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ON EXACT NUMBER OF SOLUTIONS AT INFINITY FOR AMBROSETTI-PRODI CLASS OF PROBLEMS

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1. INTRODUCTION

The following nonlinear Dirichlet problem

$$(*)\qquad \Delta u + \lambda_1 u + f(u) = g \quad \text{in } \Omega, \qquad u|_\Gamma = 0$$

where Ω is a bounded domain in $\mathbb{R}^n$, with smooth boundary Γ, λ_1 is the first eigenvalue of the Laplacian in Ω and f a real valued C^2– strictly convex function satisfying:

$$\lim_{t\to+\infty} \frac{f(t)}{t} = \alpha_+ , \alpha_+ > 0$$

$$\lim_{t\to-\infty} \frac{f(t)}{t} = -\alpha_- , \alpha_- > 0$$

with

$$0 < \lambda_1 - \alpha_- < \lambda_1 < \lambda_1 + \alpha_+ < \lambda_2$$

has been considered by many authors beginning with Ambrosetti and Prodi in [1] . See[3] for further references. These authors have characterized the range completely in the above case.

However the following problem:

$$(**)\qquad \Delta u + \lambda_k u + f(u) = g \quad \text{in } \Omega, \qquad u|_\Gamma = 0$$

with $k \neq 1$ and

$$\lambda_{k-1} < \lambda_k - \alpha_- < \lambda_k < \lambda_k + \alpha_+ < \lambda_{k+1}$$

where λ_k is the k^{th} eigenvalue, has not been tackled in such a great detail as (*) and the best result known is in a paper by Gallouet and Kavian [4] .

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Earlier to this in [6], E. Podolak obtained some partial results. The main aim of this note is to prove a theorem which gives the exact number of solutions at infinity and to the best of our knowledge this is the first result of its kind for this class of problems. Also in the case of (*) we improve the result of Ambrosetti and Prodi, in the sense we prove a theorem giving exact number of solutions at infinity without convexity condition.

2. In this section we set up the problem (**) in operator language and also put down all the assumptions we make on the nonlinearity.

We work in the Hilbert space $H_o^1(\Omega)$, the closure of $C_o^\infty(\Omega)$ functions in the norm $||u||^2 = \int_\Omega |\nabla u|^2$.

As in Podolak [6], we assume

$$\int_\Omega |\phi_k|\phi_k \neq 0 \tag{1}$$

where ϕ_k is the normalized eigenfunction corresponding to λ_k, which we assume is simple.

As regards the nonlinear map we make the following hypothesis:

$$f : \mathbb{R} \to \mathbb{R} \text{ is } C^1.$$

(H1)
$$\lim_{t \to +\infty} \frac{f(t)}{t} = \alpha \qquad \alpha > 0$$
$$\lim_{t \to -\infty} \frac{f(t)}{t} = -\alpha$$

with

$$\lambda_{k-1} < \lambda_k - \alpha < \lambda_k < \lambda_k + \alpha < \lambda_{k+1}$$

(H2)
$$|f'(t)| \leq \alpha \ \forall t$$

(H3)
$$\lim_{t \to +\infty} f'(t) = f'(\infty) = \alpha \qquad \alpha \text{ as in (H1)}$$
$$\lim_{t \to -\infty} f'(t) = f'(-\infty) = -\alpha$$

we shall assume α is small enough and this smallness of α will be made more precise as we proceeed.

We denote by $<.,.>$ the inner product in the space $H_o^1(\Omega)$, which we shall hereafter denote by V.

We define mappings $L:V \to V$ and $N:V \to V$ by

(2)

(a) $\quad <Lu, v> \quad = \int_\Omega \nabla u \cdot \nabla v - \lambda_k \int_\Omega u \cdot v$

and

(b) $\quad <Nu, v> \quad = \int_\Omega f(u)v$

we set $\tilde{g} \in V$ by requiring

$$<\tilde{g}, v> \quad = - \int_\Omega g \cdot v$$

In the above framework solving (**) is equivalent to solving

(3) $$Lu - Nu = \tilde{g}$$

Let $\{\lambda_i : i \geq 1\}$ denote the eigenvalues of the Laplacian in Ω.

We denote by V_1 the subspace of V generated by all eigenfunctions except ϕ_k. We use P_1 to denote the projection of V onto V_1. We denote by V_o the one dimensional space generated by ϕ_k and we use P_o to denote the projection of V onto V_o.

We now specify the smallness of α in (H1) by requiring that:

(H4) $$\alpha^2 < \left[\tfrac{1}{2} \operatorname{Min}\left(\frac{|P_o n(\phi_k)|}{||P_o||}, \frac{|P_o n(-\phi_k)|}{||P_o||} \right)\right] ||L^{-1}P_1||^{-1}.$$

$$\text{and} \quad \alpha < \tfrac{1}{2} ||L^{-1}P_1||^{-1}$$

where $n : V \to V$ is defined by

(H5) $$<n(\phi), v> = \alpha \int_\Omega |\phi| v \qquad \forall v \in V$$

Remark 1: Except (H3) all the hypotheses we have are the same as in Podolak [6]. However one can relax some of these to prove the results as in Podolak [6].

Remark 2: We shall assume throughout what follows that the nonlinear map N defined above is F-differentiable. Note that if Ω $\mathbb{R}^n (n \leq 4)$ then we need no further assumptions on f for N to be F-differentiable. Otherwise one can assume hypothesis as in [2].

3. In this Section we prove the main theorem of this paper.

Theorem 1. Under the above hypothesis, given $\tilde{g} \in V_1$ *there exists a* $t_o = t_o(\tilde{g})$ *such that*

$$(4) \qquad Lu - Nu = t\,\phi_k + \tilde{g}$$

has exactly two solutions for $t > t_o$ $(t < t_o)$ *depending on the sign of*

$$\int_\Omega |\phi_k| \phi_k$$

Before we prove this, we state the following result proved in [6].

Theorem 2 (Podolak [6]*): If f is as above, there exists a real number* $t_1 = t_1(\tilde{g})$ *such that for* $t < t_1$ $(t > t_1)$, *(4) has no solution and if* $t > t_1$ $(t < t_1)$, *(4) has at least two solutions depending on the sign of*

$$\int_\Omega |\phi_k| \phi_k$$

We now briefly recall the sketch of the proof of the above theorem. It is now more or less classical that to solve (4) one solves

$$(5) \qquad \begin{aligned} Lv - P\,N(v + p\phi_k) &= \tilde{g} \quad (a) \\ - P_o N(v + p\phi_k) &= t\phi_k \quad (b) \end{aligned}$$

It is easy to show (5a) is uniquely solvable for each fixed p, for a given $\tilde{g}$. Once this is done, defining

$$(6) \qquad H(p) = P_o N(\phi(p) + p\phi_k)$$

one analyzes the behaviour of H(p) to prove Theorem like 2 (see [2],[4], [6]). It is easy to see under the hypothesis we have, that, given $\tilde{g} \in V_1$

and any sequence $(p_n) \to +\infty$ then $\left(\frac{\phi(p_n)}{|p_n|}\right)$ has a convergent subsequence. The same is true if $(p_n) \to -\infty$. Moreover every convergent subsequence of $(\phi(p,\tilde{g})/|p|)$ as $|p| \to \infty$ converges to $\tilde{\phi}(\pm)$ independent of $\tilde{g}$ and $\tilde{\phi}(\pm)$ is the unique solution of (uniqueness due to hypothesis (H4)),

$$(7) \qquad L\tilde{\phi}(\pm) - P_1 n(\pm\phi_k + \tilde{\phi}(\pm)) = 0$$

Once we have this information, one can show that $H(p) \to \infty$ as $|p| \to \infty$. In fact under our hypothesis it can be shown that

$$\frac{H(p)}{|p|} \to P_o n(\pm\phi_k + \tilde{\phi}(\pm)) > 0.$$

We now proceed to prove the Theorem 1. Before proving the Theorem we state and prove the following:

Proposition 1: Let $\phi \in H^1_o(\Omega)$ *satisfy*

$$(8) \qquad \begin{aligned} &L\phi - P_1 n(\phi + \phi_k) = 0 \\ &-P_o n(\phi + \phi_k) = C\phi_k,\ C \neq 0 \end{aligned}$$

Let $E = \{x \in \Omega : (\phi + \phi_k)(x) = 0\}$. *Then measure of* $E = |E| = 0$.

Proof: (8) is the same as

$$(9) \qquad -\Delta(\phi + \phi_k) = \lambda_k(\phi + \phi_k) + \alpha\,|\phi + \phi_k| + C_1\phi_k,$$

with $C_1 \neq 0$. Clearly we have adequate regularity. Now we use the following result proved in [5, page 53] which states: Let $u \in H^{1,s}(\Omega)$. Then $\partial u/\partial x_i = 0$ a.e. in $\Omega_o = \{x \in \Omega : u = 0\}$, $1 \leq i \leq N$.

A repeated use of the above result in our case shows that $\Delta(\phi + \phi_k)$ vanishes a.e. on E. Hence $E \subset \{x \in \Omega : \phi_k(x) = 0\}$ follows from (9). But it is classical that $\{x \in \Omega : \phi_k(x) = 0\}$ is a set of measure zero. Hence the proposition.

Proof of Theorem 1: we will show that

$$H'(p) = P_o N'(\phi(p) + p\phi_k)\,(\phi'(p) + \phi_k)$$

tends to $\pm P_o n(\tilde{\phi}(\pm) \pm \phi_k)$ as $p \to \pm\infty$. It is clear that the theorem follows from this. Here $\tilde{\phi}(\pm)$ is the same as in (7).

We first claim

$$\lim_{p \to \infty} \frac{\phi(p)}{p} = \omega . \lim_{p \to \infty} \phi'(p)$$

To show this consider

$$L\phi(p) - P_1 N(\phi(p) + p\phi_k) = \tilde{g}$$

Taking derivative wrt.p, we have

$$L\phi'(p) - P_1 N'(\phi(p) + p\phi_k)\ (\phi'(p) + \phi_k) = 0$$

i.e.

(11) $$L\phi'(p) - P_1 N'(\phi(p) + p\phi_k)\ \phi'(p) = P_1 N'(\phi(p) + p\phi_k)\phi_k$$

Using the hypotheses (H2) and (H4)it follows that $(\phi'(p))$ is bounded independent of p. Let a subsequence $\phi'(p_n) \to \omega$, weakly as $p_n \to \infty$. Notice (1.1) is the same as

(12) $$< L\ \phi'(p_n), v > = \int_\Omega f'(\phi(p_n)) + p_n\ \phi_k)\ (\phi'(p_n) + \phi_k)v,$$

for all $v \in V_1$. Since we already have $\frac{\phi(p_n)}{p_n} \to \tilde{\phi}(+)$, it is easy to see, taking limit as $n \to \infty$, that

(13) $$< L\omega, v > = \alpha \int_\Omega Sgn(\phi_k + \tilde{\phi}(+))\ (\omega + \phi_k)v$$

where

$$Sgn(\phi_k + \tilde{\phi}(+) = \begin{cases} 1 & if(\phi_k + \tilde{\phi}(+)) > 0 \\ -1 & if(\phi_k + \tilde{\phi}(+)) < 0 \end{cases}$$

Notice that by proposition 1 the set $E = \{x \in \Omega : (\tilde{\phi}(+) + \phi_k)(x) = 0\}$ has measure zero. Also notice that in claiming (12) $\Rightarrow$ (13) as $n \to \infty$, we have used the hypothesis (H3), the dominates convergence theorem and the fact that if (x_n) is a sequence of rela numbers such that every subsequence has a convergent subsequence converging the same limit, then (x_n) itself converges to the same limit. Also we have used that if $v_n \to v$ strongly in $L^2(\Omega)$ then there exists a subsequence v_{n_k}, such that $v_{n_k}(x) \to v(x)$ a.e. notice that (13) implies

(14) $$L\omega - \alpha P_1 n_1(\phi_k + \tilde{\phi}(+))(\omega + \phi_k) = 0,$$

where $n_1(\phi_k + \tilde{\phi}(+))$ is defined through

$$< n_1 (\phi_k + \tilde{\phi}(+))v_1, v_2 > = \int_\Omega Sgn(\phi_k + \tilde{\phi}(+))v_1 v_2 .$$

Our hypothesis (H4)$\Rightarrow$(14) has a unique solution. But then

(15) $$L\tilde{\phi}(+) - \alpha P_1 n_1(\tilde{\phi}(+) + \phi_k)(\tilde{\phi}(+) + \phi_k) = 0$$

by (7) because (15) is the same as

$$< L\tilde{\phi}(+), v > -\alpha \int_\Omega Sgn(\phi_k + \tilde{\phi}(+))(\tilde{\phi}(+) + \phi_k)v =$$
$$= < L (+), v > \alpha \int_\Omega |\tilde{\phi}(+) + \phi_k| v \quad \forall v \in V$$
$$= 0 \text{ by } (7).$$

Hence $\omega = \tilde{\phi}(+)$ i.e.

$$S. \lim_{p \to \infty} \frac{\phi(p)}{p} = \tilde{\phi}(+) = \omega . \lim_{p \to \infty} \phi'(p)$$

Now consider

(16) $$H'(p) = \int_\Omega f'(p\phi_k + \phi(p))(\phi_k + \phi'(p))\phi_k$$

For similar reasons as (12) $\Rightarrow$ (13), we have

(17) $$\lim_{p \to \infty} H'(p) = \alpha \int_\Omega |\phi_k + \tilde{\phi}(+)| \phi_k$$

But RHS in (17) we know is positive if $\int_\Omega |\phi_k| \phi_k$ is positive. Similarly one can show,

$$\lim_{p \to \infty} H'(p) = - P_0 n(-\phi_k + \tilde{\phi}(-)) < 0$$

hence we have shown that H(p) is strictly increasing for $p > p_0 (p_0 > 0)$ and that H(p) is strictly decreasing for $p < -p_0$. This then proves the Theorem.

Remark 3: To obtain a result like Theorem (2) quoted above one does not need to assume as strong an hypothesis as in [6]. It is sufficient to assume f' is such that,

$$-\left(\frac{\lambda_k}{\lambda_{k-1}} - 1\right) h^2 < \; < f'(t)h, h > \leq k_1 h^2 \quad \forall h \neq 0,$$

where k_1 is such that $\beta = \left(1 - \frac{\lambda_k}{\lambda_{k-1}} - k_1\right) > 0$. However the proof in this case needs to be modified.

In the light of Theorem 1 proved above it is natural to ask if one can improve the theorem in the special case when $\lambda_k = \lambda_1$. In this case we prove Theorem 3 below. Before we state and prove the theorem we make the following assumptions.

$$f : \mathbb{R} \to \text{ is } C^2$$

(H'1) $\lim_{t \to \infty} f'(t) = \alpha, \alpha > 0$

$\lim_{t \to \infty} f'(t) = -\beta, \beta > 0.$

(H'2): $|f'(t)| < \lambda_2 - \lambda_1 \quad \forall t$

(H'3) $0 < \lambda_1 - \beta < \lambda_1 < \lambda_1 + \alpha < \lambda_2$

Theorem 3: Under the assumptions (H'1) – (H'3), there exists a real number $t_0 = t_0(g)$ such that, letting $P_0 g$ be the L_2 projection of g an $Ker(\Delta + \lambda_1)$, the equation (*) has exactly two solutions if $P_0 g > t_0$.

Proof: Notice that if we proceed along the same lines and with similar notations as in the proof of Theorem 1, then we have only to prove (i) $(\phi'(p))$ exists and is bounded (compare with equation (11)), (ii) Equation corrsponding to (7) has a unique solution and (iii) Equation corresponding to (14) has a unique solution.

It is easy to see that $(\phi'(p))$ exists and is bounded and hence we shall not give details. Also notice that the boundedness of $(\phi'(p)$ implies the boundedness of $\frac{\phi(p)}{|p|}$. it is essentially in proving this step that one uses (H'2).

The equation corresponding to (7) in this case is:

$$(18) \qquad \int_\Omega \nabla \tilde{\phi}(\pm) \nabla \theta - \lambda_1 \int_\Omega \tilde{\phi}(\pm)\theta - \alpha \int_\Omega (\tilde{\phi}(\pm) \pm \phi_1)^+ \theta - \beta \int_\Omega (\tilde{\phi}(\pm) \pm \phi_1)^- \theta = 0 \quad \forall \theta \in (\phi_1)^\perp$$

Observe that (18) is the same as

$$(19) \qquad -\Delta\tilde{\phi}(\pm) = P_1[(\lambda_1 + \alpha)(\tilde{\phi}(\pm) \pm \phi_1)^+ - (\lambda_1 - \beta)(\tilde{\phi}(\pm) \pm \phi_1)^-]$$

But then it is an easy consequence of contraction principle to see that $\tilde{\phi}(\pm)$ is unique. Hence we have now got over two of the three difficulties we had.

We now consider the equation corresponding to (14) and prove it has a unique solution. Observe that the equation corresponding to (14) in this case is:

$$(20) \qquad \langle L\omega, \theta \rangle = \alpha \int_{\Omega_1} (\omega + \phi_1)\theta - \beta \int_{\Omega_2} (\omega + \phi_1)\theta$$

where $\Omega_1 = \{x \varepsilon \Omega : (\tilde{\phi}(+) + \phi_1)(x) > 0\}$ and $\Omega_2 = \{x \varepsilon \Omega : (\tilde{\phi}(+) + \phi_1)(x) < 0\}$

Suppose there exists two solutions ω_1 and ω_2 for the equation (20), then

$$(21) \qquad \langle L\omega_1, \omega_1 - \omega_2 \rangle = \alpha \int_{\Omega_1} (\omega_1 + \phi_1)(\omega_1 - \omega_2) - \beta \int_{\Omega_2} (\omega_1 + \phi_1)(\omega_1 - \omega_2)$$

and

$$(22) \qquad \langle L\omega_2, \omega_1 - \omega_2 \rangle = \alpha \int_{\Omega_1} (\omega_2 + \phi_1)(\omega_1 - \omega_2) - \beta \int_{\Omega_2} (\omega_2 + \phi_1)(\omega_1 - \omega_2)$$

hold.

Subtracting (22) from (21), we have

$$(23) \qquad \langle L(\omega_1 - \omega_2), \omega_1 - \omega_2 \rangle = \alpha \int_{\Omega_1} (\omega_1 - \omega_2)^2 - \beta \int_{\Omega_2} (\omega_1 - \omega_2)^2$$

i.e.

$$\begin{aligned} 0 &= \int_{\Omega} |\nabla(\omega_1 - \omega_2)|^2 - \lambda_1 \int_{\Omega} (\omega_1 - \omega_2)^2 - \alpha \int_{\Omega_1} (\omega_1 - \omega_2) + \beta \int_{\Omega_2} (\omega_1 - \omega_2)^2 . \\ &\geq \lambda_2 \|\omega_1 - \omega_2\|^2_{L^2(\Omega)} - \lambda_1 \|\omega_1 - \omega_2\|^2_{L^2(\Omega)} - \alpha \|\omega_1 - \omega_2\|^2 . \\ &\geq (\lambda_2 - (\lambda_1 + \alpha)) \|\omega_1 - \omega_2\|^2 \end{aligned}$$

Since $\lambda_2 > (\lambda_1 + \alpha)$, the above inequality implies $\omega_1 = \omega_2$. Hence we have proved the uniqueness of solution for (20). Hence the theorem is now proved.

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