J(u) = \frac{1}{n}\|u\|^n - \int_{\Omega} F(x,u) \, \mathrm{d}x \tag{2.2}$$

$$I(u) = \frac{1}{n} \int_{\Omega} f(x,u)u^{n-1} \, \mathrm{d}x - \int_{\Omega} F(x,u) \, \mathrm{d}x \tag{2.3}$$

$$\partial B(\Omega, f) = \left\{ u \in W_0^{1,n}(\Omega) \setminus \{0\}; \ \|u\|^n = \int_{\Omega} f(x,u)u^{n-1} \, \mathrm{d}x \right\} \tag{2.4}$$

$$\frac{a(\Omega, f)^n}{n} = \inf\{J(u); \ u \in \partial B(\Omega, f)\} \tag{2.5}$$

$$\lambda_1(\Omega) = \inf\left\{ \|u\|^n; \ \int_{\Omega} |u|^n \, \mathrm{d}x = 1 \right\} \tag{2.6}$$

$$\alpha_n = n\omega_n^{1/(n-1)}, \text{ where } \omega_n = \text{ Volume of } S^{n-1}.$$

DEFINITION OF MOSER FUNCTIONS. Let $x_0 \in \Omega$ and $R \leq \mathrm{d}(x_0, \partial\Omega)$, where d denotes the distance from x_0 to $\partial\Omega$. For $0 < \ell < R$, define

$$m_{\ell,R}(x, x_0) = \frac{1}{\omega_n^{1/n}} \begin{cases} \left(\log \frac{R}{\ell}\right)^{1-\frac{1}{n}} & \text{if } 0 \leq |x - x_0| \leq \ell \\ \dfrac{\log \frac{R}{r}}{\left(\log \frac{R}{\ell}\right)^{1/n}} & \text{if } \ell \leq r = |x - x_0| \leq R \\ 0 & \text{if } |x - x_0| \geq R. \end{cases}$$

Then it is easy to see that $m_{\ell,R} \in W_0^{1,n}(\Omega)$ and $\|m_{\ell,R}\| = 1$.

For the proof of our theorem, we need the following two results whose proof is found in Moser [10] and P.L. Lions [9] respectively.

THEOREM 2.1 (Moser). 1) *Let* $u \in W_0^{1,n}(\Omega)$, *and* $p < \infty$, *then* $\exp\left(|u|^{n/(n-1)}\right) \in L^p(\Omega)$.

2) $\left(\frac{\alpha_n}{b}\right)^{n-1} = \max\left\{ c^n;\ \sup\limits_{\|w\| \leq 1} \int_\Omega \exp\left(bc^{n/(n-1)}|w|^{n/(n-1)}\right) \mathrm{d}x < \infty \right\}.$

THEOREM 2.2 (P.L. Lions). *Let* $\{u_k;\ \|u_k\| = 1\}$ *be a sequence in* $W_0^{1,n}(\Omega)$ *converging weakly to a non-zero function* u. *Then, for every* $p < (1 - \|u\|^n)^{-1/(n-1)}$,

$$\sup_k \int_\Omega \exp\left(p\alpha_n |u_k|^{n/(n-1)}\right) \mathrm{d}x < \infty.$$

3. - Proof of the Theorem

We need a few lemmas to prove the theorem. The proof of the following lemma is given in the appendix.

LEMMA 3.1. *Let* $f \in A(\Omega)$. *Then we have*

1) *If* $u \in W_0^{1,n}(\Omega)$, *then* $f(x,u) \in L^p(\Omega)$ *for all* $p \geq 0$.

2) $\left(\frac{\alpha_n}{b}\right)^{n-1} = \sup\left\{ c^n;\ \sup\limits_{\|w\| \leq 1} \int_\Omega f(x, cw) w^{n-1}\, \mathrm{d}x < \infty \right\}.$

3) *Let* $\{u_k\}$ *and* $\{v_k\}$ *be bounded sequences in* $W_0^{1,n}(\Omega)$ *converging weakly and for almost every* x *in* Ω *to* u *and* v *respectively. Further assume that*

$$\varlimsup_{k\to\infty} \|u_k\|^n < \left(\frac{\alpha_n}{b}\right)^{n-1}.$$

Then, for every integer $\ell \geq 0$,

$$\lim_{k\to\infty} \int_\Omega \frac{f(x,u_k)}{u_k} v_k^\ell\, \mathrm{d}x = \int_\Omega \frac{f(x,u)}{u} v^\ell\, \mathrm{d}x.$$

4) *Let* $\{u_k\}$ *be a sequence in* $W_0^{1,n}(\Omega)$ *converging weakly and for almost every* x *in* Ω *to* u, *such that*

$$\sup_k \int_\Omega f(x, u_k) u_k^{n-1}\, \mathrm{d}x < \infty.$$

Then, for any $0 \le \tau < 1$,

$$\lim_{k\to\infty} \int_\Omega f(x, |u_k|)\ |u_k|^{n-2+\tau}\, \mathrm{d}x = \int_\Omega f(x, |u|)\ |u|^{n-2+\tau}\, \mathrm{d}x$$

$$\lim_{k\to\infty} \int_\Omega F(x, u_k)\, \mathrm{d}x = \int_\Omega F(x, u)\, \mathrm{d}x.$$

5) $I(u) \ge 0$ *for all* u *and* $I(u) = 0$ *iff* $u \equiv 0$. *Further, there exists a constant* $M_1 > 0$ *such that, for all* $u \in W_0^{1,n}(\Omega)$,

$$\int_\Omega f(x, u) u^{n-1}\, \mathrm{d}x \le M_1(1 + I(u)).$$

LEMMA 3.2. *Let* $f = h \exp\left(b|t|^{n/(n-1)}\right) \in A(\Omega)$ *and define*

$$h_0(t) = \inf_{x\in\overline{\Omega}} h(x,t), \quad M_0 = \sup_{t\ge 0} h_0(t) t^{n-1}, \quad R_0 = \sup_{x\in\overline{\Omega}} \mathrm{d}(x, \partial\Omega),$$

and

$$k_0 = \begin{cases} \left(\frac{n}{R_0}\right)^{n/(n-1)} M_0^{-1/(n-1)} & \text{if } M_0 < \infty \\ 0 & \text{if } M_0 = \infty. \end{cases}$$

Let $a \ge 0$ *be such that*

$$\sup_{\|w\|\le 1} \int_\Omega f(x, aw) w^{n-1}\, \mathrm{d}x \le a.$$

If $\frac{k_0}{b} < 1$, *then* $a^n < \left(\frac{\alpha_n}{b}\right)^{n-1}$.

PROOF. From 2) of lemma 3.1, we have $a^n \le \left(\frac{\alpha_n}{b}\right)^{n-1}$. Suppose $a^n = \left(\frac{\alpha_n}{b}\right)^{n-1}$. Let $x_0 \in \Omega$ such that $\mathrm{d}(x_0, \partial\Omega) = R_0$ and $0 < \ell < R_0$. Let

$$m_\ell(x) = m_{\ell,R_0}(x, x_0).$$

be the Moser functions and

$$t = a\omega_n^{-1/n}\left(\log\frac{R_0}{\ell}\right)^{(n-1)/n},$$

then from (3.1) we have

$$\begin{aligned} a &\geq \int_\Omega f(x, am_\ell)m_\ell^{n-1}\,\mathrm{d}x \\ &\geq \int_{B(x_0,\ell)} h_0(am_\ell)m_\ell^{n-1}\ \exp\left(ba^{n/(n-1)}m_\ell^{n/(n-1)}\right)\,\mathrm{d}x \\ &= \frac{h_0(t)t^{n-1}\omega_n R_0^n}{na^{n-1}}. \end{aligned}$$

This implies that

$$\left(\frac{\alpha_n}{b}\right)^{n-1} = a^n \geq \frac{h_0(t)t^{n-1}\omega_n R_0^n}{n}.$$

That is, for all $t \in (0,\infty)$,

$$b \leq \left(\frac{n}{R_0}\right)^{n/(n-1)}\left(h_0(t)t^{n-1}\right)^{-1/(n-1)}$$

and hence

$$b \leq \left(\frac{n}{R_0}\right)^{n/(n-1)}\inf_{t\geq 0}\left(h_0(t)t^{n-1}\right)^{-1/(n-1)} \leq k_0$$

which contradicts the hypothesis $b > k_0$. Hence $a^n < \left(\frac{\alpha_n}{b}\right)^{n-1}$ and this proves the lemma.

LEMMA 3.3. (Compactness Lemma). *Let f be in $A(\Omega)$ and $\{u_k\}$ be a sequence in $W_0^{1,n}(\Omega)$ converging weakly and for almost every x in Ω to a non-zero function u. Further, assume that*

(i) *There exists* $C \in \left(0, \frac{1}{n}\left(\frac{\alpha_n}{b}\right)^{n-1}\right]$ *such that* $\lim_{k\to\infty} J(u_k) = C$;

(ii) $\|u\|^n \geq \int_\Omega f(x,u)u^{n-1}\,\mathrm{d}x$;

(iii) $\sup_k \int_\Omega f(x,u_k)u_k^{n-1}\,\mathrm{d}x < \infty$;

then

$$\lim_{k\to\infty}\int_\Omega f(x,u_k)u_k^{n-1}\,\mathrm{d}x = \int_\Omega f(x,u)u^{n-1}\,\mathrm{d}x.$$

PROOF. From 5) of lemma 3.1, $I(u) > 0$. Therefore, from (ii) we have $J(u) \geq I(u) > 0$ and $J(u) \leq \varliminf_{k\to\infty} J(u_k) = C$. Hence we can choose an $\epsilon > 0$ such that

$$(C - J(u))\ (1+\epsilon)^{n-1} < \frac{1}{n}\left(\frac{\alpha_n}{b}\right)^{n-1}. \tag{3.2}$$

Let $\beta = \int_\Omega F(x,u)\,\mathrm{d}x$. Then, from (iii) and 4) of lemma 3.1, we have

$$\begin{aligned}\lim_{k\to\infty} \|u_k\|^n &= n \lim_{k\to\infty}\left\{J(u_k) + \int_\Omega F(x,u_k)\,\mathrm{d}x\right\}\\ &= n(C+\beta).\end{aligned} \tag{3.3}$$

From (3.2) and (3.3) we can choose a $k_0 > 0$ such that, for all $k \geq k_0$,

$$(1+\epsilon)^{n-1}\left(\frac{b}{\alpha_n}\right)^{n-1}\|u_k\|^n < \frac{C+\beta}{C-J(u)} = \left(1 - \frac{\|u\|^n}{n(C+\beta)}\right)^{-1}. \tag{3.4}$$

Now choose p such that

$$(1+\epsilon)^{n-1}\left(\frac{b}{\alpha_n}\right)^{n-1}\|u_k\|^n \leq p^{n-1} < \frac{C+\beta}{C-J(u)}. \tag{3.5}$$

Applying theorem 2.2 to the sequence $\frac{u_k}{\|u_k\|}$ and using (3.3) and (3.5), we have

$$\sup_k \int_\Omega \exp\left[p\alpha_n\left(\frac{u_k}{\|u_k\|}\right)^{n/(n-1)}\right]\mathrm{d}x < \infty. \tag{3.6}$$

From (3.5) and (3.6), we have

$$\begin{aligned}&\sup_k \int_\Omega \exp\left((1+\epsilon)^{n-1} b|u_k|^{n/(n-1)}\right)\mathrm{d}x\\ &\qquad \leq \sup_k \int_\Omega \exp\left[p\alpha_n\left(\frac{u_k}{\|u_k\|}\right)^{n/(n-1)}\right]\mathrm{d}x < \infty.\end{aligned} \tag{3.7}$$

Let

$$M_1 = \sup_{(x,t)\in\overline{\Omega}\times\mathbb{R}} |h(x,t)t^{n-1}|\exp\left(-\epsilon\frac{b}{2}|t|^{n/(n-1)}\right)$$

and $N > 0$. Then from (3.7) we have

$$\int_{|u_k|\geq N} f(x,u_k)u_k^{n-1}\,\mathrm{d}x = \int_{|u_k|\geq N} h(x,u_k)u_k^{n-1}\exp\left(b|u_k|^{n/(n-1)}\right)\mathrm{d}x \tag{3.8}$$

$$\leq M_1 \int_{|u_k|\geq N} \exp\left(-\epsilon\frac{b}{2}|u_k|^{n/(n-1)}\right)\exp\left[(1+\epsilon)b|u_k|^{n/(n-1)}\right]\mathrm{d}x$$

$$= O\left(\exp\left(-\epsilon\frac{b}{2}N^{n/(n-1)}\right)\right).$$

Hence

$$\int_{\Omega} f(x,u_k)u_k^{n-1}\,\mathrm{d}x = \int_{|u_k|\leq N} f(x,u_k)u_k^{n-1}\,\mathrm{d}x + O\left(\exp\left(-\epsilon\frac{b}{2}N^{n/(n-1)}\right)\right).$$

Now letting $k \to \infty$, and $N \to \infty$ in the above equation, we obtain

$$\lim_{k\to\infty}\int_{\Omega} f(x,u_k)u_k^{n-1}\,\mathrm{d}x = \int_{\Omega} f(x,u)u^{n-1}\,\mathrm{d}x.$$

This proves the lemma.

LEMMA 3.4. *Let $f \in A(\Omega)$ and assume that*

(i) $\overline{\lim_{t\to\infty}}\, h_0(t)t^{n-1} = \infty$,

where $h_0(t) = \inf_{x\in\overline{\Omega}} h(x,t)$;

(ii) $\sup_{x\in\overline{\Omega}} f'(x,0) < \lambda_1(\Omega)$;

then

$$0 < a(\Omega,f)^n < \left(\frac{\alpha_n}{b}\right)^{n-1}.$$

PROOF. The lemma is proved in several steps.

STEP 1. $a(\Omega,f) > 0$.

Suppose $a(\Omega,f) = 0$. Then there exists a sequence $\{u_k\}$ in $\partial B(\Omega,f)$ such that $J(u_k) \to 0$ as $k \to \infty$. Since $J(u_k) = I(u_k)$, hence from 5) of lemma 3.1

$$\sup_k \int_{\Omega} f(x,u_k)u_k^{n-1}\,\mathrm{d}x < \infty \tag{3.9}$$

$$\sup_k \|u_k\|^n < \infty. \tag{3.10}$$

Then, by extracting a subsequence, we can assume that $\{u_k\}$ converges weakly and for almost every x in Ω to a function u. Now by Fatou's lemma,

$$0 \le I(u) \le \varliminf_{k\to\infty} I(u_k) = \varliminf_{k\to\infty} J(u_k) = 0.$$

Hence $u \equiv 0$. From (3.9) and 4) of lemma 3.1, we have

$$\lim_{k\to\infty} \|u_k\|^n = n \lim_{k\to\infty} \left\{ J(u_k) + \int_\Omega F(x, u_k)\, dx \right\} = 0. \tag{3.12}$$

Let $v_k = \frac{u_k}{\|u_k\|}$ and converging weakly to v. Using $u_k \in \partial B(\Omega, f)$, (3.12), 3) of lemma 3.1 and (ii), we have

$$\begin{aligned} 1 &= \lim_{k\to\infty} \int_\Omega \frac{f(x, u_k)}{u_k} v_k^n\, dx \\ &= \int_\Omega f'(x,0) v^n\, dx < \lambda_1(\Omega) \int_\Omega v^n\, dx \le 1, \end{aligned}$$

which is a contradiction. This prove step 1.

Step 2. For every $u \in W_0^{1,n}(\Omega)\backslash\{0\}$, there exists a constant $\gamma > 0$ such that $\gamma u \in \partial B(\Omega, f)$. Moreover, if

$$\|u\|^n \le \int_\Omega f(x,u) u^{n-1}\, dx, \tag{3.13}$$

then $\gamma \le 1$ and $\gamma = 1$ iff $u \in \partial B(\Omega, f)$.

For $\gamma > 0$, define

$$\psi(\gamma) = \frac{1}{\gamma} \int_\Omega f(x, \gamma u) u^{n-1}\, dx.$$

Then, from 3) of lemma 3.1 and (ii), we have

$$\lim_{\gamma\to 0} \psi(\gamma) = \int_\Omega f'(x,0) u^n\, dx < \|u\|^n,$$

$$\lim_{\gamma\to\infty} \psi(\gamma) = \infty.$$

Hence there exists $\gamma > 0$ such that $\psi(\gamma) = \|u\|^n$; this implies that $\gamma u \in \partial B(\Omega, f)$. From (H_1) and (H_2), it follows that $\frac{f(x,tu)}{t} u^{n-1}$ is an increasing function for $t > 0$. Hence, if u satisfies (3.13), it follows that $\gamma \le 1$ and $\gamma = 1$ iff $u \in \partial B(\Omega, f)$. This proves step 2.

STEP 3. $a(\Omega, f)^n < \left(\frac{\alpha_n}{b}\right)^{n-1}$.

Let $w \in W_0^{1,n}(\Omega)$ such that $\|w\| = 1$. From step 2, we can choose a $\gamma > 0$ such that $\gamma w \in \partial B(\Omega, f)$. Hence

$$\frac{a(\Omega, f)^n}{n} \le J(\gamma w) \le \frac{\gamma^n}{n}\|w\|^n = \frac{\gamma^n}{n};$$

this implies that $a(\Omega, f) \le \gamma$. Using again the fact that $\frac{f(x, tw)}{t} w^{n-1}$ is an increasing function of t in $(0, \infty)$ and $\gamma w \in \partial B(\Omega, f)$, we have

$$\int_\Omega \frac{f(x, a(\Omega, f)w)}{a(\Omega, f)} w^{n-1}\,\mathrm{d}x \le \int_\Omega \frac{f(x, \gamma w)}{\gamma} w^{n-1}\,\mathrm{d}x = 1.$$

This implies that

$$\sup_{\|w\| \le 1} \int_\Omega f(x, a(\Omega, f)w) w^{n-1}\,\mathrm{d}x \le a(\Omega, f). \tag{3.14}$$

Now from (i), (3.14) and lemma 3.2 we have $a(\Omega, f)^n < \left(\frac{\alpha_n}{b}\right)^{n-1}$. This proves the lemma.

LEMMA 3.5. *Let $f \in A(\Omega)$ and $u_0 \in \partial B(\Omega, f)$ such that $J'(u_0) \ne 0$ ($J'(u)$ denote the derivative of J at u). Then*

$$J(u_0) > \inf\{J(u);\ u \in \partial B(\Omega, f)\}.$$

PROOF. Choose $h_0 \in W_0^{1,n}(\Omega)$ such that $\langle J'(u_0), h_0\rangle = 1$ and, for $\alpha, t \in \mathbb{R}$, define $\sigma_t(\alpha) = \alpha u_0 - t h_0$. Then

$$\lim_{\substack{t \to 0 \\ \alpha \to 1}} \frac{\mathrm{d}}{\mathrm{d}t} J(\sigma_t(\alpha)) = -\langle J'(u_0), h_0\rangle = -1$$

and hence we can choose $\epsilon > 0$, $\delta > 0$ such that, for all $\alpha \in [1 - \epsilon,\ 1 + \epsilon]$ and $0 < t \le \delta$,

$$J(\sigma_t(\alpha)) < J(\sigma_0(\alpha)) = J(\alpha u_0). \tag{3.15}$$

Let

$$\rho_t(\alpha) = \|\sigma_t(\alpha)\|^n - \int_\Omega f(x, \sigma_t(\alpha))\ \sigma_t(\alpha)^{n-1}\,\mathrm{d}x.$$

Since $\frac{f(x,\alpha u_0)}{\alpha}u_0^{n-1}$ is an increasing function of α and using $u_0 \in \partial B(\Omega, f)$, by shrinking ϵ and δ if necessary, we have, for $0 < t \leq \delta$, $\rho_t(1-\epsilon) > 0$ and $\rho_t(1+\epsilon) < 0$. Hence there exists α_t such that $\rho_t(\alpha_t) = 0$. Thefore $\sigma_t(\alpha_t)$ is in $\partial B(\Omega, f)$. Hence from (3.15) we have

$$\inf\{J(u);\ u \in \partial B(\Omega, f)\} \leq J(\sigma_t(\alpha_t))$$
$$< J(\alpha_t u_0) \leq \sup_{t\in\mathbb{R}} J(tu_0) = J(u_0).$$

This proves the lemma.

PROOF OF THE THEOREM.

1) *Palais-Smale Condition.* Let $C \in \left(-\infty,\ \frac{1}{n}\left(\frac{\alpha_n}{b}\right)^{n-1}\right)$ and $\{u_k\}$ be a sequence such that

$$\text{(3.16)}\qquad \begin{aligned}&\lim_{k\to\infty} J(u_k) = C\\ &\lim_{k\to\infty} J'(u_k) = 0.\end{aligned}$$

Let $h \in W_0^{1,n}(\Omega)$, then we have

$$\text{(3.18)}\qquad \langle J'(u_k), h\rangle = \int_\Omega |\nabla u_k|^{n-2}\nabla u_k \cdot \nabla h \, dx - \int_\Omega f(x, u_k)u_k^{n-2}h \, dx.$$

Hence we have

$$\text{(3.19)}\qquad J(u_k) - \frac{1}{n}\langle J'(u_k), u_k\rangle = I(u_k).$$

CLAIM 1.

$$\text{(3.20)}\qquad \sup_k \|u_k\| + \sup_k \int_\Omega f(x, u_k)u_k^{n-1} \, dx < \infty.$$

Since $\{J(u_k)\}$ and $\{J'(u_k)\}$ are bounded and hence from (3.19), $I(u_k) = O(\|u_k\|)$. Now from 5) of lemma 3.1, we have $\int_\Omega f(x, u_k)u_k^{n-1} \, dx = O(\|u_k\|)$.

Now from (H_3) it follows that

$$\int_\Omega F(x, u_k) \, dx = O(\|u_k\|)$$

and, by using the boundedness of $J(u_k)$, we obtain $\|u_k\|^n = O(\|u_k\|)$. This implies (3.20) and hence the claim.

By extracting a subsequence, we can assume that

$$u_k \to u_0 \text{ weakly and for almost all } x \text{ in } \Omega. \tag{3.21}$$

CASE (I). $C \leq 0$.

From Fatou's lemma and 5) of lemma 3.1, we have

$$\begin{aligned} 0 \leq I(u_0) &\leq \underline{\lim}_{k\to\infty} I(u_k) \\ &= \underline{\lim}_{k\to\infty} \left\{ J(u_k) - \frac{1}{n}\langle J'(u_k), u_k\rangle \right\} \\ &= C. \end{aligned}$$

Hence $u_0 \equiv 0$. If $C < 0$, no Palais-Smale sequence exists. If $C = 0$, then from (3.20) and 4) of lemma 3.1 we have

$$\lim_{k\to\infty} \|u_k\|^n = n \lim_{k\to\infty} \left\{ J(u_k) + \int_\Omega F(x, u_k)\,\mathrm{d}x \right\} = 0.$$

This proves that $u_k \to 0$ strongly.

CASE (II). $C \in \left(0, \frac{1}{n}\left(\frac{\alpha_n}{b}\right)^{n-1}\right)$.

CLAIM 2. $u_0 \not\equiv 0$ and $u_0 \in \partial B(\Omega, f)$.

Suppose $u_0 \equiv 0$. Then, from (3.20) and 4) of lemma 3.1, we have

$$\begin{aligned} \lim_{k\to\infty} \|u_k\|^n &= n \underline{\lim}_{k\to\infty} \left\{ J(u_k) + \int_\Omega F(x, u_k)\,\mathrm{d}x \right\} \\ &= nC < \left(\frac{\alpha_n}{b}\right)^{n-1}. \end{aligned} \tag{3.22}$$

Hence, from 3) of lemma 3.1 and (3.22), we have

$$\lim_{k\to\infty} \int_\Omega f(x, u_k)u_k^{n-1}\,\mathrm{d}x = \int_\Omega f(x, u_0)u_0^{n-1}\,\mathrm{d}x = 0.$$

This implies that $\lim_{k\to\infty} I(u_k) = 0$ and hence from (3.19)

$$0 < C = \lim_{k\to\infty} J(u_k) = \lim_{k\to\infty} \left\{ I(u_k) + \frac{1}{n}\langle J'(u_k), u_k\rangle \right\} = 0$$

which is a contradiction. Hence $u_0 \not\equiv 0$. From (3.20) and 4) of lemma 3.1, taking $h \in C_0^\infty(\Omega)$ and letting $k \to \infty$ in (3.19), we obtain

$$\int_\Omega |\nabla u_0|^{n-2} \nabla u_0 \cdot \nabla h \, dx = \int_\Omega f(x, u_0) u_0^{n-2} h \, dx.$$

By density, the above equation holds for all $h \in W_0^{1,n}(\Omega)$. Hence, by taking $h = u_0$, we obtain

$$\|u_0\|^n = \int_\Omega f(x, u_0) u_0^{n-1} \, dx. \tag{3.23}$$

Hence $u_0 \in \partial B(\Omega, f)$ and this proves the claim.

Now from (3.20) and claim 2, $\{u_k, u_0\}$ satisfy all the hypotheses of the compactness lemma 3.3. Hence we have

$$\begin{aligned}
\|u_0\|^n &\leq \varliminf_{k\to\infty} \|u_k\|^n \\
&= n \varliminf_{k\to\infty} \left\{ J(u_k) + \int_\Omega F(x, u_k) \, dx \right\} \\
&= n \varliminf_{k\to\infty} \left\{ I(u_k) + \frac{1}{n} \langle J'(u_k), u_k \rangle + \int_\Omega F(x, u_k) \, dx \right\} \\
&= \varliminf_{k\to\infty} \left\{ \int_\Omega f(x, u_k) u_k^{n-1} \, dx + \langle J'(u_k), u_k \rangle \right\} \\
&= \int_\Omega f(x, u_0) u_0^{n-1} \, dx = \|u_0\|^n.
\end{aligned}$$

This implies that u_k converges to u_0 strongly. This proves the Palais-Smale condition.

2) *Existence of Positive Solution.* Since the critical points of J are the solutions of the equation (1.7) and $J(u) = J(|u|)$ for all u in $\partial B(\Omega, f)$ and hence in view of lemma 3.5, it is enough to prove that there exists $u_0 \not\equiv 0$ such that

$$\frac{a(\Omega, f)^n}{n} = J(u_0). \tag{3.24}$$

Let $u_k \in \partial B(\Omega, f)$ such that

$$\lim_{k\to\infty} J(u_k) = \frac{a(\Omega, f)^n}{n}.$$

Since $J(u_k) = I(u_k)$, and hence by 5) of lemma 3.1

$$\sup_k \int_\Omega f(x, u_k) u_k^{n-1}\, dx < \infty, \tag{3.25}$$

$$\sup_k \|u_k\| < \infty. \tag{3.26}$$

Hence we can extract a subsequence such that

$$u_k \to u_0 \text{ weakly and for almost all } x \text{ in } \Omega.$$

CLAIM 3. $u_0 \not\equiv 0$ and

$$\|u_0\|^n \leq \int_\Omega f(x, u_0) u_0^{n-1}\, dx. \tag{3.28}$$

Suppose $u_0 \equiv 0$, then from (3.25) and 4) of lemma 3.1

$$\begin{aligned} \lim_{k\to\infty} \|u_k\|^n &= n \lim_{k\to\infty} \left\{ J(u_k) + \int_\Omega F(x, u_k)\, dx \right\} \\ &= a(\Omega, f)^n. \end{aligned} \tag{3.29}$$

From lemma 3.4, we have $0 < a(\Omega, f)^n < \left(\frac{\alpha_n}{b}\right)^{n-1}$. Hence, from (3.29) and 3) of lemma 3.1, we have

$$\lim_{k\to\infty} \int_\Omega f(x, u_k) u_k^{n-1}\, dx = 0.$$

This implies that

$$0 < \frac{a(\Omega, f)^n}{n} = \lim_{k\to\infty} J(u_k) = \lim_{k\to\infty} I(u_k) = 0,$$

which is a contradiction. This proves $u_0 \not\equiv 0$. Suppose (3.28) is false, then

$$\|u_0\|^n > \int_\Omega f(x, u_0) u_0^{n-1}\, dx. \tag{3.30}$$

Now from (3.25), (3.30) and $0 < a(\Omega, f)^n < \left(\frac{\alpha_n}{b}\right)^{n-1}$, $\{u_k, u_0\}$ satisfy all the hypotheses of lemma 3.3. Hence

$$\lim_{k\to\infty} \int_\Omega f(x, u_k) u_k^{n-1}\, dx = \int_\Omega f(x, u_0) u_0^{n-1}\, dx.$$

This implies that

$$\|u_0\|^n \leq \varliminf_{k\to\infty} \|u_k\|^n = \lim_{k\to\infty} \int_\Omega f(x,u_k)u_k^{n-1}\,\mathrm{d}x$$
$$= \int_\Omega f(x,u_0)u_0^{n-1}\,\mathrm{d}x.$$

contradicting (3.30). This proves the claim.

Now from (3.28) and step 2 of lemma 3.4, there exists $0 < \gamma \leq 1$ such that $\gamma u_0 \in \partial B(\Omega, f)$. Hence

$$\frac{a(\Omega,f)^n}{n} \leq J(\gamma u_0) = I(\gamma u_0)$$
$$\leq I(u_0) \leq \varliminf_{k\to\infty} I(u_k)$$
$$= \varliminf_{k\to\infty} J(u_k) = \frac{a(\Omega,f)^n}{n}.$$

This implies that $\gamma = 1$ and $u_0 \in \partial B(\Omega, f)$. Hence $J(u_0) = \frac{a(\Omega,f)^n}{n}$ and this proves the Theorem.

4. Concluding Remarks

REMARK 4.1. (Regularity). From Di-Benedetto [6], Tolksdorf [12] and Gilbarg-Trudinger [8], any solution of (1.7) is in $C^{1,\alpha}(\Omega)$ if $n \geq 3$ and in $C^{2,\alpha}(\overline{\Omega})$ if $n = 2$.

REMARK 4.2. Let $f \in A(\Omega)$ and $f'(x,0) < \lambda_1(\Omega)$ for all $x \in \overline{\Omega}$. We prove the existence of a solution for (1.7) under the assumption that

$$\varlimsup_{t\to\infty} \inf_{x\in\overline{\Omega}} h(x,t)t^{n-1} = \infty. \tag{4.1}$$

The only place where it is used is to show that $a(\Omega,f)^n < \left(\frac{\alpha_n}{b}\right)^{n-1}$. But, from lemma 3.2, this inequality holds if

$$\frac{k_0}{b} < 1. \tag{4.2}$$

Hence the theorem is true under the less restrictive condition (4.2).

Now the question is what happens if $\frac{k_0}{b} \geq 1$ or the condition (4.1) is not satisfied. In this regard, we have (jointly with Srikanth - Yadava) obtained a partial result, which states that there are functions $f \in A(\Omega)$ such that

$$\lim_{t\to\infty} \inf_{x\in\overline{\Omega}} h(x,t)t^{n-1} = 0$$

for which no solution to problem (1.7) exists if Ω is a ball of sufficiently small radius. In this context, we raise the following question:

Open Problem. Let Ω be a ball and $f \in A(\Omega)$ such that $\sup\limits_{x\in\overline{\Omega}} f'(x,0) < \lambda_1(\Omega)$. Is (4.2) also a necessary condition to obtain a solution to the problem (1.7).

In the case $n = 2$, this question is related to the question of Brézis [3]: "where is the border line between the existence and non-existence of a solution of (1.7)?".

REMARK 4.3. Let $\beta \geq 0$, then by using the norm

$$\left(\int_\Omega |\nabla u|^n \, dx + \beta \int_\Omega |u|^n \, dx\right)^{1/n}$$

in $W_0^{1,n}(\Omega)$, the Theorem still holds if we replace $-\Delta_n u$ by $-\Delta_n u + \beta|u|^{n-2}u$ in the equations (1.7).

Due to this and using a result of Cherrier [5], it is possible to extend the Theorem to compact Riemann surfaces.

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5. - Appendix

PROOF OF THE LEMMA 3.1.

1) Let $f(x,t) = h(x,t) \exp\left(b|t|^{n/(n-1)}\right) \in A(\Omega)$. From (H_4), for every $\epsilon > 0$, there exists a $C(\epsilon) > 0$ such that

$$|f(x,t)| \leq C(\epsilon) \exp\left((b+\epsilon)|t|^{n/(n-1)}\right)$$

and hence, from theorem 2.1, $f(x,u) \in L^p(\Omega)$ for every $p < \infty$.

2) From (H_4), for every $\epsilon > 0$, there exist positive constants $C_1(\epsilon)$ and $C_2(\epsilon)$ such that

$$|f(x,t)t^{n-1}| \leq C_1(\epsilon) \exp\left(b(1+\epsilon)|t|^{n/(n-1)}\right) \tag{5.1}$$

$$|f(x,t)t^{n-1}| \geq C_2(\epsilon)\exp\left(b(1-\epsilon)|t|^{n/(n-1)}\right) \text{ for } |t| \geq 1. \tag{5.2}$$

Hence, if $c > 0$ such that

$$\sup_{\|w\|\leq 1}\int_\Omega f(x,cw)w^{n-1}\,\mathrm{d}x < \infty,$$

it implies that, for every $\epsilon > 0$,

$$\sup_{\|w\|\leq 1}\int_\Omega \exp\left(b(1-\epsilon)c^{n/(n-1)}|w|^{n/(n-1)}\right)\,\mathrm{d}x < \infty.$$

Therefore, from Theorem 2.1, we have

$$(1-\epsilon)^{n-1}c^n \leq \left(\frac{\alpha_n}{b}\right)^{n-1}.$$

This implies that

$$\sup\left\{c^n;\ \sup_{\|w\|\leq 1}\int_\Omega f(x,cw)w^{n-1}\,\mathrm{d}x < \infty\right\} \leq \left(\frac{\alpha_n}{b}\right)^{n-1}.$$

On the other hand, if $c^n < \left(\frac{\alpha_n}{b}\right)^{n-1}$, then by choosing $\epsilon > 0$ such that $(1+\epsilon)^{2n-1}c^n < \left(\frac{\alpha_n}{b}\right)^{n-1}$, from Theorem 2.1 and from (5.1), we have

$$\begin{aligned}&\sup_{\|w\|\leq 1}\int_\Omega f(x,(1+\epsilon)cw)w^{n-1}\,\mathrm{d}x\\ &\leq C_1(\epsilon)\sup_{\|w\|\leq 1}\int_\Omega \exp\left[b\left((1+\epsilon)c|w|\right)^{n/(n-1)}\right]\,\mathrm{d}x < \infty\end{aligned}$$

this proves

$$\sup\left\{c^n;\ \sup_{\|w\|\leq 1}\int_\Omega f(x,cw)w^{n-1}\mathrm{d}x < \infty\right\} = \left(\frac{\alpha_n}{b}\right)^{n-1}.$$

3) Since $\overline{\lim_{k\to\infty}}\|u_k\|^n < \left(\frac{\alpha_n}{b}\right)^{n-1}$, from 2) we can choose a $p > 1$ such that

$$c_1^p = \sup_k\int_\Omega |f(x,u_k)|^p\,\mathrm{d}x < \infty.$$

Let $\frac{1}{p} + \frac{1}{q} = 1$ and

$$c_2^q = \sup_k \int_\Omega |v_k|^{\ell q}\, dx.$$

Then, for any $N > 0$ and by Holder's inequality,

$$\left| \int_{|u_k|>N} \frac{f(x,u_k)}{u_k} v_k^\ell\, dx \right| \le \frac{1}{N} \int_\Omega |f(x,u_k)|\ |v_k^\ell|\, dx \le \frac{c_1 c_2}{N}.$$

Hence

$$\int_\Omega \frac{f(x,u_k)}{u_k} v_k^\ell\, dx = \int_{|u_k|\le N} \frac{f(x,u_k)}{u_k} v_k^\ell\, dx + O(1/N).$$

By dominated convergence theorem, letting $k \to \infty$ and then $N \to \infty$ in the above equation, it implies that

$$\lim_{k\to\infty} \int_\Omega \frac{f(x,u_k)}{u_k} v_k^\ell\, dx = \int_\Omega \frac{f(x,u)}{u} v^\ell\, dx.$$

4) Let $N > 0$, then

$$\int_{|u_k|>N} f(x,|u_k|)|u_k|^{n-2+\tau}\, dx \le \frac{1}{N^{1-\tau}} \int_\Omega f(x,|u_k|)|u_k|^{n-1}\, dx$$

$$= \frac{1}{N^{1-\tau}} \int_\Omega f(x,u_k) u_k^{n-1}\, dx = O\left(\frac{1}{N^{1-\tau}}\right).$$

Hence

$$\int_\Omega f(x,|u_k|)|u_k|^{n-2+\tau}\, dx = \int_{|u_k|\le N} f(x,|u_k|)|u_k|^{n-2+\tau}\, dx + O\left(\frac{1}{N^{1-\tau}}\right).$$

By dominated convergence theorem, letting $k \to \infty$ and $N \to \infty$ in the above equation, we obtain

$$\lim_{k\to\infty} \int_\Omega f(x,|u_k|)|u_k|^{n-2+\tau}\, dx = \int_\Omega f(x,|u|)|u|^{n-2+\tau}\, dx. \tag{5.3}$$

Now from (H_3),

$$|F(x,t)| \le M(1 + |f(x,t)|\ |t|^{n-2+\sigma})$$

for some $\sigma \in [0, 1)$. Hence, from (5.3) and the dominated convergence theorem,

$$\lim_{k\to\infty} \int_{\Omega} F(x, u_k)\, dx = \int_{\Omega} F(x, u)\, dx.$$

5) From (H_2) we have, for $t > 0$,

$$\frac{\partial}{\partial t}\left[f(x,t)t^{n-1} - nF(x,t)\right] = \left[f'(x,t) - \frac{f(x,t)}{t}\right] t^{n-1} > 0. \tag{5.4}$$

Therefore from (H_1) and (5.4), $f(x,t)t^{n-1} - nF(x,t)$ is an even positive function and increasing for $t > 0$. This implies that $I(u) \geq 0$ and $I(u) = 0$ iff $u \equiv 0$. From (H_3) we have

$$\begin{aligned} nI(u) &= \int_{\Omega} \left[f(x,u)u^{n-1} - nF(x,u)\right] dx \\ &\geq \int_{\Omega} \left[f(x,u)u^{n-1} - nM(1 + |f(x,u)|\ |u|^{n-2+\sigma})\right] dx \\ &\geq C_1 + \frac{1}{2} \int_{|u|\geq C_2} f(x,u)u^{n-1}\, dx \end{aligned}$$

for some constants C_1 and $C_2 > 0$. This implies that there exists a constant $M_1 > 0$ such that

$$\int_{\Omega} f(x,u)u^{n-1}\, dx \leq M(1 + I(u)).$$

This proves the lemma 3.1.

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