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Existence of positive solutions of the semilinear Dirichlet problem with critical growth for the $n$-laplacian

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Existence of Positive Solutions of the Semilinear Dirichlet Problem with Critical Growth for the $n$-Laplacian

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1. - Introduction

Let $\Omega$ be a bounded open set in $\mathbb{R}^n$ with smooth boundary. We are looking for a solution of the following problem:

Let $1 < p \leq n$, find $u \in W_0^{1,p}(\Omega) \setminus \{0\}$ such that

$$
\Delta_p u = f(x, u)|u|^{p-2} \quad \text{in } \Omega
$$

$$
u \geq 0,
$$

(1.1)

where $\Delta_p u = \text{div}(|\nabla u|^{p-2} \nabla u)$ is the $p$-Laplacian and $f : \bar{\Omega} \times \mathbb{R} \to \mathbb{R}$ is a $C^1$-function with $f(x, 0) = 0$, $f(x, t) \geq 0$ for $t \geq 0$ and of critical growth.

For $p = 2$ and $n > 3$, Brezis-Nirenberg [4] have studied the existence and non-existence of solution of (1.1) when $f$ has critical growth of the form $u^{(n+2)/(n-2)} + \lambda u$. A generalization of this result, on the same lines, for the $p$-Laplacian with $p \leq n$ and $p^2 \leq n$, has been studied by Garcia Azorero-Peral Alonso [7]. When $p = n$, in view of the Trudinger [13] imbedding, a critical growth function $f(x, u)$ behaves like $\exp \left( b|u|^{n/(n-1)} \right)$ for some $b > 0$. In this context, when $p = n = 2$ and $\Omega$ is a ball in $\mathbb{R}^2$, existence of a solution of (1.1) has been studied by Adimurthi [1], Atkinson-Peletier [2]. The method used by Atkinson-Peletier is a shooting method and hence cannot be generalized to solve (1.1) in an arbitrary domain. Whereas in Adimurthi [1], (1.1) is solved via variational method which is in the spirit of Brezis-Nirenberg [4] and, based on this method, we prove the following main result in this paper.

Let $f(x, t) = h(x, t) \exp \left( b|t|^{n/(n-1)} \right)$ be a function of critical growth and $F(x, t)$ be its primitive (see definition (2.1)). For $u \in W_0^{1,n}(\Omega)$, let

$$
J(u) = \frac{1}{n} \int_{\Omega} |\nabla u|^n \, dx - \int_{\Omega} F(x, u) \, dx
$$

(1.2)

Theorem Let \( f(x, t) = h(x, t) \exp(b|t|^{n/(n-1)}) \) be a function of critical growth on \( \Omega \). Then

1) \( J : W_0^{1,n}(\Omega) \rightarrow \mathbb{R} \) satisfies the Palais-Smale Condition on the interval \( (-\infty, \frac{1}{n} \left( \frac{\alpha_n}{b} \right)^{n-1}) \);

2) Let \( f'(x, t) = \frac{\partial}{\partial t} f(x, t) \) and further assume that

\[
\lim_{t \to \infty} \inf_{x \in \Omega} h(x, t)t^{n-1} = \infty,
\]

then there exists some \( u_0 \in W_0^{1,n}(\Omega) \setminus \{0\} \) such that

\[
\Delta_n u_0 = f(x, u_0)u_0^{n-2} \quad \text{in } \Omega
\]

\[
u_0 \geq 0
\]

\[
u_0 = 0 \quad \text{on } \partial \Omega.
\]

The method adopted to solve (1.7) in Brézis-Nirenberg [4] does not work because of the critical growth is of exponential type. Here we adapt the method of artificial constraint due to Nehari [11]. The main idea of the proof is as follows:

Define

\[
a(\Omega, f)^n = \inf \left\{ J(u); \quad \int_{\Omega} |\nabla u|^n \, dz = \int_{\Omega} f(x, u)u^{n-1} \, dx, \quad u \neq 0 \right\},
\]

then the minimizer of (1.8) is a solution of (1.7).

It has to be noted that \( \alpha_n \) is the best constant appearing in Moser’s [10] result about the Trudinger’s imbedding of \( W_0^{1,n}(\Omega) \). In view of this, one expects that \( J \) should satisfy the Palais-Smale Condition on \( \left(-\infty, \frac{1}{n} \left( \frac{\alpha_n}{b} \right)^{n-1}\right) \).

Therefore, in order to get a minimizer of (1.8), the question remains to show that

\[
a(\Omega, f)^n < \left( \frac{\alpha_n}{b} \right)^{n-1}
\]
and this has been achieved by showing the following relation

\begin{equation}
\sup_{a} \int_{|\nabla w|^{n} dx \leq 1} f(x, a(\Omega, f)w^{n-1} dx \leq a(\Omega, f).
\end{equation}

In the forthcoming paper (jointly with Yadava), we discuss the bifurcation and multiplicity results for (1.7) when \( n = 2 \).

\section{Preliminaries}

Let \( \Omega \) be a bounded domain with smooth boundary. In view of the Trudinger-Moser [13,10] imbedding, we have the following definition of functions of critical growth.

**Definition 2.1.** Let \( h : \overline{\Omega} \times \mathbb{R} \rightarrow \mathbb{R} \) be a \( C^{1} \)-function and \( b > 0 \). Let \( f(x, t) = h(x, t) \exp \left( b|t|^{n/(n-1)} \right) \). We say that \( f \) is a function of critical growth on \( \Omega \) if the following holds:

There exist constants \( M > 0, \sigma \in [0, 1) \) such that, for every \( \epsilon > 0 \), and for every \( (x, t) \in \overline{\Omega} \times (0, \infty) \),

- (H1) \( f(x, 0) = 0 \), \( f(x, t) > 0 \), \( f(x, t)t^{n-1} = f(x, -t)(-t)^{n-1} \);
- (H2) \( f'(x, t) > \frac{f(x, t)}{t} \), where \( f'(x, t) = \frac{\partial f}{\partial t}(x, t) \);
- (H3) \( F(x, t) \leq M(1 + f(x, t)t^{n-2\sigma}) \), where

\[ F(x, t) = \int_{0}^{t} f(x, s)s^{n-2} ds \]

is the primitive of \( f \);

- (H4) \( \limsup_{t \rightarrow \infty} \sup_{x \in \overline{\Omega}} h(x, t) \exp \left( -\epsilon t^{n/(n-1)} \right) = 0 \),

\[ \liminf_{t \rightarrow \infty} h(x, t) \exp \left( \epsilon t^{n/(n-1)} \right) = \infty. \]

Let \( A(\Omega) \) denote the set of all functions of critical growth on \( \Omega \).

**Examples.** In view of (H1), it is enough to define \( f \) on \( \overline{\Omega} \times (0, \infty) \).

1) For \( m \geq 1, \ b > 0, \ \beta \geq 0 \) and \( 0 \leq \alpha < \frac{n}{n-1} \), \( f(x, t) = t^{m} \exp(b\alpha t^{n/(n-1)}) \) is in \( A(\Omega) \).
2) \( f(x, t) = t^{2}e^{-t} \exp \left( t^{n/(n-1)} \right) \) is in \( A(\Omega) \).
3) Let \( f(x, t) = h(x, t) \exp(b\alpha t^{n/(n-1)}) \), satisfying (H1) and (H4).
Further assume that \( h'(x, t) \geq \frac{h(x,t)}{t} \) for \((x, t) \in \overline{\Omega} \times (0, \infty)\). Then \( f \) is in \( A(\Omega) \).

For
\[
\frac{f'(x,t)}{f(x,t)} = \frac{h'(x,t)}{h(x,t)} + \frac{nb}{n-1} t^{1/(n-1)} > \frac{1}{t}
\]
and hence \( f \) satisfy \((H_2)\).

Let \( \epsilon > 0 \), and \( \sigma = \frac{1}{n-1} \)

\[
F(x,t) - F(x,\epsilon) = \frac{n-1}{nb} \int_{\epsilon}^{t} h(x,s)s^{n-1-\sigma} \frac{d}{ds} \exp \left( bs^{n/(n-1)} \right) ds
\]

\[
\leq \frac{n-1}{nb} \left[ f(x,t)t^{n-2-\sigma} - f(x,\epsilon)t^{n-2-\sigma} \right].
\]

This implies that there exists a constant \( M > 0 \) such that \( F(x, t) \leq M[1 + f(x,t)t^{n-2-\sigma}] \) for \((x, t) \in \overline{\Omega} \times (0, \infty)\). This shows that \( f \) satisfy \((H_3)\) and hence \( f \in A(\Omega) \).

Let \( W_0^{1,n}(\Omega) \) be the usual Sobolev space and \( f(x,t) = h(x,t) \exp(\epsilon t^{n/(n-1)}) \) be in \( A(\Omega) \). For \( u \in W_0^{1,n}(\Omega) \), define

(2.1) \[\|u\|^n = \int_{\Omega} |\nabla u|^n \, dx\]

(2.2) \[J(u) = \frac{1}{n} \|u\|^n - \int_{\Omega} F(x,u) \, dx\]

(2.3) \[I(u) = \frac{1}{n} \int_{\Omega} f(x,u)u^{n-1} \, dx - \int_{\Omega} F(x,u) \, dx\]

(2.4) \[\partial B(\Omega, f) = \left\{ u \in W_0^{1,n}(\Omega) \setminus \{0\} ; \|u\|^n = \int_{\Omega} f(x,u)u^{n-1} \, dx \right\}\]

(2.5) \[\frac{a(\Omega, f)^n}{n} = \inf \{ J(u) ; u \in \partial B(\Omega, f) \}\]

(2.6) \[\lambda_1(\Omega) = \inf \left\{ \|u\|^n ; \int_{\Omega} |u|^n \, dx = 1 \right\}\]

\[\alpha_n = n\omega_n^{1/(n-1)} \text{, where } \omega_n = \text{Volume of } S^{n-1} \]
DEFINITION OF MOSER FUNCTIONS. Let $x_0 \in \Omega$ and $R \leq d(x_0, \partial \Omega)$, where $d$ denotes the distance from $x_0$ to $\partial \Omega$. For $0 < \ell < R$, define

$$m_{\ell,R}(x,x_0) = \begin{cases} (\log \frac{R}{\ell})^{1-\frac{1}{n}} & \text{if } 0 \leq |x - x_0| \leq \ell \\ \frac{\log \frac{R}{\ell}}{(\log \frac{R}{\ell})^{\frac{1}{n}}} & \text{if } \ell \leq r = |x - x_0| \leq R \\ 0 & \text{if } |x - x_0| \geq R. \end{cases}$$

Then it is easy to see that $m_{\ell,R} \in W^{1,n}_0(\Omega)$ and $\|m_{\ell,R}\| = 1$.

For the proof of our theorem, we need the following two results whose proof is found in Moser [10] and P.L. Lions [9] respectively.

**THEOREM 2.1** (Moser). 1) Let $u \in W^{1,n}_0(\Omega)$, and $p < \infty$, then $\exp \left( |u|^{n/(n-1)} \right) \in L^p(\Omega)$.

2) $\left( \frac{\alpha_n}{\beta} \right)^{n-1} = \max \left\{ e^n; \sup_{\|w\| \leq 1} \int \exp \left( b \frac{\|w\|}{n} \right) \right\}$.

**THEOREM 2.2** (P.L. Lions). Let $\{u_k; \|u_k\| = 1 \}$ be a sequence in $W^{1,n}_0(\Omega)$ converging weakly to a non-zero function $u$. Then, for every $p < (1 - \|u\|^{n-1})^{1/(n-1)}$,

$$\sup_k \int_{\Omega} \exp \left( \frac{p\alpha_n}{\beta} |u_k|^{n/(n-1)} \right) \, dx < \infty.$$

### 3. Proof of the Theorem

We need a few lemmas to prove the theorem. The proof of the following lemma is given in the appendix.

**LEMMA 3.1.** Let $f \in A(\Omega)$. Then we have

1) If $u \in W^{1,n}_0(\Omega)$, then $f(x,u) \in L^p(\Omega)$ for all $p \geq 0$.

2) $\left( \frac{\alpha_n}{\beta} \right)^{n-1} = \sup \left\{ e^n; \sup_{\|w\| \leq 1} \int f(x, cw) w^{n-1} \, dx < \infty \right\}$.

3) Let $\{u_k\}$ and $\{v_k\}$ be bounded sequences in $W^{1,n}_0(\Omega)$ converging weakly and for almost every $x$ in $\Omega$ to $u$ and $v$ respectively. Further assume that

$$\bar{\lim}_{k \to \infty} \|u_k\|^n < \left( \frac{\alpha_n}{\beta} \right)^{n-1}.$$

Then, for every integer $\ell \geq 0$,

$$\lim_{k \to \infty} \int_{\Omega} \frac{f(x,u_k)}{u_k} v_k' \, dx = \int_{\Omega} \frac{f(x,u)}{u} v' \, dx.$$
4) Let \( \{u_k\} \) be a sequence in \( W^{1,n}_0(\Omega) \) converging weakly and for almost every \( x \) in \( \Omega \) to \( u \), such that

\[
\sup_k \int_{\Omega} f(x, u_k) u_k^{n-1} \, dx < \infty.
\]

Then, for any \( 0 \leq \tau < 1 \),

\[
\lim_{k \to \infty} \int_{\Omega} f(x, |u_k|) |u_k|^{n-2+\tau} \, dx = \int_{\Omega} f(x, |u|) |u|^{n-2+\tau} \, dx
\]

and

\[
\lim_{k \to \infty} \int_{\Omega} F(x, u_k) \, dx = \int_{\Omega} F(x, u) \, dx.
\]

5) \( I(u) \geq 0 \) for all \( u \) and \( I(u) = 0 \) iff \( u \equiv 0 \). Further, there exists a constant \( M_1 > 0 \) such that, for all \( u \in W^{1,n}_0(\Omega) \),

\[
\int_{\Omega} f(x, u) u^{n-1} \, dx \leq M_1(1 + I(u)).
\]

**Lemma 3.2.** Let \( f = h\exp(\beta|t|^{n/(n-1)}) \in A(\Omega) \) and define

\[
h_0(t) = \inf_{x \in \Omega} h(x, t), \quad M_0 = \sup_{t \geq 0} h_0(t) t^{n-1}, \quad R_0 = \sup_{x \in \partial \Omega} d(x, \partial \Omega),
\]

and

\[
k_0 = \begin{cases} 
\left( \frac{\alpha a}{\beta} \right)^{n/(n-1)} M_0^{-1/(n-1)} & \text{if } M_0 < \infty \\
0 & \text{if } M_0 = \infty.
\end{cases}
\]

Let \( a \geq 0 \) be such that

\[
\sup_{||w|| \leq 1} \int_{\Omega} f(x, aw) w^{n-1} \, dx \leq a.
\]

If \( k_0 \beta \leq 1 \), then \( a^n < \left( \frac{\alpha a}{\beta} \right)^{n-1} \).

**Proof.** From 2) of lemma 3.1, we have \( a^n \leq \left( \frac{\alpha a}{\beta} \right)^{n-1} \). Suppose \( a^n = \left( \frac{\alpha a}{\beta} \right)^{n-1} \). Let \( x_0 \in \Omega \) such that \( d(x_0, \partial \Omega) = R_0 \) and \( 0 \leq t < R_0 \). Let

\[
m_t(x) = m_{t, R_0}(x, x_0).
\]
be the Moser functions and
\[ t = a \omega_{n-1/n} \left( \log \frac{R_0}{\ell} \right)^{(n-1)/n}, \]
then from (3.1) we have
\[ a \geq \int_{\Omega} f(x, am_\ell) m_\ell^{n-1} \, dx \]
\[ \geq \int_{B(x_0, \ell)} h_0(am_\ell)m_\ell^{n-1} \exp \left( ba^{n/(n-1)}m_\ell^{n/(n-1)} \right) \, dx \]
\[ = \frac{h_0(t)t^{n-1}\omega_n R_0^n}{na^{n-1}}. \]
This implies that
\[ \left( \frac{\alpha_n}{b} \right)^{n-1} = a^n \geq \frac{h_0(t)t^{n-1}\omega_n R_0^n}{n}. \]
That is, for all \( t \in (0, \infty) \),
\[ b \leq \left( \frac{n}{R_0} \right)^{n/(n-1)} \left( h_0(t)t^{n-1} \right)^{-1/(n-1)} \]
and hence
\[ b \leq \left( \frac{n}{R_0} \right)^{n/(n-1)} \inf_{t \geq 0} \left( h_0(t)t^{n-1} \right)^{-1/(n-1)} \leq k_0 \]
which contradicts the hypothesis \( b > k_0 \). Hence \( a^n < \left( \frac{\alpha_n}{b} \right)^{n-1} \) and this proves the lemma.

**LEMMA 3.3.** (Compactness Lemma). Let \( f \) be in \( A(\Omega) \) and \( \{u_k\} \) be a sequence in \( W_0^{1,n}(\Omega) \) converging weakly and for almost every \( x \in \Omega \) to a non-zero function \( u \). Further, assume that

(i) There exists \( C \in \left( 0, \frac{1}{n} \left( \frac{\alpha_n}{b} \right)^{n-1} \right) \) such that \( \lim_{k \to \infty} J(u_k) = C; \)

(ii) \( \|u\|^n \geq \int \Omega f(x, u)u^{n-1} \, dx; \)

(iii) \( \sup_k \int \Omega f(x, u_k)u_k^{n-1} \, dx < \infty; \)

then
\[ \lim_{k \to \infty} \int \Omega f(x, u_k)u_k^{n-1} \, dx = \int \Omega f(x, u)u^{n-1} \, dx. \]
PROOF. From 5) of lemma 3.1, \( I(u) > 0 \). Therefore, from (ii) we have
\[
J(u) \geq I(u) > 0 \quad \text{and} \quad J(u) \leq \lim_{k \to \infty} J(u_k) = C.
\]
Hence we can choose an \( \epsilon > 0 \) such that
\[
(C - J(u)) (1 + \epsilon)^{n-1} < \frac{1}{n} \left( \frac{\alpha_n}{b} \right)^{n-1}.
\]

Let \( \beta = \int_{\Omega} F(x, u) \, dx \). Then, from (iii) and 4) of lemma 3.1, we have
\[
\lim_{k \to \infty} \|u_k\|^n = n \lim_{k \to \infty} \left\{ J(u_k) + \int_{\Omega} F(x, u_k) \, dx \right\}
= n(C + \beta).
\]

From (3.2) and (3.3) we can choose a \( k_0 > 0 \) such that, for all \( k \geq k_0 \),
\[
(1 + \epsilon)^{n-1} \left( \frac{b}{\alpha_n} \right)^{n-1} \|u_k\|^n < \frac{C + \beta}{C - J(u)} = \left( 1 - \frac{\|u\|^n}{n(C + \beta)} \right)^{-1}.
\]

Now choose \( p \) such that
\[
(1 + \epsilon)^{n-1} \left( \frac{b}{\alpha_n} \right)^{n-1} \|u_k\|^n \leq p^{n-1} < \frac{C + \beta}{C - J(u)}.
\]

Applying theorem 2.2 to the sequence \( \frac{u_k}{\|u_k\|} \) and using (3.3) and (3.5), we have
\[
\sup_k \int_{\Omega} \exp \left[ p\alpha_n \left( \frac{u_k}{\|u_k\|} \right)^{n/(n-1)} \right] \, dx < \infty.
\]

From (3.5) and (3.6), we have
\[
\sup_k \int_{\Omega} \exp \left[ (1 + \epsilon)^{n-1} b|u_k|^{n/(n-1)} \right] \, dx
\leq \sup_k \int_{\Omega} \exp \left[ p\alpha_n \left( \frac{u_k}{\|u_k\|} \right)^{n/(n-1)} \right] \, dx < \infty.
\]

Let
\[
M_1 = \sup_{(x,t) \in \overline{\Omega} \times \mathbb{R}} |h(x,t)t^{n-1}| \exp \left( -\epsilon^2 |t|^{n/(n-1)} \right).
\]
and $N > 0$. Then from (3.7) we have

\[
\int \frac{f(x, u_k)u_k^{n-1}}{|u_k| \geq N} \, dx = \int \frac{h(x, u_k)u_k^{n-1}}{|u_k| \geq N} \exp \left( b|u_k|^n/(n-1) \right) \, dx
\]

\[
\leq M_1 \int \exp \left( -\frac{b}{2}|u_k|^n/(n-1) \right) \exp \left[ (1 + \epsilon)b|u_k|^n/(n-1) \right] \, dx
\]

\[
= O \left( \exp \left( -\frac{b}{2}N^n/(n-1) \right) \right).
\]

Hence

\[
\int \frac{f(x, u_k)u_k^{n-1}}{|u_k| \leq N} \, dx = \int f(x, u_k)u_k^{n-1} \, dx + O \left( \exp \left( -\frac{b}{2}N^n/(n-1) \right) \right).
\]

Now letting $k \to \infty$, and $N \to \infty$ in the above equation, we obtain

\[
\lim_{k \to \infty} \int \frac{f(x, u_k)u_k^{n-1}}{|u_k| \leq N} \, dx = \int f(x, u)u^{n-1} \, dx.
\]

This proves the lemma.

**LEMMA 3.4.** Let $f \in A(\Omega)$ and assume that

(i) $\lim_{t \to \infty} h_0(t)t^{n-1} = \infty$,

where $h_0(t) = \inf \{ h(x, t) : x \in \Omega \}$;

(ii) $\sup_{x \in \partial \Omega} f'(x, 0) < \lambda_1(\Omega)$;

then

\[
0 < a(\Omega, f) = \left( \frac{\alpha_n}{b} \right)^{n-1}.
\]

**PROOF.** The lemma is proved in several steps.

**STEP 1.** $a(\Omega, f) > 0$.

Suppose $a(\Omega, f) = 0$. Then there exists a sequence $\{u_k\}$ in $\partial B(\Omega, f)$ such that $J(u_k) \to 0$ as $k \to \infty$. Since $J(u_k) = I(u_k)$, hence from (5) of lemma 3.1

\[
\sup_k \int_{\Omega} f(x, u_k)u_k^{n-1} \, dx < \infty
\]

(3.9)

\[
\sup_k \|u_k\|^n < \infty.
\]

(3.10)
Then, by extracting a subsequence, we can assume that \( \{u_k\} \) converges weakly and for almost every \( x \) in \( \Omega \) to a function \( u \). Now by Fatou's lemma,

\[
0 \leq I(u) \leq \lim_{k \to \infty} I(u_k) = \lim_{k \to \infty} J(u_k) = 0.
\]

Hence \( u \equiv 0 \). From (3.9) and 4) of lemma 3.1, we have

\[
\lim_{k \to \infty} \|u_k\|^n = n \lim_{k \to \infty} \left\{ J(u_k) + \int_{\Omega} F(x, u_k) \, dx \right\} = 0.
\]

Let \( v_k = \frac{u_k}{\|u_k\|} \) and converging weakly to \( v \). Using \( u_k \in \partial B(\Omega, f) \), (3.12), 3) of lemma 3.1 and (ii), we have

\[
1 = \lim_{k \to \infty} \int_{\Omega} f(x, u_k) \frac{\psi_k}{u_k} \, dx = \int_{\Omega} f'(x, 0) u^n \, dx < \lambda_1(\Omega) \int_{\Omega} v^n \, dx \leq 1,
\]

which is a contradiction. This prove step 1.

**STEP 2.** For every \( u \in W^{1,n}_0(\Omega) \setminus \{0\} \), there exists a constant \( \gamma > 0 \) such that \( \gamma u \in \partial B(\Omega, f) \). Moreover, if

\[
\|u\|^n \leq \int_{\Omega} f(x, u) u^{n-1} \, dx,
\]

then \( \gamma \leq 1 \) and \( \gamma = 1 \) iff \( u \in \partial B(\Omega, f) \).

For \( \gamma > 0 \), define

\[
\psi(\gamma) = \frac{1}{\gamma} \int_{\Omega} f(x, \gamma u) u^{n-1} \, dx.
\]

Then, from 3) of lemma 3.1 and (ii), we have

\[
\lim_{\gamma \to 0} \psi(\gamma) = \int_{\Omega} f'(x, 0) u^n \, dx < \|u\|^n,
\]

\[
\lim_{\gamma \to \infty} \psi(\gamma) = \infty.
\]

Hence there exists \( \gamma > 0 \) such that \( \psi(\gamma) = \|u\|^n \); this implies that \( \gamma u \in \partial B(\Omega, f) \). From \( (H_1) \) and \( (H_2) \), it follows that \( f(\frac{x}{t}, u) u^{n-1} \) is an
increasing function for \( t > 0 \). Hence, if \( u \) satisfies (3.13), it follows that \( \gamma \leq 1 \) and \( \gamma = 1 \) iff \( u \in \partial B(\Omega, f) \). This proves step 2.

**STEP 3.** \( a(\Omega, f)^n < \left( \frac{\alpha_n}{b} \right)^{n-1} \).

Let \( w \in W_0^{1,n}(\Omega) \) such that \( \|w\| = 1 \). From step 2, we can choose a \( \gamma > 0 \) such that \( \gamma w \in \partial B(\Omega, f) \). Hence

\[
\frac{a(\Omega, f)^n}{n} \leq J(\gamma w) \leq \frac{\gamma^n}{n} \|w\|^n = \frac{\gamma^n}{n};
\]

this implies that \( a(\Omega, f) \leq \gamma \). Using again the fact that \( f(x, tw) w^{n-1} \) is an increasing function of \( t \) in \((0, \infty)\) and \( \gamma w \in \partial B(\Omega, f) \), we have

\[
\int_{\Omega} \frac{f(x, a(\Omega, f)w)}{a(\Omega, f)} w^{n-1} \, dx \leq \int_{\Omega} \frac{f(x, \gamma w)}{\gamma} w^{n-1} \, dx = 1.
\]

This implies that

\[
(3.14) \quad \sup_{\|w\| \leq 1} \int_{\Omega} f(x, a(\Omega, f)w) w^{n-1} \, dx \leq a(\Omega, f).
\]

Now from (i), (3.14) and lemma 3.2 we have \( a(\Omega, f)^n < \left( \frac{\alpha_n}{b} \right)^{n-1} \). This proves the lemma.

**LEMMA 3.5.** Let \( f \in A(\Omega) \) and \( u_0 \in \partial B(\Omega, f) \) such that \( J'(u_0) \neq 0 \) (\( J'(u) \) denote the derivative of \( J \) at \( u \)). Then

\[
J(u_0) > \inf \{J(u); u \in \partial B(\Omega, f)\}.
\]

**PROOF.** Choose \( h_0 \in W_0^{1,n}(\Omega) \) such that \( (J'(u_0), h_0) = 1 \) and, for \( \alpha, t \in \mathbb{R} \), define \( \sigma_t(\alpha) = \alpha u_0 - t h_0 \). Then

\[
\lim_{\alpha \to 1} \lim_{t \to 0} \frac{d}{dt} J(\sigma_t(\alpha)) = -(J'(u_0), h_0) = -1
\]

and hence we can choose \( \epsilon > 0, \delta > 0 \) such that, for all \( \alpha \in [1 - \epsilon, 1 + \epsilon] \) and \( 0 < t \leq \delta \),

\[
(3.15) \quad J(\sigma_t(\alpha)) < J(\sigma_0(\alpha)) = J(\alpha u_0).
\]

Let

\[
\rho_t(\alpha) = ||\sigma_t(\alpha)||^n - \int_{\Omega} f(x, \sigma_t(\alpha)) \sigma_t(\alpha)^{n-1} \, dx.
\]
Since \( f(x, \alpha u_0) u_0^{n-1} \) is an increasing function of \( \alpha \) and using \( u_0 \in \partial B(\Omega, f) \), by shrinking \( \epsilon \) and \( \delta \) if necessary, we have, for \( 0 < t \leq \delta \), \( \rho(t(1-\epsilon)) > 0 \) and \( \rho(t(1+\epsilon)) < 0 \). Hence there exists \( \alpha_t \) such that \( \rho_t(\alpha_t) = 0 \). Therefore \( \sigma_t(\alpha_t) \) is in \( \partial B(\Omega, f) \). Hence from (3.15) we have

\[
\inf\{J(u); \ u \in \partial B(\Omega, f)\} \leq J(\sigma_t(\alpha_t)) \\
< J(\alpha_t u_0) \leq \sup_{t \in \mathbb{R}} J(tu_0) = J(u_0).
\]

This proves the lemma.

**Proof of the Theorem.**

1) **Palais-Smale Condition.** Let \( C \in \left( -\infty, \frac{1}{n} \left( \frac{\alpha_n}{\beta} \right)^{n-1} \right) \) and \( \{u_k\} \) be a sequence such that

\[
\lim_{k \to \infty} J(u_k) = C \\
\lim_{k \to \infty} J'(u_k) = 0.
\]  
(3.16)

Let \( h \in W_0^{1,n}(\Omega) \), then we have

\[
(J'(u_k), h) = \int_{\Omega} |\nabla u_k|^{n-2} \nabla u_k \cdot \nabla h \, dx - \int_{\Omega} f(x, u_k) u_k^{n-2} h \, dx.
\]  
(3.18)

Hence we have

\[
J(u_k) - \frac{1}{n} (J'(u_k), u_k) = I(u_k).
\]  
(3.19)

**Claim 1.**

\[
\sup_k \|u_k\| + \sup_k \int_{\Omega} f(x, u_k) u_k^{n-1} \, dx < \infty.
\]  
(3.20)

Since \( \{J(u_k)\} \) and \( \{J'(u_k)\} \) are bounded and hence from (3.19), \( I(u_k) = O(\|u_k\|) \). Now from 5) of lemma 3.1, we have \( \int_{\Omega} f(x, u_k) u_k^{n-1} \, dx = O(\|u_k\|) \).

Now from \((H_2)\) it follows that

\[
\int_{\Omega} F(x, u_k) \, dx = O(\|u_k\|)
\]

and, by using the boundedness of \( J(u_k) \), we obtain \( \|u_k\|^n = O(\|u_k\|) \). This implies (3.20) and hence the claim.
By extracting a subsequence, we can assume that

\[ u_k \rightharpoonup u_0 \text{ weakly and for almost all } x \text{ in } \Omega. \]

CASE (I). \( C \leq 0 \).

From Fatou's lemma and 5) of lemma 3.1, we have

\[
0 \leq I(u_0) \leq \lim_{k \to \infty} I(u_k) = \lim_{k \to \infty} \left\{ J(u_k) - \frac{1}{n} \langle J'(u_k), u_k \rangle \right\} = C.
\]

Hence \( u_0 \equiv 0 \). If \( C < 0 \), no Palais-Smale sequence exists. If \( C = 0 \), then from (3.20) and 4) of lemma 3.1 we have

\[
\lim_{k \to \infty} \|u_k\|^n = n \lim_{k \to \infty} \left\{ J(u_k) + \int_{\Omega} F(x, u_k) \, dx \right\} = 0.
\]

This proves that \( u_k \to 0 \) strongly.

CASE (II). \( C \in \left( 0, \frac{1}{n} \left( \frac{\alpha_n}{b} \right)^{n-1} \right) \).

CLAIM 2. \( u_0 \neq 0 \) and \( u_0 \in \partial B(\Omega, f) \).

Suppose \( u_0 \equiv 0 \). Then, from (3.20) and 4) of lemma 3.1, we have

\[
\lim_{k \to \infty} \|u_k\|^n = n \lim_{k \to \infty} \left\{ J(u_k) + \int_{\Omega} F(x, u_k) \, dx \right\} = nC < \left( \frac{\alpha_n}{b} \right)^{n-1}.
\]

Hence, from 3) of lemma 3.1 and (3.22), we have

\[
\lim_{k \to \infty} \int_{\Omega} f(x, u_k)u_k^{n-1} \, dx = \int_{\Omega} f(x, u_0)u_0^{n-1} \, dx = 0.
\]

This implies that \( \lim_{k \to \infty} I(u_k) = 0 \) and hence from (3.19)

\[
0 < C = \lim_{k \to \infty} J(u_k) = \lim_{k \to \infty} \left\{ I(u_k) + \frac{1}{n} \langle J'(u_k), u_k \rangle \right\} = 0.
\]
which is a contradiction. Hence \( u_0 \neq 0 \). From (3.20) and 4) of lemma 3.1, taking \( h \in C_0^\infty(\Omega) \) and letting \( k \to \infty \) in (3.19), we obtain

\[
\int_\Omega |\nabla u_0|^{n-2} \nabla u_0 \cdot \nabla h \, dx = \int_\Omega f(x, u_0)u_0^{n-2}h \, dx.
\]

By density, the above equation holds for all \( h \in W_0^{1,n}(\Omega) \). Hence, by taking \( h = u_0 \), we obtain

\[
\|u_0\|^n = \int_\Omega f(x, u_0)u_0^{n-1} \, dx.
\]

Hence \( u_0 \in \partial B(\Omega, f) \) and this proves the claim.

Now from (3.20) and claim 2, \( \{u_k, u_0\} \) satisfy all the hypotheses of the compactness lemma 3.3. Hence we have

\[
\|u_0\|^n \leq \lim_{k \to \infty} \|u_k\|^n
\]

\[
= n \lim_{k \to \infty} \left\{ J(u_k) + \int_\Omega F(x, u_k) \, dx \right\}
\]

\[
= n \lim_{k \to \infty} \left\{ J(u_k) + \frac{1}{n} \langle J'(u_k), u_k \rangle + \int_\Omega F(x, u_k) \, dx \right\}
\]

\[
= \lim_{k \to \infty} \left\{ \int_\Omega f(x, u_k)u_k^{n-1} \, dx + \langle J'(u_k), u_k \rangle \right\}
\]

\[
= \int_\Omega f(x, u_0)u_0^{n-1} \, dx = \|u_0\|^n.
\]

This implies that \( u_k \) converges to \( u_0 \) strongly. This proves the Palais-Smale condition.

2) Existence of Positive Solution. Since the critical points of \( J \) are the solutions of the equation (1.7) and \( J(u) = J(|u|) \) for all \( u \) in \( \partial B(\Omega, f) \) and hence in view of lemma 3.5, it is enough to prove that there exists \( u_0 \neq 0 \) such that

\[
\frac{a(\Omega, f)^n}{n} = J(u_0).
\]

Let \( u_k \in \partial B(\Omega, f) \) such that

\[
\lim_{k \to \infty} J(u_k) = \frac{a(\Omega, f)^n}{n}.
\]
Since $J(u_k) = I(u_k)$, and hence by 5) of lemma 3.1

\[ (3.25) \quad \sup_k \int f(x, u_k)u_k^{n-1} \, dx < \infty, \]

\[ (3.26) \quad \sup_k \|u_k\| < \infty. \]

Hence we can extract a subsequence such that

$u_k \rightharpoonup u_0$ weakly and for almost all $x$ in $\Omega$.

CLAIM 3. $u_0 \not= 0$ and

\[ (3.28) \quad \|u_0\|^n \leq \int f(x, u_0)u_0^{n-1} \, dx. \]

Suppose $u_0 = 0$, then from (3.25) and 4) of lemma 3.1

\[ (3.30) \quad \lim_{k \to \infty} \|u_k\|^n = n \lim_{k \to \infty} \left\{ J(u_k) + \int \frac{F(x, u_k)}{n} \, dx \right\} \]

\[ = a(\Omega, f)^n. \]

From lemma 3.4, we have $0 < a(\Omega, f)^n < \left(\frac{\alpha_n}{b} \right)^{n-1}$. Hence, from (3.29) and 3) of lemma 3.1, we have

\[ \lim_{k \to \infty} \int f(x, u_k)u_k^{n-1} \, dx = 0. \]

This implies that

\[ 0 < \frac{a(\Omega, f)^n}{n} = \lim_{k \to \infty} J(u_k) = \lim_{k \to \infty} I(u_k) = 0, \]

which is a contradiction. This proves $u_0 \not= 0$. Suppose (3.28) is false, then

\[ (3.30) \quad \|u_0\|^n > \int f(x, u_0)u_0^{n-1} \, dx. \]

Now from (3.25), (3.30) and $0 < a(\Omega, f)^n < \left(\frac{\alpha_n}{b} \right)^{n-1}$, \{u_k, u_0\} satisfy all the hypotheses of lemma 3.3. Hence

\[ \lim_{k \to \infty} \int f(x, u_k)u_k^{n-1} \, dx = \int f(x, u_0)u_0^{n-1} \, dx. \]
This implies that

$$||u_0||^n \leq \lim_{k \to \infty} ||u_k||^n = \lim_{k \to \infty} \int_{\Omega} f(x, u_k)u_k^{n-1} \, dx$$

$$= \int_{\Omega} f(x, u_0)u_0^{n-1} \, dx,$$

contradicting (3.30). This proves the claim.

Now from (3.28) and step 2 of lemma 3.4, there exists $0 < q \leq 1$ such that $\gamma u_0 \in \partial B(\Omega, f)$. Hence

$$\frac{a(\Omega, f)^n}{n} \leq J(\gamma u_0) = I(\gamma u_0)$$

$$\leq I(u_0) \leq \lim_{k \to \infty} I(u_k)$$

$$= \lim_{k \to \infty} J(u_k) = \frac{a(\Omega, f)^n}{n}.$$

This implies that $\gamma = 1$ and $u_0 \in \partial B(\Omega, f)$. Hence $J(u_0) = \frac{a(\Omega, f)^n}{n}$ and this proves the Theorem.

### 4. Concluding Remarks

**REMARK 4.1. (Regularity).** From Di-Benedetto [6], Tolksdorf [12] and Gilbarg-Trudinger [8], any solution of (1.7) is in $C^{1,\alpha}(\Omega)$ if $n > 3$ and in $C^{2,\alpha}(\Omega)$ if $n = 2$.

**REMARK 4.2.** Let $f \in A(\Omega)$ and $f'(x, 0) < \lambda_1(\Omega)$ for all $x \in \overline{\Omega}$. We prove the existence of a solution for (1.7) under the assumption that

$$\lim_{t \to \infty} \inf_{x \in \overline{\Omega}} h(x, t)t^{n-1} = \infty. \tag{4.1}$$

The only place where it is used is to show that $a(\Omega, f)^n < \left( \frac{a_n}{b} \right)^{n-1}$. But, from lemma 3.2, this inequality holds if

$$\frac{k_0}{b} < 1. \tag{4.2}$$

Hence the theorem is true under the less restrictive condition (4.2).
Now the question is what happens if \( \frac{k_0}{t} \geq 1 \) or the condition (4.1) is not satisfied. In this regard, we have (jointly with Srikanth - Yadava) obtained a partial result, which states that there are functions \( f \in A(\Omega) \) such that

\[
\liminf_{t \to \infty} h(x, t)t^{n-1} = 0
\]

for which no solution to problem (1.7) exists if \( \Omega \) is a ball of sufficiently small radius. In this context, we raise the following question:

**Open Problem.** Let \( \Omega \) be a ball and \( f \in A(\Omega) \) such that \( \sup_{x \in \Omega} f'(x, 0) < \lambda_1(\Omega) \). Is (4.2) also a necessary condition to obtain a solution to the problem (1.7).

In the case \( n = 2 \), this question is related to the question of Brézis [3]: “where is the border line between the existence and non-existence of a solution of (1.7)?”

**REMARK 4.3.** Let \( \beta > 0 \), then by using the norm

\[
\left( \int_{\Omega} |\nabla u|^n \, dx + \beta \int_{\Omega} |u|^n \, dx \right)^{1/n}
\]

in \( W^{1,n}_0(\Omega) \), the Theorem still holds if we replace \(-\Delta u \) by \(-\Delta u + \beta |u|^{n-2}u \) in the equations (1.7).

Due to this and using a result of Cherrier [5], it is possible to extend the Theorem to compact Riemann surfaces.

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5. - Appendix

**PROOF OF THE LEMMA 3.1.**

1) Let \( f(x, t) = h(x, t) \exp\left(b|t|^{n/(n-1)}\right) \in A(\Omega) \). From \((H_4)\), for every \( \epsilon > 0 \), there exists a \( C(\epsilon) > 0 \) such that

\[
|f(x, t)| \leq C(\epsilon) \exp\left((b + \epsilon)|t|^{n/(n-1)}\right)
\]

and hence, from theorem 2.1, \( f(x, u) \in L^p(\Omega) \) for every \( p < \infty \).

2) From \((H_4)\), for every \( \epsilon > 0 \), there exist positive constants \( C_1(\epsilon) \) and \( C_2(\epsilon) \) such that

\[
|f(x, t)t^{n-1}| \leq C_1(\epsilon) \exp\left((b(1 + \epsilon)|t|^{n/(n-1)}\right)
\]
Hence, if \( c > 0 \) such that
\[
\sup_{\|w\| \leq 1} \int_\Omega f(x, cw) w^{n-1} \, dx < \infty,
\]
it implies that, for every \( \epsilon > 0 \),
\[
\sup_{\|w\| \leq 1} \int_\Omega \exp \left( b(1 - \epsilon)c^n/(n-1) |w|^{n/(n-1)} \right) \, dx < \infty.
\]

Therefore, from Theorem 2.1, we have
\[
(1 - \epsilon)^{n-1} c^n \leq \left( \frac{a_n}{b} \right)^{n-1}.
\]

This implies that
\[
\sup \left\{ c^n, \sup_{\|w\| \leq 1} \int_\Omega f(x, cw) w^{n-1} \, dx < \infty \right\} \leq \left( \frac{a_n}{b} \right)^{n-1}.
\]

On the other hand, if \( c^n < \left( \frac{a_n}{b} \right)^{n-1} \), then by choosing \( \epsilon > 0 \) such that
\[
(1 + \epsilon)2^{n-1} c^n < \left( \frac{a_n}{b} \right)^{n-1},
\]
from Theorem 2.1 and from (5.1), we have
\[
\sup_{\|w\| \leq 1} \int_\Omega f(x, (1 + \epsilon)cw) w^{n-1} \, dx
\leq C_1(\epsilon) \sup_{\|w\| \leq 1} \int_\Omega \exp \left[ b \left( (1 + \epsilon)c |w| \right)^{n/(n-1)} \right] \, dx < \infty
\]
this proves
\[
\sup \left\{ c^n, \sup_{\|w\| \leq 1} \int_\Omega f(x, cw) w^{n-1} \, dx < \infty \right\} = \left( \frac{a_n}{b} \right)^{n-1}.
\]

3) Since \( \lim_{k \to \infty} \|u_k\|^n < \left( \frac{a_n}{b} \right)^{n-1} \), from 2) we can choose a \( p > 1 \) such that
\[
c_1^p = \sup_k \int_\Omega |f(x, u_k)|^p \, dx < \infty.
\]
Let \( \frac{1}{p} + \frac{1}{q} = 1 \) and 
\[
c_2^q = \sup \int_{\Omega} |v_k|^q \, dx.
\]

Then, for any \( N > 0 \) and by Holder's inequality,
\[
\left| \int_{|u_k| > N} f(x, u_k) v_k^q \, dx \right| \leq \frac{1}{N} \int_{\Omega} |f(x, u_k)| v_k^q \, dx \leq \frac{c_1 c_2}{N}.
\]

Hence
\[
\int_{\Omega} f(x, u_k) v_k^q \, dx = \int_{|u_k| \leq N} f(x, u_k) v_k^q \, dx + O(1/N).
\]

By dominated convergence theorem, letting \( k \to \infty \) and then \( N \to \infty \) in the above equation, it implies that
\[
\lim_{k \to \infty} \int_{\Omega} f(x, u_k) v_k^q \, dx = \int_{\Omega} f(x, u) v^q \, dx.
\]

4) Let \( N > 0 \), then
\[
\int_{|u| > N} f(x, |u_k|) |u_k|^{n-2+r} \, dx \leq \frac{1}{N^{1-r}} \int_{\Omega} f(x, |u_k|) |u_k|^{n-1} \, dx
\]
\[
= \frac{1}{N^{1-r}} \int_{\Omega} f(x, u_k) u_k^{n-1} \, dx = O \left( \frac{1}{N^{1-r}} \right).
\]

Hence
\[
\int_{\Omega} f(x, |u_k|) |u_k|^{n-2+r} \, dx = \int_{|u_k| \leq N} f(x, |u_k|) |u_k|^{n-2+r} \, dx + O \left( \frac{1}{N^{1-r}} \right).
\]

By dominated convergence theorem, letting \( k \to \infty \) and \( N \to \infty \) in the above equation, we obtain
\[
(5.3) \quad \lim_{k \to \infty} \int_{\Omega} f(x, |u_k|) |u_k|^{n-2+r} \, dx = \int_{\Omega} f(x, |u|) |u|^{n-2+r} \, dx.
\]

Now from \((H_3)\),
\[
|F(x, t)| \leq M (1 + |f(x, t)| |t|^{n-2+r})
\]
for some \( u \in [0, 1) \). Hence, from (5.3) and the dominated convergence theorem,
\[
\lim_{k \to \infty} \int_{\Omega} F(x, u_k) \, dx = \int_{\Omega} F(x, u) \, dx.
\]

5) From (H2) we have, for \( t > 0 \),
\[
(5.4) \quad \frac{\partial}{\partial t} \left[ f(x, t)t^{n-1} - nF(x, t) \right] = \left[ f'(x, t) - \frac{f(x, t)}{t} \right] t^{n-1} > 0.
\]
Therefore from (H1) and (5.4), \( f(x, t)t^{n-1} - nF(x, t) \) is an even positive function and increasing for \( t > 0 \). This implies that \( I(u) \geq 0 \) and \( I(u) = 0 \) iff \( u \equiv 0 \). From (H3) we have
\[
nI(u) = \int_{\Omega} \left[ f(x, u)u^{n-1} - nF(x, u) \right] \, dx
\]
\[
\geq \int_{\Omega} \left[ f(x, u)u^{n-1} - nM(1 + |f(x, u)| |u|^{n-2+}) \right] \, dx
\]
\[
\geq C_1 + \frac{1}{2} \int_{|u| \geq C_2} f(x, u)u^{n-1} \, dx
\]
for some constants \( C_1 \) and \( C_2 > 0 \). This implies that there exists a constant \( M_1 > 0 \) such that
\[
\int_{\Omega} f(x, u)u^{n-1} \, dx \leq M(1 + I(u)).
\]
This proves the lemma 3.1.

REFERENCES


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