Temperature Dependence of Gluon and Quark Condensates as from Linear Confinement

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Abstract

The gluon and quark condensates and their temperature dependence are investigated within QCD premises. The input for the former is a gauge invariant gg kernel made up of the direct (D), exchange (X) and contact(C) QCD interactions in the lowest order, but with the perturbative propagator k^{-2} replaced by a 'non-perturbative k^{-4} form obtained via two differentiations: $\mu^2 \partial_m^2 (m^2 + k^2)^{-1}$, ($\mu$ a scale parameter), and then setting $m = 0$, to simulate linear confinement. Similarly for the input $q\bar{q}$ kernel the gluon propagator is replaced by the above k^{-4} form. With these 'linear' simulations, the respective condensates are obtained by 'looping' up the gluon and quark lines in the standard manner. Using Dimensional regularization (DR), the necessary integrals yield the condensates plus temperature corrections, with a common scale parameter μ for both. For gluons the exact result is

$$< GG >= 36\mu^4 \pi^{-3} \alpha_s(\mu^2)[2 - \gamma - 4\pi^2 T^2/(3\mu^2)]$$

Evaluation of the quark condensate is preceded by an approximate solution of the SDE for the mass function $m(p)$, giving a recursive formula, with convergence achieved at the third iteration. Setting the scale parameter μ equal to the universal Regge slope $1 GeV^2$, the gluon and quark condensates at $T = 0$ are found to be $0.586 Gev^4$ and $(240 - 260 MeV)^3$ respectively, in fair accord with QCD sum rule values. Next, the temperature corrections (of order $-T^2$ for both condensates) is determined via finite-temperature field theory a la Matsubara.

Keywords: Gluon Condensate, mass tensor, gauge invariance, linear confinement, finite-temperature, contour-closing.

PACS: 11.15.Tk ; 12.38.Lg ; 13.20.Cz

1   Introduction

The thermal behaviour of QCD parameters has acquired considerable relevance in recent times in the context of global experimentation on heavy ion collisions as a means of

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accessing the quark-gluon plasma (QGP) phase. There are two distinct aspects to the effects of finite temperature, viz., deconfinement and restoration of chiral symmetry at high enough temperatures, but the question of which one of these two should occur before the other, is probably one of technical details of the theory [1]. We are concerned here not so much with the similarity of these phenomena as with the general manner of their onset, to the extent that the temperature variation of certain condensates carries this information. Indeed it is generally believed that the thermal behavour of condensates provides a fairly reliable index to the deconfinement/chiral symmetry restoration phases, but the precise mechanism through which this occurs often gets obscured in the technical details of the models employed. To that end, the quark and gluon condensates, by virtue of their basic nature, offer themselves as prime candidates for a study of the problem, to the extent that such phase transitions can be inferred from their thermal behaviour. In this respect, QCD-SR has been a leading candidate for such investigations for more than two decades [2], yet it cannot be regarded as a full-fledged substitute for a non-perturbative treatment in a *closed* form since the OPE underlying it stems from the high energy end. Prima facie it appears that the QCD-SR- prediction of temperature correction to $< GG >$ of order T^4 [3,4], instead of T^2, is rather 'flat' unless justified by more convincing arguments. On the other hand, the chiral perturbation theory, with non-perturbative information of a different type built-in [5], gives temperature corrections of $O(T^2)$ for $< q\bar{q} >_0$ [6]), while the non-perturbative content of instanton theories [7] is of a different nature. In view of such wide variations in the different predictions it should be worth exploring still other non-perturbative approaches to QCD, especially ones which incorporate *confinement* more transparently.

With this idea in mind, we propose an alternative non-perturbative approach to QCD and test it initially against the problem of the two basic QCD condensates. The precise content of this approach is expressed by the replacement of the o.g.e. propagator k^{-2} by a non-perturbative form k^{-4} which corresponds to *linear* confinement (in an 'asymptotic' sense) according to conventional wisdom, as follows:

$$lim_{m \to 0}(\mu \partial_m)^2 \frac{1}{k^2 + m^2} = \frac{2\mu^2}{k^4} \tag{1.1}$$

where μ is a scale parameter to be specified further below. While the full ramifications of this form are still being developed (see Section 6), the present study is limited only to its predictions on the condensates $< GG >$ and $< q\bar{q} >$ plus f_π. Consider first the gluon case where the dominant interaction is provided by the gluons themselves, with quark effects playing a secondary role. Now the standard QCD Lagrangian defines both the 3-and 4-gluon vertices, from which the lowest order gg interaction, via gluon exchange (both the 'direct'(D) and 'exchange' (X) terms), as well as the 4-gluon 'contact'(C) term can be generated as a *gauge invariant* package proportional to the QCD coupling constant $\alpha_s(\mu)$ [8,9](see fig.1):

$$\frac{g_s^2(\mu)F_1.F_2}{k^2}(1 + P_{12})[V^{(1)}.V^{(2)} - \frac{k^2}{2}\Delta(12)] \tag{1.2}$$

where P_{12} a permutation operator, and the other symbols are defined later in Sect. 2. This structure was employed in [8,9] as a kernel of a BSE for the gg wave function for the calculation of glueball spectra, but with the perturbative propagator replaced by harmonic type confinement, on similar lines to $q\bar{q}$ spectroscopy [10]. Alternative BSE
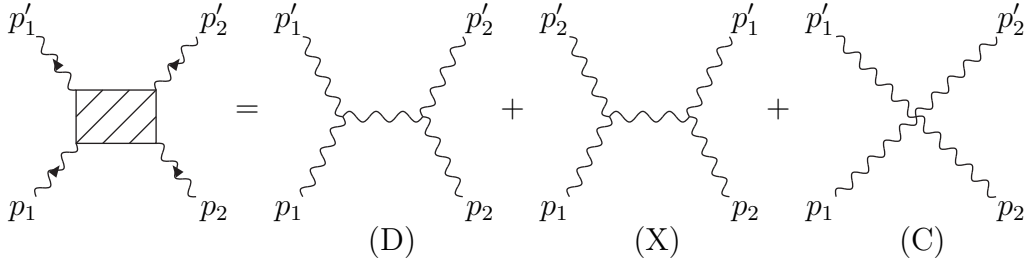
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Figure 1: A gauge invariant package of gg interaction in second order of QCD. D, X, C represent direct, exchange and contact interactions respectively.

treatments for glueballs also exist in the literature [11]. In this paper, on the other hand, we seek specifically to examine the precise point of departure of Eq.(1.2) from perturbative QCD by replacing the o.g.e propagator k^{-2} in front by its $non-perturbative$ form (1,1), (with a conscious simulation of $linear$ confinement), while retaining both Lorentz and gauge invariance of the full $(D + X + C)$ package. This gg interaction is proposed as a candidate for non-perturbative treatment of a few leading QCD amplitudes, with the burden of variation from perturbative QCD resting on the mathematical properties of a simple differential operator. Among possible applications,this interaction may serve as the kernel of a BSE on the lines of (8-11), but even in the $lowest$ order this non-perturbative quantity also offers other interesting possibilities. For, in the absence of quark (fermion) interactions, there is no coupling of the (vector) gluon field to $axial$ currents [12], hence no possibility of 'poles' in the gluon self-energy tensor [12] via the Schwinger mechanism [13]. Therefore it makes sense to calculate the gluon self-energy tensor by 'looping up' two of the gluon lines, and as a next step, the gluon condensate by letting the other two gluon lines disappear into the vacuum, a la Fig. 2. The derivative structure of Eq. (1.1-2) is ideally suited to the 't Hooft- Veltman method [14] of dimensional regularization (DR), while the temperature dependence a la Matsubara [15] may be introduced by an elegant method due to Kislinger-Morley [16]. The final result for the gluon condensate, including temperature effects, is extremely simple:

$$< GG >= \frac{36\mu^4}{\pi^3}(2 - \gamma)\alpha_s(\mu^2)[1 - (2\pi T/\mu)^2/(6 - 3\gamma)] \tag{1.3}$$

where $\gamma = 0.5772$ is the Euler-Mascheroni constant. It is rather amusing that this value $(0.586GeV^4)$ comes fairly close to the QCD-SR value of (0.480) [3] if the DR parameter μ^2 (the only free parameter in the theory) is identified with the universal Regge slope of $1GeV^2$. However the temperature correction of $O(T^2)$ seems to be larger than the QCD-SR value [4], while comparison with other models is postponed till Section 6.

As a second test of our non-perturbative ansatz (1.1), we shall consider the $q\bar{q}$ interaction via o.g.e., wherein the factor $g_s^2 F_1.F_2/k^2$ in Eq. (1.2) is replaced by $g_s^2 f_1.f_2\mu^2/k^4$, and generate the mass function $m(p)$ via SDE, and thence the quark condensate [17], via this new non-perturbative interaction. A third test is to the pionic constant f_π by the same method. In Sect. 2 we give a derivation of the analytic structure of the gluon condensate
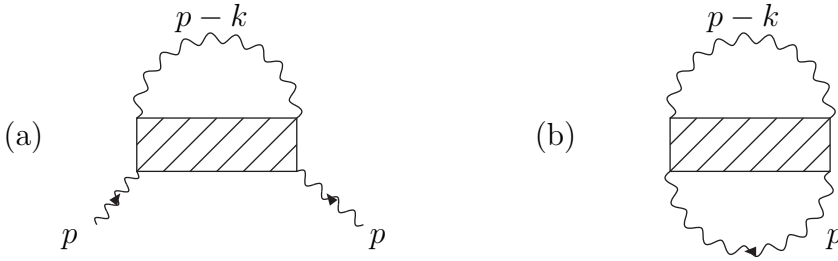
Figure 2: Looping up of gluon lines to give (a) self-energy tensor and (b) gluon condensate.

based on the diagrams of Figs. 1 and 2. In Sect. 3 we evaluate this quantity a la ref. [14], with explicit gauge invariance built in, and also introduce the temperature dependence a la ref. [16], to obtain Eq. (3). Sects. 4 and 5 give a corresponding derivations of the quark condensate and pionic constant respectively, showing equally promising results with the same scale parameter $\mu = 1 GeV$. Sect. 6 concludes with a short discussion on the non-perturbative ansatz (1.1) vis-a-vis the more standard approaches.

2 Structure of the Gluon Condensate

We first spell out the structure of the gg scattering amplitude arising from the interaction (1.1-2), in accordance with the figures 1(a), 1(b), 1(c) corresponding to direct (D), exchange (X) and contact (C) processes respectively. To that end the momenta and polarization indices of the incoming gluon lines are denoted by $(p_1, \mu_1; p_2, \mu_2)$, and of the outgoing ones by $p'_1, \nu_1; p'_2, \nu_2$ respectively [8]. In this notation, the various quantities are expressed as follows [8]:

$$V_\mu^{(1,2)} = (p_{1,2} + p'_{1,2})_\mu + 2iS_{\mu\nu}^{(1,2)}(p'_{1,2} - p_{1,2})_\nu; \quad -i[S_{\mu\nu}^{(1)}]_{\mu_1\nu_1} = \delta_{\mu\mu_1}\delta_{\nu\nu_1} - \delta_{\mu\nu_1}\delta_{\mu_1\nu} \quad (2.1)$$

and a similar value for $S_{\mu\nu}^{(2)}$. The spin dependence of the contact term is given by [8]:

$$\Delta_{\mu_1\mu_2}^{\nu_1\nu_2} = \delta_{\mu_1\nu_1}\delta_{\mu_2\nu_2} + \delta_{\mu_1\nu_2}\delta_{\mu_2\nu_1} - 2\delta_{\mu_1\mu_2}\delta_{\nu_1\nu_2} \quad (2.2)$$

The permutation operator P_{12} in Eq. (1.2) provides the exchange process by simultaneously interchanging these collective indices for the *outgoing* lines, while holding the incoming one fixed. However, unlike the case of glueball spectra [8-9], the effect of the permutation operator is not merely a factor of two, but a non-trivial sum of two different groups of terms related by the above interchanges. Defining the total (P) and relative (q, q') momenta as [8,9]

$$P = p_1 + p_2 = p'_1 + p'_2; \quad 2q = p_1 - p_2; \quad 2q' = p'_1 - p'_2; \quad P^2 = -M^2 \quad (2.3)$$

The momentum and spin dependence of the 'direct' part plus *half* the 'contact' part of (1.2) may now be adapted from [8] to give

$$A(p_1, p_2; p'_1 p'_2)_{\mu_1\mu_2}^{\nu_1\nu_2} = \quad (2.4)$$
$$[-M^2 - (q + q')^2]\delta_{\mu_1\nu_1}\delta_{\mu_2\nu_2} + [(p_1 + p'_1)_{\mu_2}(-2p_2)_{\nu_2} +$$

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$$(p_1 + p'_1)_{\nu_2}(-2p'_2)_{\mu_2}]\delta_{\mu_1\nu_1} + [(p_2 + p'_2)_{\mu_1}(-2p_1)_{\nu_1}$$
$$+ (p_2 + p'_2)_{\nu_1}(-2p'_1)_{\mu_1}]\delta_{\mu_2\nu_2} + (-2p_1)_{\nu_1}(-2p_2)_{\nu_2}\delta_{\mu_1\mu_2}$$
$$+ (-2p_1)_{\nu_1}(-2p'_2)_{\mu_2}\delta_{\mu_1\nu_2} + (-2p'_1)_{\mu_1}(-2p_2)_{\nu_2}\delta_{\nu_1\mu_2}$$
$$+ (-2p'_1)_{\mu_1}(-2p'_2)_{\mu_2}\delta_{\nu_1\nu_2} - \frac{1}{2}(q - q')^2\Delta_{\mu_1\mu_2}^{\nu_1\nu_2}$$

The corresponding 'exchange' part is then given by $A(p_1, p_2; p'_2 p'_1)_{\mu_1\mu_2}^{\nu_2\nu_1}$. We now tie up the two legs p'_1, ν_1 and p'_2, ν_2 in the above expression into a propagator (in the simple Feynman gauge), in accordance with Fig. 2(a), together with a similar expression for the corresponding exchange part. Before writing this down, we relabel the momenta and polarizations in (2.4) as follows

$$p_1 = -p_2 = p; \quad p'_1 = -p'_2 = p - k; \quad q + q' = 2p - k; \quad q - q' = k; \quad \mu_1, \mu_2 = \mu, \nu$$

For the exchange part, simply change the sign of q' which will interchange k with $2p - k$, and ν_1 with ν_2. The resultant quantity simplifies after several vital cancellations to

$$N_{\mu\nu}(p, k) = [(2M^2 + 2k^2 + 2(2p - k)^2)\delta_{\mu\nu} - 8(p - k)_\mu(p - k)_\nu + 8(p_\mu p_\nu - p^2\delta_{\mu\nu}) \quad (2.5)$$

which, apart from constant factors, can be related to the gluon self-energy tensor $\Pi_{\mu\nu}(p)$ by

$$\Pi_{\mu\nu}(p) = [\mu^2\partial_m^2]_{m=0}\int \frac{-id^4k}{(2\pi)^4} \frac{N_{\mu\nu}(p, k)}{(m^2 + (p - k)^2)(m^2 + k^2)} \quad (2.6)$$
$$\equiv (p_\mu p_\nu - p^2\delta_{\mu\nu})\Pi(p^2)$$

We have used the same mass for the propagator of the internal gluon line $p - k$. Since this quantity must be explicitly gauge invariant, we have anticipated its proportionality to $(p_\mu p_\nu - p^2\delta_{\mu\nu})$ in the second line. The final stage involves the looping up of the remaining two gluon lines into the vacuum, in the standard fashion [18] which includes taking a 4D 'curl' of the (non-perturbative) gluon propagator to get $< G_{\mu\nu}G_{\nu\mu} >$. These routine operations on Eq. (2.6) yield

$$< G_{\mu\nu}^a G_{\nu\mu}^b >= [6C]\delta^{ab}\int \frac{-id^4p}{(2\pi)^4}\Pi(p^2)] \quad (2.7)$$

where $\Pi(p^2)$ is defined in Eq. (2.6). The factor [6] in front arises from the effect of 'curl' on the gluon propagator to obtain the LHS, while [C] represents the remaining constants from the lowest order QCD amplitude of Eq. (1.1-2), viz.,

$$C = 4\pi\alpha_s(\mu^2)F_1.F_2 \quad (2.8)$$

$F_1.F_2 = -3$ being the color Casimir of the $g - g$ interaction [9] which should be distinguished from the factor $\delta^{ab} = 8$ arising from the last gluon propagator before disappearing into the vacuum.

3   Dimensional Regularization for Integrals

We first consider the integral of Eq. (2.6) for which we shall follow closely the DR method [14]. Introducing the Feynman variable $0 \le u \le 1$, giving the usual shift $k \to p + ku$ and

dropping the odd terms in k_μ upstairs, the resultant integral reduces to

$$
\begin{aligned}
\Pi_{\mu\nu} = & [\mu^2 \partial_m^2]_{m=0}]\mu^{(4-n)} \int \frac{d^n k}{(2\pi)^n} \int_0^1 du \\
& \frac{[(8p_m u p_\nu - 8p^2 \delta_{\mu\nu})(2u - u^2) + (4p^2(2u - u^2) - 2M^2 + 2k^2)\delta_{\mu\nu}]}{[m^2 + p^2 u(1 - u) + k^2]^2}
\end{aligned} \tag{3.1}
$$

The result of DR may be expressed by the formula [14]:

$$
\mu^{(4-n)} \int d^n k \frac{[k^2; 1]}{[k^2 + A]^2} = \frac{\pi^{n/2} \Gamma(2 - n/2)}{(A/\mu^2)^{(2-n/2)}} [\frac{n}{2 - n}; 1] \tag{3.2}
$$

Subtracting in the usual manner the residue-cum-pole parts corresponding to $n = 4$, it is easily seen that gauge invariance is satisfied except for the portion $2(p^2 - M^2)\delta_{\mu\nu}$ in Eq. (3.1), which vanishes for $p^2 = M^2$ only! We are unable to offer an interpretation of it, but we see little alternative to 'regularizing' it before proceeding further. The final result after taking out $(p_\mu p_\nu - p^2 \delta_{\mu\nu})$ is

$$
\Pi(p^2) = [\mu^2 \partial_m^2] \int_0^1 du 8(1/2 + u(1 - u)) \frac{1}{2\pi^2} [\gamma + ln(A/\mu^2)]; \quad A = m^2 + p^2 u(1 - u) \tag{3.3}
$$

where we followed the precaution advocated in ref. [14], viz., dropping terms odd in $(2u - 1)$, leading to the replacement $u(2 - u) \to 1/2 + u(1 - u)$ in the numerator of Eq. (3.3).

3.1   Gluon Condensate at $T = 0$

The gluon condensate may be obtained by substituting Eq. (3.3) in Eq. (2.7), where the second derivative w.r.t. m (multipled by μ^2), gives

$$
< G^a G^b > = [6C]\delta^{ab} \int \frac{-id^4 p}{(2\pi)^4} \int_0^1 du \frac{(1/2 + u(1 - u))}{\pi^2 [m^2 + p^2 u(1 - u)]} \tag{3.4}
$$

Once again the p-integration is done a la DR [14], Eq. (3.2) and the result of p-integration is

$$
< G^a G^b > = [6C]\delta^{ab} \mu^2 \int_0^1 \frac{du m^2}{u^2 (1 - u)^2 \pi^2} (1/2 + u(1 - u))[\gamma - 1 - ln(u\mu^2/m^2)] \tag{3.5}
$$

The u-integration is now routine, and in the limit $m = 0$ only the $(1/2)$ part survives, and as a check on the consistency of the calculation, the $ln(\mu^2/m^2)$ cancel out exactly. The final result using Eq. (2.8) is

$$
< GG >_0 = \frac{[36\mu^4 \alpha_s(\mu^2)]}{\pi^3} [2 - \gamma] \tag{3.6}
$$

where the common scale parameter μ^2 has been employed throughout. If one uses the universal Regge slope value of $\mu^2 \approx 1 GeV^2$, then $< GG >_0 = 0.586 GeV^4$, slightly on the high side of the QCD-SR value [3], without any fine-tuning.

3.2 Temperature Dependence of Gluon Condensate

So far our formalism has been fully Lorentz and gauge invariant. To incorporate the temperature dependence of $< GG >$, a convenient starting point is the quantity $\Pi(p^2)$, Eq. (3.3), in which the time-like component p_0 is analytically continued, a la Matsubara [15], to the imaginary axis, with discrete eigenvalues $ip_0 = 2\pi Tn$ (for bosons). The p-dependence of $\Pi(p^2)$ may be recognized through the denominator $(m^2 + up^2)^{-1}$ in the integral for $< GG >$ in Eq. (3.4) where the substitution $p^2 \to \hat{p}^2 + (2\pi Tn)^2$ will give a 'discreteness' to the time-like momentum. To evaluate $< GG >$ via Eq. (3.4), we now convert it into a contour integral over p_0, a la Kislinger-Morley [16]. To that end we reproduce their Eq. (2), except for $K^0 \to p_0$:

$$2i\pi T \sum f(\nu_n) = \int_{-i\infty+\epsilon}^{+i\infty+\epsilon} \frac{dp_0 f(p_0)}{\exp(p_0/T) - 1}$$
$$+ \int_{-i\infty-\epsilon}^{+i\infty-\epsilon} \frac{dp_0 f(p_0)}{\exp(-p_0/T) - 1} + \int dp_0 f(p_0) \tag{3.7}$$

where the last term (independent of T) is precisely identifiable with our Eq. (3.4) *after* integration over $d^3\hat{p}$. On the other hand, the first two terms which are temperature dependent, are most easily evaluated by their method of 'contour closing' [16] in the variable p_0, which amounts to the replacement of p_0 by its 'pole' values $\pm\omega_u = \pm\sqrt{(m^2/u(1-u) + \hat{p}^2)}$ in the two Bose-Einstein functions respectively, giving exactly equal contributions. [This result agrees with the one found in ref. [16] for the one-loop temperature-dependent correction to $mass^2$ in the ϕ^4 theory, since the analytic structures of both are similar]. As to the u-integration, the pieces $1/2$ and $u(1-u)$ have rather complementary roles: The former had given a finite non-zero limit Eq. (3.6) for $T = 0$, only due to DR [14], but now in the absence of (a second stage) DR for $T > 0$, this term gives a divergence for $m = 0$, and there is little alternative to its "regularization". On the other hand the $u(1-u)$ term which had vanished for the $T = 0$ case, now gives a perfectly finite value for $T > 0$ in the $m = 0$ limit. The remaining integrations are all convergent and elementary on setting $m = 0$ at the outset, so that the T-dependent part of $< GG >$, via Eq. (3.4) becomes:

$$< GG >_T = [4C]\frac{\mu^2 T^2}{\pi^2} = - < GG >_0 \frac{4\pi^2 T^2}{\mu^2(6 - 3\gamma)} \tag{3.8}$$

4 The Quark Mass Function and Condensate

We now come to our next item, the quark condensate, which however requires a prior determination of the mass function as the crucial ingredient. Thus must be done dynamically via the(non-perturbative) SDE for which we follow the treatment of ref. [17], now adapted to the confining interaction of Eq. (1.1):

$$m(p) = 3g_s^2 f_1.f_2[\mu^2 \partial_m^2] \int \frac{-id^4 k}{(2\pi)^4} \frac{m(p-k)}{(m^2 + k^2)[m^2(p-k) + (p-k)^2]} \tag{4.1}$$

where $g_s^2 = 4\pi\alpha_s$, $f_1.f_2 = -4/3$ is the color Casimir; the Landau gauge has been employed [19], and $m = 0$ after differentiation. Our defence of the Landau gauge is essentially one of practical expediency, since this gauge usually offers the safest and quickest route to

a gauge invariant result, even without a detailed gauge check, for there has been no conscious violation of this requirement at any stage in the input assumptions. For the solution of this equation, unfortunately no exact analytic solution is available in this case (unlike the previous case of harmonic confinement [17]), so we have developed an iterative analytical procedure as follows. As a first step, we replace the mass function inside the integral by $m(p)$. Then the DR method [14] of Sect.3 may be used almost verbatim to evaluate the integral on the RHS of Eq. (4.1), by subtracting the pole contribution and carrying out the indicated differentiation w.r.t. m. Eq. (4.1) now reads :

$$m(p) = -3f_1.f_2 m(p)\frac{\pi^2 g_s^2}{(2\pi)^4}\frac{\mu^2}{m^2(p)/2 + p^2/4}$$

which gives a first iterative solution (valid for momenta $p^2 \le \mu^2$):

$$m^2(p) = m_0^2 - \frac{1}{2}p^2; \quad m_0^2 \equiv m_q^2 = \frac{\alpha_s \mu^2}{\pi} \tag{4.2}$$

after substituting for $f_1.f_2 = -4/3$ and using $g_s^2 = 4\pi\alpha_s$. Then the first iteration for the self-energy operator is

$$\Sigma_1(p) = \frac{-ig_s^2 f.f_2}{(2\pi)^4}\int d^4k \frac{\gamma_\mu(m_1 - i\gamma.(p - k))\gamma_\mu}{(m^2 + k^2)[(m_1^2 + (p - k)^2)/2]} \tag{4.3}$$

where the second factor in the denominator on the RHS is the result of substitution of the first iteration Eq. (4.2) for the mass function in $m^2(p-k)+(p-k)^2$, giving $(m_1^2+(p-k)^2)/2$, and $m_1^2 = 2m_0^2$ represents the first iteration to m_0^2, Eq. (4.2), which also goes into the numerator of Eq. (4.3). Now proceed exactly as in Sect. 3: introduce the Feynman variable u, symmetrize wrt $u \leftrightarrow 1 - u$, integrate via DR [14], and differentiate twice wrt m. Then up to $O(p^2)$, the u integration gives:

$$\Sigma_1(p) = \frac{2m_0^2}{3}\frac{4m_1 + i\gamma.p}{(m_2^2 + p^2)/4}; \quad m_2^2 = 2m_1^2 = 4m_0^2 \tag{4.4}$$

To account for the factor $1/4$ in the denominator on right, a factor 2 each is absorbed in the two flanking $S_{F1}(p)$ functions which are needed to define the quark condensate, in this order of iteration, which is given by

$$< q\bar{q} >_1 = N_c Tr \int d^4p S_{F1}(p)\Sigma_1(p)S_{F1}(p); \quad S_{F1} = (m_1 + i\gamma.p)^{-1} \tag{4.5}$$

where $\Sigma_1(p)$ of Eq. (4.4) must now be read without the said factor 4. This structure is quite general and reproduces itself at every stage of iteration, which however must stop when the iterated mass m_n gets near μ, due to the approximate nature of Eq. (4.2). Eq. (4.5) now simplifies to:

$$< q\bar{q} >_1 = \frac{m_q^2}{4\pi^2}\int \frac{-id^4p 2m_1}{(2\pi)^4}\frac{m_2^2 - p^2}{(m_1^2 + p^2)^2(m_2^2 + p^2)} \tag{4.6}$$

The recursive law is now clear: In the next step, $< q\bar{q} >_2$ will be given by replacing m_1, m_2 in Eq. (4.6) with m_2, m_3 respectively, where $m_3^2 = 2m_2^2$; see Eq. (4.4). And so on till the 'limit' μ is reached. Evaluation of (4.6) by the DR method [14] now yields the result:

$$< q\bar{q} >_1 = \frac{m_q^2 m_1}{2\pi^2}\int[4/3 + \gamma - ln(\mu^2/m_1^2)] \tag{4.7}$$

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from which the successive iterations can be directly written down (including the 'zero' order) since the 'mass' increases by $\sqrt{2}$ at each step. The numerical values (using the universal $\mu = 1GeV$ as before) for the 0,1,2 iterations are $(132MeV)^3$, $(204MeV)^3$ and $(270MeV)^3$ respectively, beyond which our simple formula (4.2) breaks down.

4.1 Temperature Corrections to Quark Condensate

To calculate the temperature dependence of the quark condensate, a convenient starting point is Eq. (4.6), where the formalism of ref [16] applies a la Eq. (4.7), with the function $f(p_0)$ identified as

$$\frac{m_2^2 - p^2}{(m_1^2 + p^2)^2(m_2^2 + p^2)}; \quad p^2 = \hat{p}^2 - p_0^2$$

As in the gluon case the last term of (3.7) will give the $T = 0$ condensate (4.6), while the temperature dependent integrals will give equal contributions, with the Bose-Einstein functions replaced by the Fermi-Dirac functions. The contour closing now gives rise to the residues at the poles $p_0 = \omega_1$ (double) and $p_0 = \omega_2$ (single) respectively, where $\omega_n^2 = m_n^2 + \hat{p}^2$. Substituting the residues and simplifying yields the net 3D integral:

$$< q\bar{q} >_T = \frac{10m_q^2}{4\pi^2 m_2} \int \frac{d^3 p}{(2\pi)^3}[g(\omega_2/T) - g(\omega_1/T)] \tag{4.8}$$

where

$$g(y) = \frac{1}{y}\frac{1}{e^y + 1} \tag{4.9}$$

To evaluate (4.8), the substitutions $\hat{p} = Tx$ and $\omega_n = T\sqrt{x^2 + \lambda_n^2}$, where $m_n = T\lambda_n$, lead to the result:

$$< q\bar{q} >_T = \frac{10m_q^2 T^2}{\pi m_2}[F(\lambda_2) - F(\lambda_1)] \tag{4.10}$$

where

$$F(\lambda) = \int_0^\infty \frac{x^2 dx}{\sqrt{\lambda^2 + x^2}}[1 + \exp\sqrt{\lambda^2 + x^2}]^{-1} \tag{4.11}$$

This integral can be rapidly evaluated by the substitution $x = \lambda \cosh(\theta/\lambda)$, and noting that terms of $O(\theta^2)$ (gaussian) give the main contributions. The final result for $F(\lambda)$ is

$$F(\lambda) \approx \sqrt{\pi\lambda/2}e^{-\lambda}[1 - \frac{\exp(-\lambda)}{\sqrt{8}}] \tag{4.12}$$

Substitution of numerical values shows that unlike the gluon case, the decrease with temperature is rather slow, being less than half percent at $150MeV$, despite the T^2 dependence of the correction, as in chiral perturbation theory [6].

5 Calculation of the Pionic Constant

As a third and final test of this formalism, the evaluation of the pionic constant requires only an extra ingredient which stems from the Ward identity for chiral symmetry breaking [12], viz., in the 'chiral limit' (i.e., when the pion mass vanishes), the self-energy operator $\Sigma(p)$ and the pion-quark vertex function $\Gamma p_1, p_2$ are identical [20-22]. Although

originally discovered for the contact interaction [20], its validity was found for more general interactions [21] which allowed an alternative derivation of the mass function by the Bethe-Salpeter route for the pion-quark vertex function [17]. In a more elegant fashion, the content of this identity was expressed by Pagels-Stokar [22] as follows:

$$2f_\pi \Gamma(p_1, p_2) = \Sigma(p_1) + \Sigma(p_2); \quad f_\pi = 93(MeV) \tag{5.1}$$

where for each p_i, $\Sigma(p)$ is given by (4.3) and simplifies to (4.4) after k-integration. The pionic constant is formally defined (with $P = p_1 + p_2$) [17] by

$$f_\pi P_\mu = \int \frac{-iN_c d^4 q}{(2\pi)^4} Tr[\gamma_5 \Gamma(p_1, p_2) S_F(p_1) i\gamma_\mu \gamma_5 S_F(-p_2)] \tag{5.2}$$

where $p_{1,2} = P/2 \pm q$, and $S_F(p_i)$ are given by eq.(4.5) in the same order of iteration as in Sect.4. The trace evaluation is routine, allowing for calcellation of P_μ from both sides. Further, in the $P^2 = 0$ limit, the integrand simplifies greatly to give (with $N_c = 3$):

$$f_\pi^2 = 8m_q^2 \int \frac{-id^4 q}{(2\pi)^4} \frac{3m_1^2 + (m_1^2 + q^2)}{[(m_2^2 + q^2)(m_1^2 + q^2)^2]} \tag{5.3}$$

where the various symbols are the same as in Sect 4. The first term on the RHS is convergent as it is, while the second one amenable to a DR treatment [14], exactly as in the two previous sections.. The final result, in the chiral limit $P^2 = 0$ is

$$f_\pi^2 = \frac{m_q^2}{2\pi^2}[1 - \gamma + ln(\mu^2/3m_q^2)] \tag{5.4}$$

where m_q is related to m_1 and m_0 by $m_1^2 = 2m_0^2 = 2m_q^2$, leading to

$$f_\pi^2 = 0.00929 GeV^2 = (96.4 MeV)^2 \tag{5.5}$$

in rather close agreement with the standard value $93MeV$, with no adjustable parameters. The temperature correction is again proportional to T^2, which seems to be a rather general result arising out of the first two terms in the contour integral representation, eq.(3.7), a la Kislinger-Morley [16] of the Matsubara formalism [15], but we omit the result for brevity.

6   Summary and Conclusion

In retrospect, we have proposed a simple form of non- perturbative QCD, obtained by two differentiations of the o.g.e. propagator which, in conventional wisdom (almost since the inception of QCD), amounts to a Lorentz-invariant generalization of a *linear* potential. [This is a rather qualitative statement, with not more than asymptotic validity, if one expects a *quantitative* behaviour like $1/k^4$ of the gluon propagator for finite k^2]. The proposal has been motivated mainly by a desire for an alternative approach for understanding non-perturbative QCD, since the predictions of some of the well-known 'standard' methods like QCD-SR [2-4] and chiral perturbation theory [5,6] indicate considerable variations on the temperature dependence of QCD condensates. While the fuller implications of this proposal(which is characterized by a process of differentiation w.r.t. a small mass parameter) are yet to emerge, we have offered some preliminary tests through the calculation of

two basic QCD condensates, plus the pionic constant f_π, together with their temperature dependence. For these simple applications the non-perturbative proposal merely amounts to the replacement of the o.g.e. propagator k^{-2} of perturbative QCD to $2\mu^2/k^4$, but more complex amplitudes would presumably need a more microscopic formulation in terms of the gluon fields themselves at the level of the QCD Lagrangian. Such a formulation requires each gluon field A_μ appearing in the Lagrangian to be replaced as follows:

$$A_\mu \to A_\mu[1 + \mu\partial_m] \tag{6.1}$$

with a suitable limiting process $m \to 0$ on the (small) gluon mass m after the requisite differentiations on the o.g.e. propagators in the resulting equations of motion and/or Feynman amplitudes have been performed. The second term is supposed to be an effective substitute for an *exact* QCD treatment, although at this stage it is not clear how much of the latter it accounts for, while avoiding 'double counting' with the first. The simplest scenario is one in which the derivative term is substituted for the o.g.e. term in a lowest order Feynman diagram for a specified process, somewhat akin to the Pagels-Stokar [22] 'dynamical perturbation theory'. Although the precise connection of this term to a $closed-form$ solution of the QCD Lagrangian is not yet in sight, the parameter μ (which sets the scale of the theory) has, by virtue of the structure of (6.1), the *potential* for a deeper understanding of non-perturbative QCD, on analogous lines to QCD-SR [2] or chiral perturbation theory [5]. One may also add, by way of precaution, that (at this stage) there is no tangible relation of this parameter with the QCD dimensional constant which must obey the constraints of renormalization group theory.

For purposes of the present applications, the gg interaction kernel is defined via fig.1, showing the relative contributions of 'direct' (D), 'exchange' (X) and 'contact' (C) terms arising from lowest order QCD Lagrangian to give a gauge-invariant package, except for the replacement of the perturbative o.g.e. propagator k^{-2} by $2\mu^2 k^{-4}$, corresponding to a non-perturbative (linear) form. The relation of the condensate to this kernel is shown in fig.2 via looping up of the gluon lines in two successive stages. This lowest order treatment in the gluon case is justified by the absence of a Schwinger mechanism [13] since, as explained in the introduction, our gluon does not couple to an axial current [12]. The ∂_m^2 method lends itself readily to the DR method [14], leading to the rather simple formulae (3.6-7), where the identification of the scale parameter with the universal Regge slope seems to provide a rather welcome accord with QCD-SR [3], although we are not inclined at this stage no to attach much significance to this coincidence. In a similar way, the corresponding result on the $q\bar{q}$ condensate, has necessitated a prior derivation of the mass function $m(p)$ via the solution of the SDE. This treatment has been admittedly approximate, but we have found a simple recursive procedure which converges rapidly. Here again the same universal constant μ has been employed, and the QCD-SR value [2] has been almost reproduced.

The more interesting result is the temperature dependence of the corresponding condensates which shows corrections of order T^2 for both the gluon and quark condensates, with a somewhat bigger coefficient for the former than for the latter. In this context, as already noted at the end of Section 5, the order T^2 effect is a rather general feature of the Kislinger-Morley [16] contour integral representation, eq.(3.7), of the Matsubara formalism [15] wherein, after contour closing, the temperature dependence that arises from first two terms has proportionality to a T^2 behaviour in terms of the 3D integral over the momenta. Now for the quark condensate the $O(T^2)$ correction, eqs.(4.8-10), agrees

with the prediction of chiral perturbation theory (χPT)[6], albeit with a slower variation, but the result for the gluon condensate is up against a whole spectrum of predictions! Even QCD-SR seems to predict both T^2 [23] and T^4 [3-4] variations, of which only one [23] agrees with our prediction. On the other hand, χPT seems to predict [24] an even stronger dependence on ('flat' rise with) T, viz., a $O(T^8/\pi^4)$ correction. In the absence of a 'standard' criterion, it is difficult to decide on which of these alternatives corresponds to the truth, except that lattice calculations [25] predict an almost flat $< GG >$, with rise in temperature more in line with the χPT prediction. This suggests that the gluon condensate, which is related to the anomaly of the energy momentum tensor $T_{\mu\nu}$, is *not* the relevant order parameter for the description of chiral restoration with temperature [25] ! If the lattice criterion is taken seriously, it would seem to indicate a verdict for χPT over QCD-SR [3-4, 23] predictions, at least one of which [23] agrees with our result. Corrections to QCD-SR have been suggested [26], arising from non-diagonal condensates [26] at finite temparatures, which may play a role in resolving the discrepancy [25].

As to the effect of this discussion on our result showing a T^2 dependence, there is a case for inclusion of neglected effects, albeit within the contours of the Kislinger-Morley representation, eq.(3.7). The first correction that is yet to be investigated is the effect of the quark-gluon interaction on the pure gluonic self-couplings whose dominance over the quark-gluon couplings stems from the relative strengths of the color Casimir factors. A related question concerns the role of the pion which is almost a separate identity in the χPT scenario [5]. On the other hand, the logic of the present investigation does not permit the pion to play such a central role, (except as one of the many $q\bar{q}$ composites!), but this is *not* to deny its strong dynamical link with the theory, one of whose many manifestations is the *correct* prediction of f_π^2, vide eqs.(5.4-5), within the strict premises of the theory. Indeed the chiral character of the pion is a natural consequence of the (vector-like) quark-gluon interaction in the QCD Lagrangian, a simple check being the prediction of its very small mass as a *dynamical* consequence of such interaction. Although we have not carried out this check within the present formulation, analogous treatments with the same QCD premises, albeit with harmonic confining interactions [10, 27], automatically predict a very small mass [10, 27] for this unique pseudoscalar composite, by virtue of the chiral character of the (vector-like) interaction. Finally, the status of the quantity μ at this stage is that of an effective scale parameter analogous to f_π of χPT, and *not* that of the QCD scale parameter a la renormalization group theory! A proper study of this quantity can only be done in the context of the structure of eq.(6.1) at the level of the gluon fields themselves. These and other related applications like the pion form factor (for further calibration) and three-gluon condensates are currently being studied, before extending the ideas to the full hadronic sector with its wider ramifications.

7   Acknowledgement

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