#### A Dynamical Principle For The Salpeter Equation

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#### Abstract

The 'Salpeter Equation' which has long been known as the 3D version of the 4D Bethe-Salpeter Equation under the Instantaneous Approximation, has a well-defined rationale that stems from the half-century old Markov-Yukawa Transversality Principle (MYTP) which not only effects an exact 3D reduction from the original (4D) BS form, but also provides an equally exact *reconstruction* of the 4D BS amplitude in terms of 3D ingredients. The second aspect which is *new*, opens up a vista of applications to transition amplitudes as 4D loop integrals.

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## 1. Introduction: Historical Background

The problem of probability interpretation of the well-known Bethe-Salpeter Equation (BSE) [1] has for long been a much discussed issue in the Physics literature, and has led to various attempts [2-5] towards its 3D reduction, as well as for intrinsic 3D formulations of various kinds [6]. On the other hand, the issue should be viewed against the theoretical perspective of the BSE as only an *approximate* description (in the so-called ladder approximation) of the equations of motion that follow strictly from the standard QED Lagrangian, viz., the Schwinger-Dyson Equations (SDE) which are an infinite chain of equations connecting successively higher order vertex functions. Thus a conceptual problem like lack of probability interpretation for a *truncated* two-body system need hardly cause undue surprise. Nevertheless, the approximate nature of the BSE [1] has never been a detracting issue from its practical usefulness which has lent considerable credence to the attempts at its 3D reduction designed among other things to restore the probability interpretation to the BS amplitude. Of these attempts, the Salpeter Equation [3] is perhaps the oldest, and has claimed considerable attention in the contemporary literature, from the atomic two-body problem to the QCD context of heavy quarkonia. Another possible motivation for the 3D reduction of the BSE is the observed O(3)-like spectra of the respective energy levels.

To meet this dual requirement of both the probability interpretation as well as the observed spectra, the "Instantaneous Approximation" was historically [2] the earliest ansatz to be invoked for a 3D reduction of the full 4D BSE, leading to the Salpeter equation [3]. However, the ansatz (which also suffered from lack of Lorentz Covariance)

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seemed to beg a solid underlying principle. On the other hand, the extensive use of the Salpeter equation in the contemporary literature demands a firmer theoretical foundation.

#### 1.1 The Markov-Yukawa Transversality Principle

In the quest for a theoretical basis for the Instantaneous Approximation, success seems to have come from a rather unexpected quarter, viz., the half-century old Markov-Yukawa Transversality Principle (MYTP) [7], as was to be discovered by the Dubna Group [8]. Now the MYTP [7] presupposes a dependence of the field variable on both x and p. While this is unacceptable for an elementary particle description (and is probably the reason why the Yukawa non-local field thery [7b] was found unattractive), it seems to be ideally suited to a *composite* particle description, wherein the momentum dependence comes from the direction of the total 4-momentum  $P_{\mu}$ . For a bilocal field  $\mathcal{M}(z, X)$ , the Transversality condition [7] was shown by Lukierski et al [9] to be equivalent to a 'gauge principle' which expresses the redundance of the *longitudinal* component of the relative momentum for the physical interaction between the two constituents. As will be shown in Sect.2, this condition amounts to a covariant 3D support to the input 4-quark Lagrangian, whence follows the 4D BSE with a 3D kernel support governed by Covariant Instantaneity. This Principle was first invoked by the Dubna group [8] to show that the 3D Salpeter equation [3] follows as an exact consequence of the covariant 3D support to the Bethe-Salpeter kernel, with the preferred direction as  $P_{\mu}$ . This gave a firm theoretical basis to the 3D Salpeter equation.

The other side of the coin, apparently missed by the Dubna Group [8], concerns the question of whether the information on the 4D content of the original BSE is retrievable, after the 3D reduction. As was to be found soon afterwards [10], the inbuilt structure of MYTP [7] ensures that the original 4D BSE is exactly recovered by retracing the steps ! This two-way interconnection [10] between the 3D and 4D BSE forms was initially proved [10] for an idealized spinless fermion problem, but, as will be shown in this paper, the logic goes through equally well for spinor fermions, thus facilitating an exact reconstruction of the original 4D BS amplitude in terms of the 3D ingredients of the Salpeter equation. This is, surprisingly enough, a *new* result, considering the fact that this aspect of the Salpeter equation has never seen the light of the day despite its half century old existence. A generalization of the 3D-4D interconnection of Bethe-Salpeter amplitudes under MYTP [7] to the 3-body problem has been given recently [11].

The paper is organized as follows. In Sect.2, starting with the logic of the MYTP [7] which mandates a covariant 3D support to the kernel of a BSE, we first recapulate the main steps [10] that lead to an exact 3D-4D interconnection between the corresponding BS amplitudes for a 'spinless' two-particle system. In Sect.3 we outline a corresponding derivation for the Salpeter equation by recalling the main steps leading from the 4D to the 3D form [3], and reversing these steps. Sect.4 concludes with some comments on the significance of this results vis-a-vis the Markov-Yukawa Principle, especially the applicability of the 'Salpeter Vertex fn' to transition amplitudes as 4D loop integrals.

### 2. MYTP As A Gauge Principle

The logic of MYTP [7] may be traced to Yukawa's non-local field theory [7b], characterized by the field dependence on both coordinate and momentum. As this violates local microcausality, this concept as a basic theory of elementary particles did not find much favour within the physics community. However this (limited) perspective had to change with the advent of QCD [12] which pushed the status of hadrons from the elementary to a composite level, and gave rise to the concept of bilocal fields [13]. Within such a bilical scenario, the total 4-momentum  $P_{\mu}$  of the composite hadron provides a naturally preferred direction which forms the basis for a covariant 3D support to the interaction kernel [8,10].

An important feature of bilocal dynamics is the redundance [9] of the relative 'time' variable  $x_0$ ,  $(x = x_1 - x_2)$ , whose covariant definition is just the longitudinal component of  $x_{\mu}$  in the direction of  $P_{\mu}$ , viz.,  $x.PP_{\mu}/P^2$ . This 'redundance' is expressed by the statement that a translation of the relative coordinate [9]  $x_{\mu} \to x'_{\mu} + \xi P_{\mu}$  on the bilocal field  $\mathcal{M}(x, P)$ :

$$\mathcal{M}(x_{\mu}, P_{\mu}) \to \mathcal{M}_{\xi}(x_{\mu}, P_{\mu}) = \mathcal{M}(x_{\mu} + \xi P_{\mu}, P_{\mu})$$

, which is a sort of 'gauge transformation' of the bilocal field [9], should leave this quantity *invariant*. This invariance is just the content of the Markov-Yukawa subsidiary condition [7] which, under an interchange of the relative coordinates and the momenta reads as [9, 8b]

$$P_{\mu}\frac{\partial}{\partial x_{\mu}}\mathcal{M}(x_{\mu}, P_{\mu}) = 0 \tag{1}$$

where the direction  $P_{\mu}$  guarantees an irreducible representation of the Poincare' group for the bilocal field  $\mathcal{M}$  [9]. An equation of type has been used in other approaches to bilocal field dynamics (see ref [9] for other references), but this 'gauge' interpretation of the subsidiary condition [9] provides a more transparent view of the same condition which we have abbreviated as MYTP above.

Eq.(1) amounts to an effective 3D support to the interaction between the constituents of the bilocal field, which may be alternatively postulated directly for the pairwise BSE kernel K [10] by demanding that it be a function of only  $\hat{q}_{\mu} = q - q \cdot P P_{\mu} / P^2$ , which implies that  $\hat{q} \cdot P \equiv 0$ . In this approach, the propagators are left untouched in their full 4D forms. This is somewhat complementary to the 3D BSE reduction methods [4-6] (propagators manipulated but kernel left untouched), so that the resulting equations [8,10] look rather unfamiliar vis-a-vis 3D BSE's [4-6], but it has the advantage of allowing a *simultaneous* use of both 3D and 4D BSE forms via their interlinkage. Indeed what distinguishes the Covariant Instantaneity Ansatz [10] from the more familiar 3D BSE reductions [4-6] is its capacity for a 2-way linkage: an exact 3D BSE reduction, and an equally exact reconstruction of the original 4D BSE form without extra charge [10]. In contrast the more familiar methods of BSE reduction [4-6] give at most a one-way connection, viz., a  $4D \rightarrow 3D$  reduction, but not vice versa. This is a plus point for MYTP, and may well have a wider significance than the mere BSE context above, as an effective dynamics for strong interactions.

#### 2.1 3D-4D Interconnection: Spinless Particles

To demonstrate the basic 3D-4D interconnection under MYTP [7], consider a system of two dentical spinless particles, with the BSE [10]

$$i(2\pi)^{4}\Phi(q,P) = (\Delta_{1}\Delta_{2})^{-1} \int d^{3}\hat{q}' M d\sigma' K(\hat{q},\hat{q}')\Phi(q',P); [\Delta_{1,2} = m_{q} + p_{1,2}^{2}]$$
(2)

where the 3D support to the kernel K is implied in its 'hatted' structure:

$$\hat{q}_{\mu} = q_{\mu} - \sigma P_{\mu}; \sigma = q.P/P^2; \hat{q}.P \equiv 0.$$
 (3)

The relative and total 4-momenta are related by

$$p_1 + p_2 = P = p'_1 + p'_2; 2q = p_1 - p_2; 2q' = p'_i - p'_2.$$

The 3D wave function  $\phi(\hat{q})$  is defined by [10]

$$\phi(\hat{q}) = \int M d\sigma \Phi(q, P) \tag{4}$$

When (4) is substituted on the RHS of (2) one gets

$$i(2\pi)^{4}\Phi(q,P) = (\Delta_{1}\Delta_{2})^{-1} \int d^{3}\hat{q}' K(\hat{q},\hat{q}')\phi(\hat{q}')$$
(5)

Now integrate both sides of this equation wrt  $\sigma$  to get an explicit 3D equation

$$(2\pi)^{3} D(\hat{q}) \phi(\hat{q}) = \int d^{3} \hat{q}' K(\hat{q}, \hat{q}') \phi(\hat{q}')$$
(6)

where the 3D denominator function is given by

$$2i\pi D^{-1}(\hat{q}) = \int M d\sigma (\Delta_1 \Delta_2)^{-1} \tag{7}$$

A comparison of (5) with (6) via (7) gives the 3D-4D interconnection

$$21\pi\Delta_1\Delta_2\Phi(q,P) = D(\hat{q})\phi(\hat{q}) \tag{8}$$

which directly identifies the RHS as the hadron-quark Vertex Function

$$\Gamma = D \times \phi/2i\pi. \tag{9}$$

# 3. Salpeter Eqn: 3D-4D Interlinkage

Let us now look at the Salpeter equation for the relativistic hydrogen atom problem, which in the notation of the original paper [3] reads as

$$i\pi^2 F(q_\mu)\psi(q) = \alpha \int d^4k \mathbf{k}^{-2}\psi(q+k)$$
(10)

A comparison of this equation with eq.(6) shows a precise correspondence, except for certain technicalities arisin from its fermionic content. Indeed it stems from an equation of the form (2), where the 3D kernel support is due to the (non-covariant) instantaneous

(adiabatic) assumption [3], manifesting from its dependence on the 3-vector **k**, while the quantity  $F(q_{\mu})$  plays just the role of the product of the two 4D propagators  $\Delta_1$  and  $\Delta_2$  in (2):

$$F(q) = (\mu_1 E - H_1(\mathbf{q}) + \epsilon)(\mu_2 E - H_1(\mathbf{q}) - \epsilon)$$
(11)

with the time-like components identified as the  $\epsilon$  terms ! Next, define the 3D wave function  $\phi(\mathbf{q})$  by

$$\phi(\mathbf{q}) = \int d\epsilon \psi(\mathbf{q}, \epsilon) \tag{12}$$

which is the counterpart of (4), and use this result to integrate both sides of (10) wrt  $\epsilon$ , after dividing by F(q), so as to get the 3D Salpeter equation [3]

$$[E - H_1(\mathbf{q}) - H_2(\mathbf{q})]\phi_{\pm\pm} = \pm \Lambda_{\pm}^{(1)}\Lambda_{\pm}^{(2)}(-2i\pi\Gamma(\mathbf{q}) = (-4i\alpha)\int d^3k\mathbf{k}^2\phi(\mathbf{q}+\mathbf{k})$$
(13)

where the  $\pm$  components are associated with the energy projection operators  $\Lambda$  which however do not involve the time-like  $\epsilon$ .

The new aspect, on the other hand, is the 3D-4D interconnection which is obtained by substituting the second part of eq.(13) on the RHS of (10), after making use of (12):

$$F(q)\psi(q) = \Gamma(\mathbf{q}) \tag{14}$$

where  $\Gamma(\mathbf{q})$  is the 3D BS vertex function. It is the precise fermionic counterpart of the scalar eq.(9), since the F(q) function is the product of the two 4D propagators. The form (14) is not formally covariant, but this is a mere technicality which can be remedied by standard methods; see e.g., ref [14].

The more interesting thing about this demonstration is the exciting prospect of using the reconstructed 4D 'Salpeter vertex function' (14) as a basic ingredient for the calculation of various types of transition amplitudes as 4D loop integrals by standard Feynman techniques without having to face the usual problems of probability interpretation and/or spectroscopy, both of which are now subsumed in the 3D equation (13). This gives a sort of 'two-tier' description, the 3D form (13) just right for spectroscopy, energy levels, etc, while the 4D form (14) provides the proper vehicle for 4D loop integrals. It is only the first (3D) part of the Salpeter Equation [3] that has so far been evidenced in the contemporary literature, but the second (4D) aspect is entirely *new*.

#### 4. Retrospect And Summary

In retrospect, we have attempted to project an aspect of the well-known Salpeter equation [3], which had remained obscured from view for decades, viz., a theoretical basis for its underlying Instantaneous Approximation, offered by the Markov-Yukawa Transversality Principle [7]: An in-built MYTP [7] in the 4D BSE structure leads to an exact 3D reduction which, as first shown by the Dubna group [8], is a covariant generalization of the Salpeter Equation [3]. The (new) complementary aspect of MYTP is its in-built capacity to *reconstruct* with equal ease [10], the 4D vertex function, (9) or (14), in terms of 3D ingredients, which allows access to transition amplitues of diverse types as 4D loop integrals. This offers a two-tier description for the Salpeter Equation, analogously to the quark-level hadronic BSE problem that has been under study for several years, with its 3D form providing access to spectroscopy [15], and the 4D form offering applications to processes like

e.m. form factors [14], within a single framework. This dual feature distinguishes MYTP from most other 3D approaches to strong interaction dynamics [4-6] which give at most a one-way connection (4D to 3D). This remarkable property of 3D-4D interlinkage enjoyed by the Salpeter equation [3], by virtue of its compliance with MYTP, should hopefully offer new incentives for its (second stage) applications to 4D loop integrals in a covariant manner.

The 3D-4D interlinkage offered by MYTP is also generalizable to a 3-body BSE with pairwise kernels under covariant 3D support [11]. A second type of generalization of MYTP is to the covariant null-plane [14] which facilitates trouble-free evaluation of form factors with triangle loops.

To summarise, the instantaneous approximation which characterizes the Salpeter equation, comes as a mere consequence of the Markov-Yukawa Transversality Principle which by its very definition gives a precise 3D support to the BSE kernel. Secondly, MYTP allows reconstruction of the 4D Salpeter amplitude in terms of 3D ingredients, a property which had remained obscured from view so far. Thus the Salpeter equation is amenable to a two-tier formalism, the 3D form for spectroscopy, and the reconstructed 4D vertex function for 4D loop integrals. Finally, the non-covariance of the Salpeter equation [3] is a mere technicality which is easily remedied by standard techniques [14].

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### REFERENCES

- [1] E.E.Salpeter and H.A.Bethe, Phys.Rev.84, 1232 (1951).
- [2] M.M.Levy, Phys.Rev.88, 72 (1952).
- [3] E.E.Salpeter, *ibid* 87, 328 (1952)
- [4] (a) A. Logunov and A.N.Tavkhelidze, Nuovo Cimento 29, 380 (1963); (b)
   V.R.Garsevanishvili, et al, Phys.Lett.29B, 191 (1968); (c) R.N.Faustov, Ann.
   Phys.(N.Y.)78, 176 (1973).
- [5] R. Blankenbecler and R. Sugar, Phys.Rev.142, 105 (1966); F.Gross, *ibid*, 1408 (1966)
- [6] Some typical examples: (a) S.Weinberg, Phys.Rev.150, 1313 (1966); (b) V.
   Kadychevsky, Nucl.Phys.B6, 125 (1968); (c) I.Todorov, Phys.Rev.D3, 2351 (1971)
- [7] (a) M.A. Markov, Sov.J.Phys.3, 452 (1940); (b) H. Yukawa, Phys.Rev.77, 219 (1950).
- [8] Yu. L. Kalinowski et al, Phys.Lett.**B231**, 288 (1989).
- [9] J. Lukierski and M. Oziewicz, Phys.Lett.**B69**, 339 (1977).
- [10] A.N. Mitra and S. Bhatnagar, Int.J.Mod.Phys.A7, 121 (1992).
- [11] A.N.Mitra, Int.J.Mod.Phys.A14, 4781 (1999).
- [12] H.D.Politzer, Phys.Rev.Lett.30,1346 (1973); D.J.Gross and F.Wilczek, Phys.Rev.D8,3633 (1973).

- [13] (a) J.M. Cornwall et al, Phys.Rev.D10, 2428 (1974); (b) H. Kleinert, Phys.Lett.B26, 429 (1976); (c) D.Ebert and V.N.Pervushin, Teor.Mat.Fiz.(USSR) 36, 313 (1978); (d) D.W. McKay et al, Phys.Rev.D37, 195 (1988).
- [14] A.N.Mitra, Phys.Lett.**B463**, 293 (1999).
- [15] (a) qq̄: A.Mittal and A.N.Mitra, Phys.Rev.Lett.57, 290 (1986); K.K.Gupta et al, Phys.Rev.D42, 1604 (1990);
  (b) qqq: A.Sharma et al, Phys.Rev.D50, 454 (1994).