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Relation between atomic coherent-state representation, state multipoles, and generalized phase-space distributions

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The relationship between the atomic coherent-state representation of Arecchi *et al.* [Phys. Rev. A 6, 2211 (1972)] and the state multipoles is established. The state multipoles are used to develop a theory of generalized phase-space distributions for angular momentum (collective atomic) systems. The general theory for angular momentum systems is shown to have many features in common with the general theory for boson systems [Phys. Rev. D 2, 2161 (1970)]. These generalized phase-space distributions contain as a special case the coherent-state representation of Arecchi *et al*. The applications of the generalized phase-space distributions and state multipoles to the dynamical problems and to the calculation of multitime correlations are given. State-multipole techniques are used to give a brief discussion of the master equation describing cooperative resonance fluorescence.

I. INTRODUCTION

This paper unifies and establishes the equivalence of two independent theoretical developments (i) the atomic coherent-state representation[1] in the field of quantum optics and (ii) the use of the state multipoles[2] in the study of the properties of radiation emitted by atomic or nuclear systems. The atomic coherent-state representation has been used with great success in the study of the cooperative phenomena[1,3,4] in atomic systems, such as superradiance[5] and resonance fluorescence[6] from a collection of atoms. The atomic coherent states are now beginning to be applied in a variety of fields[7] such as in the nuclear many-body problem and ferromagnetism. On the other hand, state multipoles[2] have been used extensively, earlier in the context of angular correlations in the radiation emitted by nuclear systems, and now in the study of the distribution of radiation emitted by atomic systems. Such state multipoles characterize the properties of the excited states belonging to a multiplet of states. In what follows, we show the intimate relationship between the atomic coherent-state representation and the idea of state multipoles. The organization of this paper is as follows: After a brief summary of the important properties of the atomic coherent states and the state multipoles, we study, in Sec. II, the connection between the two. In Sec. III, we develop a theory of generalized phase-space distributions. In Sec. IV, we study some algebraic properties of the state-multipole operators. These properties are then used to obtain, from the master equation for the density matrix of the atomic system, the equations for the state multipoles. The multitime-correlation functions of a radiating system are then computed in Sec. V. We begin by summarizing the relevant properties of the coherent states and state multipoles, that are needed in subsequent sections.

A. Properties of atomic coherent states

The atomic coherent states for a system with angular momentum j are defined by[1]

$$|\theta,\phi\rangle=\sum_{m=-j}^{+j}\binom{2j}{m+j}^{1/2}\left(\sin\frac{\theta}{2}\right)^{j+m}\times\left(\cos\frac{\theta}{2}\right)^{j-m}e^{-i(j+m)\phi}\,|jm\rangle\,. \tag{1.1}$$

These states are not orthogonal but overcomplete, i.e.,

$$|\langle\theta,\phi|\theta';\phi'\rangle|^2=[\tfrac{1}{2}+\tfrac{1}{2}\cos\theta\cos\theta'+\tfrac{1}{2}\sin\theta\sin\theta'\cos(\phi-\phi')]^{2j}\,, \tag{1.2}$$

$$\frac{2j+1}{4\pi}\int\sin\theta\,d\theta\,d\phi\,|\theta,\phi\rangle\langle\theta,\phi|=1\,. \tag{1.3}$$

The diagonal representation for the density operator in terms of atomic coherent states is

$$\rho=\int P(\theta,\phi)|\theta,\phi\rangle\langle\theta,\phi|\sin\theta\,d\theta\,d\phi\,. \tag{1.4}$$

Another useful function in the study of the properties of quantum systems using atomic coherent states is provided by

$$R(\theta,\phi)=\langle\theta,\phi|\rho|\theta,\phi\rangle\,. \tag{1.5}$$

The dynamics of a system[3-6] can be studied by using the dynamical equations for $P(\theta,\phi)$.

B. Properties of state multipoles

The state-multipole operators are defined[2,8] by

$$T_{KQ}=\sum_{mm'}(-1)^{j-m}(2K+1)^{1/2}\begin{pmatrix} j & K & j\\ -m & Q & m'\end{pmatrix}|jm\rangle\langle jm'|\,, \tag{1.6}$$

where

$$\begin{pmatrix} j & K & j\\ -m & Q & m'\end{pmatrix}$$

is the Wigner $3j$ symbol.[9] Note that K is an integer taking values $0,1,2,\ldots,2j$ and $-K\le Q\le +K$. The state multipoles have the following important orthogonality property:

$$\mathrm{Tr}(T^{+}_{K_1Q_1}T_{K_2Q_2})=\delta_{K_1K_2}\delta_{Q_1Q_2},\quad T^{+}_{KQ}=(-1)^{Q}T_{K,-Q}\,, \tag{1.7}$$

and hence any function of angular momentum operators can be expanded in terms of T_{KQ}. In particular, for the density matrix, we get the expansion

$$\rho=\sum_{KQ}\mathrm{Tr}(\rho T^{+}_{KQ})\,T_{KQ}$$

$$=\sum_{KQ}\langle T^{+}_{KQ}\rangle\, T_{KQ}\,. \tag{1.8}$$

Thus the density matrix is completely characterized by the state multipoles $\langle T^{+}_{KQ}\rangle$. These expectation values are closely related to the moments of the angular momentum operators:

$$\langle T^{+}_{10}\rangle=\left(\frac{3}{j(2j+1)(j+1)}\right)^{1/2}\langle J_z\rangle\,,$$

$$\langle T^{+}_{11}\rangle=-\left(\frac{3}{2j(j+1)(2j+1)}\right)^{1/2}\langle J_-\rangle\,, \tag{1.9}$$

$$\langle T^{+}_{20}\rangle=\left(\frac{5}{(2j+3)(j+1)(2j+1)j(2j-1)}\right)^{1/2}\times[3\langle J_z^2\rangle-j(j+1)]\,.$$

The tensor $\langle T^{+}_{2Q}\rangle$ is also known as the alignment tensor. It may also be noticed that the properties of the optical coherences will be reflected in the properties of $\langle T^{+}_{KQ}\rangle$ with $Q\neq 0$.

II. RELATION BETWEEN THE ATOMIC COHERENT-STATE REPRESENTATION AND THE STATE MULTIPOLES

In this section we will establish the relation between the diagonal representation (1.4) and the state multipoles $\langle T^{+}_{KQ}\rangle$. From Eq. (1.4), it follows that

$$\langle T^{+}_{KQ}\rangle=\int P(\theta,\phi)\langle\theta,\phi|T^{+}_{KQ}|\theta,\phi\rangle\sin\theta\,d\theta\,d\phi\,, \tag{2.1}$$

which on using (1.1) and (1.6) reduces to

$$\langle T^{+}_{KQ}\rangle=\int P(\theta,\phi)f_{KQ}(\theta,\phi)\sin\theta\,d\theta\,d\phi\,, \tag{2.2}$$

where

$$f_{KQ}(\theta,\phi)=e^{-iQ\phi}\left(\tan\frac{\theta}{2}\right)^{Q}\sum_m\binom{2j}{m+j}^{1/2}\binom{2j}{m+j+Q}^{1/2}\left(\sin\frac{\theta}{2}\right)^{2j+2m}\left(\cos\frac{\theta}{2}\right)^{2j-2m}\times(-1)^{j-m-Q}(2K+1)^{1/2}\begin{pmatrix} j & j & K\\ m & -m-Q & Q\end{pmatrix}\,, \tag{2.3}$$

and where we have also used the property that the Wigner $3j$ symbol

$$\begin{pmatrix} j_1 & j_2 & j_3\\ m_1 & m_2 & m_3\end{pmatrix}$$

vanishes if $m_1+m_2+m_3\neq 0$. Next we expand $P(\theta,\phi)$ in spherical harmonics

$$P(\theta,\phi)=\sum_{KQ}Y_{KQ}(\theta,\phi)P_{KQ}\,. \tag{2.4}$$

Then (2.2) reduces to

$$\langle T^{+}_{KQ}\rangle=\sum_{K_1Q_1}P_{K_1Q_1}\int Y_{K_1Q_1}(\theta,\phi)\times f_{KQ}(\theta,\phi)\sin\theta\,d\theta\,d\phi\,. \tag{2.5}$$

We will now show that the function f_{KQ} is proportional to $Y^{*}_{KQ}(\theta,\phi)$:

$$f_{KQ}(\theta,\phi)\equiv Y^{*}_{KQ}(\theta,\phi)(-1)^{K-Q}\sqrt{4\pi}\times\frac{2j!}{[(2j-K)!(2j+K+1)!]^{1/2}}\,. \tag{2.6}$$

To prove (2.6) we first note the relation[10]

$$\sum_{m'n'} \begin{pmatrix} j & k & l \\ m' & n' & p' \end{pmatrix} D^{j}_{m'm}(A) D^{k}_{n'n}(A) = \sum_{p} \begin{pmatrix} j & k & l \\ m & n & p \end{pmatrix} [D^{l}_{p'p}(A)]^{*}, \tag{2.7}$$

with

$$D^{j}_{mn}(A) = i^{m-n} \sum_{q} (-1)^{q} \frac{[(j+m)!(j+n)!(j-n)!(j-m)!]^{1/2}}{(j+m-q)!(q+n-m)!q!(j-n-q)!} (\cos\theta)^{2j+m-n-2q} (\sin\theta)^{2q+n-m} e^{-i(m+n)\phi} e^{i(n-m)\psi}. \tag{2.8}$$

In the special case when $k=j, n' \to -n', m=j, n=-j$, the relation (2.7) on simplification leads to

$$\sum_{m'} \begin{pmatrix} j & j & l \\ m' & -m'-Q & Q \end{pmatrix} \left(\frac{2j!\,2j!}{(j+m')!(j-m')!(j-m'-Q)!(j+m'+Q)!} \right)^{1/2} (-1)^{j-m'} (\cos\theta)^{2j-2m'} (\sin\theta)^{2j+2m'}$$

$$= (-1)^{-l+Q} \begin{pmatrix} j & j & l \\ j & -j & 0 \end{pmatrix} \sum_{q} \frac{[(l-Q)!(l+Q)!l!l!]^{1/2}}{(l-Q-q)!(l-q)!q!(q+Q)!} (-1)^{q} (\cos\theta)^{2l-2q} (\sin\theta)^{2q} \tag{2.9}$$

$$= e^{iQ\phi} Y^{*}_{lQ}(2\theta, \phi) (\tan\theta)^{-Q} \left(\frac{4\pi}{2l+1} \right)^{1/2} (-1)^{l} \frac{2j!}{[(2j-l)!(2j+l+1)!]^{1/2}}. \tag{2.10}$$

In deriving (2.10) we have used the following expression for spherical harmonics[11]

$$Y_{lm}(\theta, \phi) = (-1)^{m} \left(\frac{2l+1}{4\pi} \right)^{1/2} e^{im\phi} \sum_{q} \frac{(-1)^{q} [l!\,l!(l-m)!(l+m)!]^{1/2}}{(l-q)!q!(q+m)!(l-m-q)!} \left(\cos\frac{\theta}{2} \right)^{2l-m-2q} \left(\sin\frac{\theta}{2} \right)^{2q+m}. \tag{2.11}$$

The relation (2.6) now follows by using the identity (2.10) in (2.3).

On using (2.6) and the orthogonality of the spherical harmonics, we find that

$$\langle T^{+}_{KQ} \rangle = P_{KQ} (-1)^{K-Q} \sqrt{4\pi} \frac{2j!}{[(2j-K)!(2j+K+1)!]^{1/2}}. \tag{2.12}$$

The relation (2.12) is the desired relation between the atomic coherent-state representation and the state multipoles. We can also relate the state multipoles $\langle T^{+}_{KQ} \rangle$ to the representation (1.5) by using the well-known relation[12] between $P(\theta, \phi)$ and $R(\theta, \phi)$:

$$R(\theta, \phi) = \sum_{KQ} R_{KQ} Y_{KQ}(\theta, \phi), \tag{2.13}$$

$$P_{KQ} = \left(\frac{2j+1}{4\pi} \right) \frac{(2j-K)!(2j+K+1)!}{(2j)!(2j+1)!} R_{KQ}, \tag{2.14}$$

and hence

$$\langle T^{+}_{KQ} \rangle = (-1)^{K-Q} R_{KQ} \frac{1}{\sqrt{4\pi}} \frac{[(2j-K)!(2j+K+1)!]^{1/2}}{(2j)!}. \tag{2.15}$$

III. GENERALIZED PHASE-SPACE DISTRIBUTIONS ASSOCIATED WITH THE SPIN SYSTEM

In this section we demonstrate how a generalized theory of phase-space distributions could be formulated for angular momentum operators. The results we present here are analogous to the earlier ones[13] for boson systems. We will show how a whole class of generalized phase-space distributions could be generated using the completeness of T_{KQ} operators.

Any arbitrary operator $\mathcal{G}$ can be expanded as

$$\mathcal{G} = \sum_{KQ} \mathcal{G}_{KQ} T_{KQ}, \tag{3.1}$$

where

$$\mathcal{G}_{KQ} = \mathrm{Tr}(T^{+}_{KQ} \mathcal{G}). \tag{3.2}$$

We now introduce the operators $\Delta^{(\Omega)}(\theta, \phi)$ defined by

$$\Delta^{(\Omega)}(\theta, \phi) = \sum_{KQ} T_{KQ} Y^{*}_{KQ}(\theta, \phi) \Omega_{KQ}, \tag{3.3}$$

where Ω_{KQ} is a function of the integers K, Q whose properties will be shortly determined. Also introduce the operator

$$\Delta^{(\tilde{\Omega})}(\theta, \varphi) = \sum_{KQ} T_{KQ} Y^{*}_{KQ}(\theta, \phi) \tilde{\Omega}_{KQ}, \quad \tilde{\Omega}_{KQ} = \Omega^{-1}_{K,-Q}. \tag{3.4}$$

It can now be easily shown from (3.3), (3.4), (1.7), and the completeness of spherical harmonics that

$$\mathrm{Tr}[\Delta^{(\Omega)}(\theta_1, \phi_1) \Delta^{(\tilde{\Omega})}(\theta_2, \phi_2)] = \delta(\phi_1 - \phi_2) \delta(\cos\theta_1 - \cos\theta_2). \tag{3.5}$$

The relation (3.5) is the key relation in the development of the generalized theory of phase-space distributions. It is now clear from (3.1) and the property (3.5) that

$$\mathcal{S} = \int F^{(\Omega)}(\theta, \phi)\, \Delta^{(\Omega)}(\theta, \phi) \sin\theta \, d\theta \, d\phi \,, \tag{3.6}$$

where $F^{(\Omega)}(\theta, \phi)$ is the generalized phase-space distribution given by

$$F^{(\Omega)}(\theta, \phi) = \mathrm{Tr}[\mathcal{S}\Delta^{(\tilde{\Omega})}(\theta, \phi)] \,. \tag{3.7}$$

From (3.6) it is clear that

$$\mathrm{Tr}(\mathcal{S}_1 \mathcal{S}_2) = \int F_1^{(\Omega)}(\theta, \phi) F_2^{(\tilde{\Omega})}(\theta, \phi) d\theta \, d\phi \sin\theta \,, \tag{3.8}$$

and hence the expectation values in terms of generalized phase-space distributions are given as

$$\langle \mathcal{S} \rangle = \mathrm{Tr}(\rho \mathcal{S}) = \int F_\rho^{(\Omega)}(\theta, \phi) F_{\mathcal{S}}^{(\tilde{\Omega})}(\theta, \phi) \sin\theta \, d\theta \, d\phi \,. \tag{3.9}$$

One can easily prove the following properties of $\Delta^{(\Omega)}(\theta, \phi)$:

$$\int \Delta^{(\Omega)}(\theta, \phi) \sin\theta \, d\theta \, d\phi = \left(\frac{4\pi}{2j+1}\right)^{1/2} \Omega_{0,0} \,, \tag{3.10}$$

$$\Delta^{(\Omega)\dagger}(\theta, \phi) = \sum_{KQ} T_{KQ} Y^*_{KQ} \Omega^*_{K,-Q} \,, \tag{3.11}$$

and hence $\Delta^{(\Omega)}$'s are Hermitian provided

$$\Omega_{K,Q} = \Omega^*_{K,-Q} \,. \tag{3.12}$$

Note also the property

$$\mathrm{Tr}[\Delta^{(\Omega)}(\theta, \phi)] = \left(\frac{(2j+1)}{4\pi}\right)^{1/2} \Omega_{0,0} \,. \tag{3.13}$$

In view of (3.13), we find the following normalization relation:

$$\mathrm{Tr}\rho = 1 = \frac{\Omega_{0,0}}{\sqrt{4\pi}} \int F_\rho^{(\Omega)}(\theta, \phi) \sin\theta \, d\theta \, d\phi \, (2j+1)^{1/2} \,. \tag{3.14}$$

It is clear from the foregoing, that different choices of the functions Ω_{KQ} lead to different phase-space distributions. Using (3.7), each of these can be expanded in terms of spherical harmonics as

$$F^{(\Omega)}(\theta, \phi) = \sum_{KQ} Y_{KQ}(\theta, \phi) F^{(\Omega)}_{KQ} \,, \tag{3.15}$$

$$F^{(\Omega)}_{KQ} = \Omega^{-1}_{KQ} \mathrm{Tr}(\mathcal{S} T^\dagger_{KQ}) \,. \tag{3.16}$$

Equation (3.16) also yields immediately the relationship between two different phase-space distributions corresponding to the different choices of Ω functions

$$F^{(\Omega_1)}_{KQ} = F^{(\Omega_2)}_{KQ} (\Omega^{(2)}_{KQ}/\Omega^{(1)}_{KQ}) \,. \tag{3.17}$$

The integral form of the relation between two phase-space distributions corresponding to two different choices of Ω functions also follows from (3.15)

$$F^{(\Omega_2)}(\theta, \phi) = \int F^{(\Omega_1)}(\theta', \phi') d\theta' d\phi' \sin\theta' K_{21}(\theta, \phi; \theta', \phi') \,, \tag{3.18}$$

where the kernel K_{21} is given by

$$K_{21}(\theta, \phi, \theta', \phi') = \sum_{KQ} Y_{KQ}(\theta, \phi) Y^*_{KQ}(\theta'\phi') \Omega^{(1)}_{KQ}/\Omega^{(2)}_{KQ} \,. \tag{3.19}$$

Having formulated the general theory of phase-space distributions, we now establish the connection with the phase-space distributions introduced by Arecchi *et al.*[1] For this purpose we use relation (2.6), namely,

$$\langle \theta, \phi | T^\dagger_{KQ} | \theta, \phi \rangle = Y^*_{KQ}(\theta, \phi)(-1)^{K-Q} \times \sqrt{4\pi} \frac{2j!}{[(2j-K)!(2j+K+1)!]^{1/2}} \,. \tag{3.20}$$

We will now find the Ω function such that $\Delta^{(\Omega)} \times (\theta, \phi)$ will reduce to the projector $|\theta, \phi\rangle\langle\theta, \phi|$, i.e.,

$$\Delta^{(\Omega)}(\theta, \phi) = \sum_{KQ} Y^*_{KQ}(\theta, \phi) T_{KQ} \Omega_{KQ} = |\theta, \phi\rangle\langle\theta, \phi| \,. \tag{3.21}$$

From (3.21), (1.7), and (3.20) we see that

$$\begin{aligned}\Omega_{KQ} Y^*_{KQ}(\theta, \phi) &= \mathrm{Tr}\, T^\dagger_{KQ} |\theta, \phi\rangle\langle\theta, \phi| = \langle\theta, \phi| T^\dagger_{KQ} |\theta, \phi\rangle \\ &= Y^*_{KQ}(\theta, \phi)(-1)^{K-Q} \\ &\quad \times \sqrt{4\pi} \frac{2j!}{[(2j-K)!(2j+K+1)!]^{1/2}} \,,\end{aligned}$$

and hence the Ω function that reduces $\Delta^{(\Omega)}$ to $|\theta, \phi\rangle\langle\theta, \phi|$ is given by

$$\Omega_{KQ} = (-1)^{K-Q} \sqrt{4\pi} \frac{2j!}{[(2j-K)!(2j+K+1)!]^{1/2}} \,. \tag{3.22}$$

With the particular choice (3.22), relation (3.6) reduces to the well-known diagonal coherent-state representation for atomic systems

$$\mathcal{S} = \int F^{(\Omega)}(\theta, \phi) |\theta, \phi\rangle\langle\theta, \phi| \sin\theta d\theta \, d\phi \,, \tag{3.23}$$

with

$$F^{(\Omega)}(\theta,\phi)=\sum_{KQ}Y_{KQ}(\theta,\phi)(\mathrm{Tr}\mathcal{G}\,T^{\dagger}_{KQ})\times\frac{(-1)^{K-Q}}{\sqrt{4\pi}}\left(\frac{(2j-K)!\,(2j+K+1)!}{2j!\,2j!}\right)^{1/2}. \tag{3.24}$$

In the special case $\mathcal{G}\to\rho$, (3.23) goes over to (1.4). Similarly if we were to use $\Delta^{(\Omega)}$ for the expansion in Eq. (3.6) with Ω given by (3.22), then we would find

$$\mathcal{G}=\int F^{(\bar{\Omega})}(\theta,\phi)\Delta^{(\bar{\Omega})}(\theta,\phi)d\theta\,d\phi\,\sin\theta\,, \tag{3.25}$$

with

$$F^{(\bar{\Omega})}(\theta,\phi)=\mathrm{Tr}[\mathcal{G}\Delta^{(\Omega)}(\theta,\phi)]=\langle\theta,\phi\,|\mathcal{G}\,|\theta,\phi\rangle=\sum_{KQ}Y_{KQ}(\theta,\phi)\,\mathrm{Tr}(\mathcal{G}\,T^{\dagger}_{KQ})(-1)^{K-Q}\times\sqrt{4\pi}\left(\frac{2j!\,2j!}{(2j-K)!\,(2j+K+1)!}\right)^{1/2}. \tag{3.26}$$

We have thus obtained the general relations of (1.4) and (1.5) of Arecchi *et al.* as a special case of our theory. Note also that our general relation (3.17) relating any two phase-space distributions in the special case, also yields Gilmore's relation (2.14). It is perhaps of interest to note that all the general relations of this section are the analogs of the general relations obtained by Agarwal and Wolf[13] in the context of boson systems. This has been possible because the operator T_{KQ} is found to play the role analogous to the operator $D(\alpha)=(e^{a^{\dagger}\alpha-a\alpha^*})$ for boson systems. In the next section we show how to formulate the product theorem and how the equations of motion can be transcribed into equations for phase-space distributions.

IV. EXPANSION OF THE PRODUCT OF TWO MULTIPOLE OPERATORS IN TERMS OF MULTIPOLE OPERATORS

For dynamical problems which have to be mapped onto phase space, we have to know how the product of two multipole operators behaves. It is clear that such a product can be expanded in terms of other T_{KQ} operators, i.e.,

$$T_{K_1Q_1}T_{K_2Q_2}=\sum_{K_3Q_3}\mathrm{Tr}(T^{\dagger}_{K_3Q_3}T_{K_1Q_1}T_{K_2Q_2})T_{K_3Q_3}\,, \tag{4.1}$$

where (1.7) has been used. The coefficient appearing in (4.1) can be obtained by using the definition of (1.6) and the following relationship between Wigner $3j$ and Wigner $6j$ symbols:

$$\sum_{\mu_1\mu_2\mu_3}(-1)^{l_1+l_2+l_3+\mu_1+\mu_2+\mu_3}\begin{pmatrix}j_1 & l_2 & l_3\\ m_1 & \mu_2 & -\mu_3\end{pmatrix}\begin{pmatrix}l_1 & j_2 & l_3\\ -\mu_1 & m_2 & \mu_3\end{pmatrix}\begin{pmatrix}l_1 & l_2 & j_3\\ \mu_1 & -\mu_2 & m_3\end{pmatrix}=\begin{pmatrix}j_1 & j_2 & j_3\\ m_1 & m_2 & m_3\end{pmatrix}\begin{Bmatrix}j_1 & j_2 & j_3\\ l_1 & l_2 & l_3\end{Bmatrix}, \tag{4.2}$$

with the result that[14]

$$T_{K_1Q_1}T_{K_2Q_2}=\sum_{K_3Q_3}(-1)^{K_1+K_2+K_3-2j-Q_3}[(2K_1+1)(2K_2+1)(2K_3+1)]^{1/2}\begin{pmatrix}K_1 & K_2 & K_3\\ Q_1 & Q_2 & -Q_3\end{pmatrix}\begin{Bmatrix}K_1 & K_2 & K_3\\ j & j & j\end{Bmatrix}T_{K_3Q_3}. \tag{4.3}$$

The range of K_3 values is $|K_1-K_2|,\ldots,(K_1+K_2)$. It is obvious from the symmetry property of Wigner $3j$ symbols that

$$T_{K_2Q_2}T_{K_1Q_1}=\sum_{K_3Q_3}(-1)^{2K_1+2K_2+2K_3-2j-Q_3}[(2K_1+1)(2K_2+1)(2K_3+1)]^{1/2}\begin{pmatrix}K_1 & K_2 & K_3\\ Q_1 & Q_2 & -Q_3\end{pmatrix}\begin{Bmatrix}K_1 & K_2 & K_3\\ j & j & j\end{Bmatrix}T_{K_3Q_3}. \tag{4.4}$$

Using (4.4) and (4.3) one can prove the following important relations involving the product of angular momentum operators and T_{KQ} operators:

$$J_{\mp} T_{KQ} = \pm \left(\frac{2j(j+1)(2j+1)}{3}\right)^{1/2} T_{1,\mp 1} T_{KQ}$$
$$= \tfrac{1}{2}[(K\pm Q)(K\mp Q+1)]^{1/2} T_{K,Q\mp 1} \pm \frac{1}{2}\left(\frac{(2j-K)(2j+K+2)(K\mp Q+1)(K\mp Q+2)}{(2K+1)(2K+3)}\right)^{1/2} T_{K+1,Q\mp 1}$$
$$\mp \frac{1}{2}\left(\frac{(2j+K+1)(2j-K+1)(K\pm Q)(K\pm Q-1)}{(2K+1)(2K-1)}\right)^{1/2} T_{K-1,Q\mp 1}$$
$$= T_{KQ}J_{\mp} + [(K\pm Q)(K\mp Q+1)]^{1/2} T_{K,Q\mp 1} \,. \tag{4.5}$$

Relation (4.3) enables us to express the phase-space distribution of the product of two operators in the form (3.15), i.e., if

$$F_i^{(\Omega)}(\theta,\phi) = \sum_{KQ} Y_{KQ}(\theta,\phi) F_{iKQ}^{(\Omega)},$$
$$F_{iKQ}^{(\Omega)} = \Omega_{KQ}^{-1} \operatorname{Tr}(G_i T_{KQ}^{\dagger}), \quad i=1,2 \tag{4.6}$$

then

$$F_{12}^{(\Omega)}(\theta,\phi) = \operatorname{Tr}[G_1 G_2 \Delta^{(\bar\Omega)}(\theta,\phi)] = \sum_{KQ} Y_{KQ}(\theta,\phi) F_{12KQ}^{(\Omega)}, \tag{4.7}$$

$$F_{12KQ}^{(\Omega)} = \sum_{K_1Q_1K_1Q_2} F_{1K_1Q_1}^{(\Omega)} F_{2K_2Q_2}^{(\Omega)} (-1)^{K_1+K_2+K-2j-Q} [(2K_1+1)(2K_2+1)(2K_2+1)]^{1/2}\, \Omega_{KQ}^{-1}\Omega_{K_1Q_1}\Omega_{K_2Q_2}$$
$$\times \begin{pmatrix} K_1 & K_2 & K \\ Q_1 & Q_2 & -Q \end{pmatrix} \begin{Bmatrix} K_1 & K_2 & K \\ j & j & j \end{Bmatrix}. \tag{4.8}$$

The dynamical equation for the density matrix could be easily transcribed into an equation for $F_{KQ}^{(\Omega)}$. We first note that the density matrix equation

$$\frac{\partial \rho}{\partial t} = \mathcal{L}\rho \tag{4.9}$$

implies that

$$\frac{\partial \langle T_{KQ}\rangle}{\partial t} = \langle \mathcal{L}^{+} T_{KQ}\rangle , \tag{4.10}$$

where $\mathcal{L}^{+}$ is the adjoint of the operator $\mathcal{L}$. By the use of the product theorem (4.3) and the commutation relations

$$\begin{aligned}[]
[J_{\pm}, T_{KQ}] &= [(K\mp Q)(K\pm Q+1)]^{1/2} T_{K,Q\pm 1}, \\
[J_z, T_{KQ}] &= Q\, T_{KQ} .
\end{aligned} \tag{4.11}$$

The expectation value $\langle \mathcal{L}^{+} T_{KQ}\rangle$ can be expressed as a linear superposition of $\langle T_{K'Q'}\rangle$. The equations for $F_{KQ}^{(\Omega)}$ can be obtained from the equation for $\langle T_{KQ}\rangle$ and from (3.16).

As an example, we consider the important master equation[6] describing such processes as superradiance[15] and cooperative resonance fluorescence

$$\frac{\partial \rho}{\partial t} = -\gamma(J_+J_-\rho - 2J_-\rho J_+ + \rho J_+J_-) - i\Omega(J_+ + J_-,\ \rho)\,. \tag{4.12}$$

From (4.12) and (4.11) we evidently have

$$\langle \dot T_{KQ}\rangle = i\Omega[(K-Q)(K+Q+1)]^{1/2}\langle T_{K,Q+1}\rangle + i\Omega[(K+Q)(K-Q+1)]^{1/2}\langle T_{K,Q-1}\rangle$$
$$+\gamma[(K-Q)(K+Q+1)]^{1/2}\langle T_{K,Q+1}J_-\rangle - \gamma[(K+Q)(K-Q+1)]^{1/2}\langle J_+T_{K,Q-1}\rangle , \tag{4.13}$$

which, on using product relations (4.5), simplifies to

$$\langle \dot T_{KQ}\rangle = i\Omega[(K-Q)(K+Q+1)]^{1/2}\langle T_{K,Q+1}\rangle + i\Omega[(K+Q)(K-Q+1)]^{1/2}\langle T_{K,Q-1}\rangle$$
$$-\gamma[K(K+1)-Q^2]\langle T_{K,Q}\rangle + \gamma K\{[(K+1)^2-Q^2](2j-K)(2j+K+2)/(2K+1)(2K+3)\}^{1/2}$$
$$\times \langle T_{K+1,Q}\rangle - \gamma(K+1)\left(\frac{(K^2-Q^2)(2j+K+1)(2j-K+1)}{(2K+1)(2K-1)}\right)^{1/2}\langle T_{K-1,Q}\rangle . \tag{4.14}$$

From (4.14) and (3.16), we obtain the equation for $F_{KQ}^{(\Omega)}$:

$$\dot{F}_{KQ}^{(\Omega)} = -\frac{\Omega_{K,Q+1}}{\Omega_{KQ}} i\Omega[(K-Q)(K+Q+1)]^{1/2} F_{K,Q+1}^{(\Omega)} - \gamma[K(K+1)-Q^2]F_{KQ}^{(\Omega)} - \frac{\Omega_{K,Q-1}}{\Omega_{KQ}}$$
$$\times i\Omega[(K+Q)(K-Q+1)]^{1/2} F_{K,Q-1}^{(\Omega)} + \gamma K \frac{\Omega_{K+1,Q}}{\Omega_{KQ}} \left(\frac{[(K+1)^2-Q^2](2j-K)(2j+K+2)}{(2K+1)(2K+3)}\right)^{1/2} F_{K+1,Q}^{(\Omega)}$$
$$-\gamma(K+1)\frac{\Omega_{K-1,Q}}{\Omega_{KQ}}\left(\frac{(K^2-Q^2)(2j+K+1)}{(2K+1)(2K-1)}(2j-K+1)\right)^{1/2} F_{K-1,Q}^{(\Omega)} . \tag{4.15}$$

In the special case if we choose Ω_{KQ} to be given by (3.22), then (4.15) can be shown to be equivalent[16] to Eq. (3.5) of Ref. 6. We also mention here, that in Ref. 17, the steady-state solution of (4.12) in the limit of intense fields was shown to have the structure

$$\rho = \frac{1}{(2j+1)} \sum_{-j}^{+j} |j\, m\rangle\langle j\, m| , \quad j = \tfrac{1}{2}N . \tag{4.16}$$

For the state (4.16), only $\langle T_{0,0}^{+}\rangle$ is nonvanishing. The steady state is thus characterized by a uniform phase-space distribution.

V. EXPANSION OF TIME CORRELATION FUNCTIONS IN TERMS OF STATE MULTIPOLES

We now show how the multitime correlation functions of a cooperative system could be expressed simply in terms of the state multipoles. Such multitime correlations determine, for example, the absorption and emission spectra of the radiation from a cooperative system.

Using the equations of motion for $\langle T_{KQ}^{+}\rangle$, one can obviously write

$$\langle T_{KQ}^{+}(t)\rangle = \sum_{K_1 Q_1} g_{K_1 Q_1}^{KQ}(t,t')\langle T_{K_1 Q_1}^{+}(t')\rangle ,$$
$$g_{K_1 Q_1}^{KQ} = \delta_{KK_1}\delta_{QQ_1} \quad \text{if} \quad t=t' . \tag{5.1}$$

Using (5.1) and the quantum regression theorem[18] for a Markovian system, one gets for the multitime correlations

$$\langle T_{K_3Q_3}(t'')\, T_{KQ}^{+}(t)\, T_{K_2Q_2}(t')\rangle = \sum_{K_1Q_1} g_{K_1Q_1}^{KQ}(t,t')\langle T_{K_3Q_3}(t'')\, T_{K_1Q_1}^{+}(t')\, T_{K_2Q_2}(t')\rangle , \quad t>t'>t'' . \tag{5.2}$$

By successive use of the relations of the type (5.2) and the product relation (4.3) one can reduce the n-time correlation function to the one-time expectation values which are determined from (5.1). The function $g_{K_1Q_1}^{KQ}$ plays the role of Green's function for the difference differential Eq. (4.10). We now give some examples of the calculation of multitime correlation functions. Let us consider the correlation which determines the spectrum of the radiation scattered by a cooperative system

$$\chi(\tau) = \langle J_+(\tau) J_-(0)\rangle = \left(\frac{2j(j+1)(2j+1)}{3}\right)^{1/2} \langle T_{1,-1}^{+}(\tau) J_-(0)\rangle . \tag{5.3}$$

On using (5.2), the correlation function $\chi(\tau)$ becomes

$$\chi(\tau) = \left(\frac{2j(j+1)(2j+1)}{3}\right)^{1/2} \sum_{KQ} g_{KQ}^{1,-1}(\tau)\langle T_{K,Q}^{+} J_-\rangle , \tag{5.4}$$

which on using (4.5) reduces to

$$\chi(\tau) = \left(\frac{2j(j+1)(2j+1)}{3}\right)^{1/2} \sum_{KQ} g_{KQ}^{1,-1}(\tau) \left[\tfrac{1}{2}[(K-Q)(K+Q+1)]^{1/2}\langle T_{K,Q+1}^{+}\rangle\right.$$
$$-\frac{1}{2}\left(\frac{(2j-K)(2j+K+2)(K+Q+1)(K+Q+2)}{(2K+1)(2K+3)}\right)^{1/2}\langle T_{K+1,Q+1}^{+}\rangle$$
$$\left.+\frac{1}{2}\left(\frac{(2j+K+1)(2j-K+1)(K-Q)(K-Q-1)}{(2K+1)(2K-1)}\right)^{1/2}\langle T_{K-1,Q+1}^{+}\rangle\right] \tag{5.5}$$

$$\xrightarrow[\tau\to 0]{} \left(\frac{j(j+1)(2j+1)}{3}\right)^{1/2}\left[\langle T_{1,0}^{+}\rangle - \left(\frac{(2j-1)(2j+3)}{15}\right)^{1/2}\langle T_{2,0}^{+}\rangle + \left(\frac{2j(2j+2)}{3}\right)^{1/2}\langle T_{0,0}^{+}\rangle\right] . \tag{5.6}$$

A simple exercise again shows that (5.5) is equivalent to relation[19] (3.14) obtained in Ref. 6 by a much more elaborate procedure. The above example shows clearly how simply the state multipoles can be used in the calculation of correlation functions. The correlation function $\chi(\tau)$ can also be expressed in terms of $F_{KQ}^{(\Omega)}$ by using relation (3.16) in (5.5). The correlation function giving the absorption spectra has a much simpler structure in terms of state multipoles

$$\begin{aligned}\langle [J_+(\tau), J_-(0)]\rangle &= \left(\frac{2j(j+1)(2j+1)}{3}\right)^{1/2} \sum_{KQ} g_{KQ}^{1,-1}(\tau)\langle [T_{KQ}^{\dagger}, J_-]\rangle \\ &= \left(\frac{2j(j+1)(2j+1)}{3}\right)^{1/2} \sum_{KQ} g_{KQ}^{1,-1}(\tau)\langle T_{K,Q+1}^{\dagger}\rangle[(K-Q)(K+Q+1)]^{1/2}. \end{aligned} \tag{5.7}$$

In conclusion we have shown the deep relationship between the atomic coherent-state representation of Arecchi *et al.* and the state multipoles as introduced by Fano.[2] The use of state multipoles enables us to formulate a general theory of phase-space distributions and to generate a class of phase-space distributions for angular momentum (collective atomic) systems. Such state multipoles are shown to yield in a rather straightforward manner the dynamical equations and multitime correlation functions.

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[8] More general definitions of multipole operators exist for a system consisting of multiplet of states with each multiplet characterized by a different value of the angular momentum.

[9] For a summary of the properties of $3j$ symbols and $6j$ symbols as well as their tables, see, for example, I. I. Sobelman, *Atomic Spectra and Radiative Transitions* (Springer, Berlin, 1979), Chap. 4.

[10] See, for example, J. D. Talman, *Special Functions, A Group Theoretic Approach* (Benjamin, New York, 1968), Eqs. (8.11), (8.13), (8.18), and (8.38); similar relations are implicitly contained in Appendix D of the paper by Arecchi *et al.* (Ref. 1).

[11] Reference 10; Eqs. (9.20) and (9.39).

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[14] Note that the corresponding relation

$$D(\alpha)D(\beta) = D(\alpha+\beta)\exp\tfrac{1}{2}(\alpha\beta^* - \alpha^*\beta)$$

for the boson case is much simpler.

[15] More general relaxation behavior in the framework of phase-space distributions is discussed in Ref. 4 and in L. M. Narducci, C. M. Bowden, V. Bluemel, and G. Patricio Carrazana, Phys. Rev. A 11, 280 (1975).

[16] Such equations were obtained earlier by first getting a differential equation for $F^{(\Omega)}(\theta,\phi)$ using the techniques developed in Refs. 3, 5, and 15.

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[19] A minor printing error in (3.14) of Ref. 6 may be noted—replace the coefficient of $p_{l+1,\,m+1}(\infty)$ within the square brackets by

$$-\tfrac{1}{2}(N-l)\left(\frac{(l+m+1)}{(2l+1)}\frac{(l+m+2)}{(2l+3)}\right)^{1/2}.$$