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QCD-MOTIVATED BSE-SDE FRAMEWORK FOR QUARK-DYNAMICS UNDER MARKOV-YUKAWA TRANSVERSALITY —A UNIFIED VIEW OF $Q\bar{Q}$ AND QQQ SYSTEMS

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This article aims at an integrated formulation of BSE's for 2- and 3-quark hadrons under the Markow-Yukawa Transversality Principle (MYTP) which provides a deep interconnection between the 3D and 4D BSE forms, and hence offers a unified treatment of 3D spectroscopy with 4D quark-loop integrals for hadronic transitions. For the actual dynamics, an NJL-type realization of *DBXS* is achieved via the interplay of Bethe-Salpeter (BSE) and Schwinger-Dyson (SDE) equations, which are simultanelously derivable from a chiral Lagrangian with a gluonic (Vector-exchange) 4-fermion interaction of 'current' *uds* quarks, specifically addressing the non-perturbative regime. A prior critique of the literature on various aspects of the non-perturbative QCD problem, on the basis of some standard criteria, helps converge on a BSE-SDE framework with a 3D-4D interconnection based on MYTP. This framework is then employed for a systematic self-contained presentation of 2- and 3-quark dynamics on the lines of MYTP-governed *DBXS*, with enough calculational details illustrating the techniques involved. Specific topics include: 3D-4D interconnection of $q\bar{q}$ and qqq wave functions by Green's Function methods; pion form factor; 3-hadron form factors with unequal mass loops; $SU(2)$ mass splitting; Vacuum condensates (direct and induced); Complex H.O. techniques and $SO(2,1)$
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1 Introduction: QCD-Type Confinement Models

One of the biggest challenges in physics to-day is a viable theory of strong interactions for which QCD is the leading candidate. Unfortunately, despite many of its extremely attractive features, this theory is not yet available in a sufficiently tractable form so as to appeal instantly to all its practitioners in as universal a manner as, e.g. in QED. The bone of contention in this regard is the non-perturbative sector of QCD which shows up as the phenomenon of 'confinement' at low and moderate energies: As yet there is no visible evidence of a sort of minimum consensus on a common dynamical framework to incorporate this physical effect in QCD applications in the strong interaction sector in a sufficiently convincing yet doable manner. As a result, there exist a multiplicity of approaches which, while incorporating the QCD ideas in varying degrees of sophistication, nevertheless often need to resort to additional parametric assumptions to calculate various low energy hadronic properties. Some of the principal approaches are:

Bag models[1]; QCD-sum rules[2]; later adaptations[3] of the Nambu-Jona-lasino model[4]; QCD bosonization approaches[5]; Instanton methods[6]; Vacuum self-dual gluon fields[7]; Quark confinement models[8]; Schwinger-Dyson and Bethe-Salpeter models[9]; adaptation of BSE-SDE to a 3D-4D hybrid form[10-11] in a 'two-tier' fashion (to incorporate the spectroscopy sector). To this list one should also add QCD-motivated 'quarkonia' models which is one of the oldest types in existence, and those state of the art may be found in a fairly recent collection[12]. Last not least, one must pay homage to "Lattice QCD" which addresses confinement at a more fundamental level, and has grown into a self-contained field of study by itself. However its philosophy and methodology have so little in common with the less ambitious approaches listed above[1-12], that it does not fall within the scope of the present study.

1.1 A Short Critique of Models[1-12]

If the pretence of implementing confinement through an exact solution of the QCD equations of motion is given up in favour of an "effective confinement' programme, the central issue boils down to the extent to which the same can be formulated in a manner which is both physically convinving as well as mathematically tractable enough to warrant wide-ranging applications, all the way from the low-energy spectroscopy[12] to deep inelastic processes amenable to perturbative QCD[2]. Such a philosophy is reminiscent of Bethe's "Second Principle Theory" for effective nucleon-nucleon interactions, now reborn at the level of quark-quark interactions, with confinement addressed in a semi-empirical manner which incorporates the main features of QCD structure. Unlike Lattice-QCD, such programmes are not meant to address confinement directly, but rather to take its role more or less for granted in anticipation of future developments. It is from this angle that most of the approaches listed above may be viewed, from putting in the QCD feature by hand[1,8-12] to a conscious effort to 'derive' its content more explicitly[5-7]. Of the last group, the model that comes closest to tackling confinement is perhaps ref. [7], but its methodology has understandably more limitations for wider applicational purposes. A complementary role is that of ref. [5] which is characterized by a 'chiral perturbation' approach to the mechanism of formation of hadronic states in QCD, and in the process gives rise to an effective chiral Lagrangian for low energy hadron physics. However the *perturbative* expansion in the momenta, deemed small in the low momentum limit, robs such a Lagrangian of a vital property: its capacity to predict the *bound* (confined) states of hadrons in the low momentum regime, due to the lack of a 'closed form' approach. (A closed form approach is best exhibited by some sort of form factors characterized by a confinement scale, a feature that gets lost in any expansion in the momenta).

A 'two-tier' 3D-4D BSE approach like ref. [10,11] is meant for a 'more conscious' incorporation of the spectroscopy sector, i.e., an explicit recognition of the fact, often ignored in the more usual formulations of BSE-cum-SDE methods[9], that the observed hadronic spectra are O(3)-like ref. [13], while a literal BSE formulation in euclidean form, with a standard 4D support to the kernel leads to O(4)-like spectra[14]. In this respect, the two-tier strategy[11] invokes the Markov-Yukawa Transversality Principle (MYTP)[15] wherein the quark-quark interaction is in a hyperplane which is *transverse* to the 4-momentum P_μ of the composite hadron, so that the modified BSE has a (covariant) 3D support to its kernel. This feature in turn leads to an exact 3D reduction of the BSE from its 4D form, and an equally exact *reconstruction* of the 4D wave function in terms of 3D ingredients[16], thus implying an *exact interconnection* between the 3D and 4D BSE forms[16]. (A parallel formulation of MYTP[15] by the Pervushin group[17] gave rise to the 3D reduction from the 4D BSE form, but the *inverse* connection (from 3D to 4D) was missing in their paper[17]. The 3D BSE form makes contact with the observed O(3)-like spectra[13], while the reconstructed 4D BSE form provides a natural language for evaluating transition amplitudes via quark loops[16,11].

A 3D BSE form has its own logical basis which receives support from several independent angles in view of its crucial role in the understanding of physical processes in general and the theme of the present article in particular (see below). We list three supporting themes which have been developed over the decades from entirely different premises, all converging to a basically 3D picture for the effective $q\,q/q\,\bar{q}$ interaction:

i) It gives a physical meaning to the interaction of the quark constituents in their respective mass shells, consistently with the tenets of local field theory[18]; ii) It arises from the concept of instantaneous interaction[17,16] among the quark constituents in accordance with the Markov-Yukawa picture[15] of transversality to the composite 4-momentum P_μ; iii) It is the only structure of the BSE kernel which makes this equation compatible with a pair of Dirac equations for two particles under their mutual interaction[19].

1.2 Bethe's 'Second Principle' Criteria for Model Selection

Next we consider certain guiding principles (criteria) which form the basis of this study, before identifying the specific model/models for a more detailed and self-contained exposure. A clue is provided by the observation that most of these approaches[1-12], irrespective of their individual theoretical premises, have a common characteristic: Applicability to hadronic processes viewed as quark composites, limited only by their individual predictive powers, while a deeper understanding of the underlying models themselves is best left to future investigations. Perhaps the painfully slow progress of Lattice QCD results gives an inkling of this scenario: The formidable dimensions of the quark-gluon strong interaction physics leaves little alternative to the respective practitioners of QCD but to settle for a less ambitious approach to the problem. Thus the different models[2-12] are best regarded as alternative strategies, each with its own methodology and parametric limitations, aimed at selected sectors of hadron physics that are suited to their structural budgets before the 'final' theory unfolds itself, leaving their successes or otherwise to be judged in the interim by the depth and range of their respective predictions *vis-a-vis* the data. This is once again Bethe's 'second principle' philosophy in retrospect, which would presumably continue to operate perhaps as long as QCD remains a partially solved theory. Within this restricted philosophy, some obvious criteria for theme selection from among the available candidates[1-12] could be the folowing partial shopping list (not mutually exclusive):

A) Maximum number of mutually compatible as well as time-tested ideas that can be extracted from out of ref. [4-12] in the sense of an "HCF"; B) Close proximity to QCD as the ideal theory which stands on the three pillars of Lorentz-, Gauge- and Chiral- invariance; C) Formal capacity to address several sectors of physics simultaneously, all the way from hadron spectroscopy to quark loop integrals for different types of transition amplitudes within a common dynamical framework; D) Sufficient flexibility of the core dynamical framework to permit smooth on-line incorporation of mutually compatible ideas, like Markov-Yukawa transversality MYTP[15], and Dynamical Breaking of Chiral Symmetry ($DB\chi S$)[4], and other similar principles if need be, without causing any structural (or parametric) damage to the basic framework itself; E) Natural capacity of the *conceptual premises* to include both 2- and 3-quark hadrons within a common dynamical framework, to give concrete shape to the (widely accepted) principle of meson-baryon duality; F) A built-in microcausality in the dynamical framework which takes in its stride sensitive items like the structure of the vacuum in strong interaction physics, without the need for fresh ansatze/parametrizations.

Although these criteria are neither exhaustive nor mutually exclusive, their collective effect is nevertheless focussed enough to eliminate many prospective candidates in favour of a chosen few that would survive the tests (A)-(F) for a reasonably self-contained account of strong interaction physics within the tenets of the 'Second Principle' philosophy. Thus the too simplistic premises of Quarkonia models[11] which played a crucial role in the early stages of QCD-motivated investigations, would not stand the tests of (B), (C) and (D). Bag models [1] which had also played a similar role in the early phases, would not qualify under (B), (C) and (F). QCD sum rules[2] represent perhaps one of the most successful applications of perturbative QCD by relating the high energy quark-gluon sector to the low energy hadron sector through the principle of an FESR-like duality discovered in the Sixties[20]. Even to-day it is extensively used for many hadronic investigations. Yet it fails on count (C) mainly because of its failure to satisfy (F): The lack of microcausality in this model can be traced to the 'matching condition' between the quark-level and hadron-level amplitudes whose solution is far from unique. Indeed, while the borelisation technique suffices for the prediction of ground state hadron masses, that very mechanism also causes it to lose information on the spectra of the *excited* states, thus reducing its predictability on this vital (low energy) front.

As for models[5-8], the Quark Confinement model[8] lacks enough microcausality – condition (F) – which shows up through a relatively poor satisfaction of (C), since the spectroscopy sector is badly neglected. QCD bosonization methods[5] have made very impressive strides in respect of *transition amplitudes* through the powerful technique of construction of effective Lagrangians in terms of the hadronic fields, which make them ideally suited for 'tree-diagrams'. However such structures hide from view the *composite* character of the hadrons in the 'soft' QCD regime, which is best exhibited in 'closed' form via quark-hadron form factors – the vehicle for sensitivity to various non-perturbative features of the theory. For the baryon dynamics too, the inadequacy of the formalism[5] to depict correctly the 3-quark form factors shows up through its excessive dependence on the quark-diquark description with a *rigid* diquark structure which results in an inevitable loss of information on the true 3-quark content of the baryon. Finally, while ref. [6-7] satisfy the conditions (B), (C), (F) on separate counts, there is not enough published evidence of support from the other quarters (A), (D), (E). Hence while their comparison with others is useful for a comparative discussion of different models, their claims to a primary 'theme' status fall rather short of a good starting point. This leaves the BSE-SDE framework[9-11] for a more detailed scrutiny to follow.

1.3 BSE-SDE Framework: DBχS and 'Soft'-QCD

The last group[9-11] is characterized by an interplay of BSE and SDE, both derivable from a suitably chosen 4-fermion Lagrangian as input. It has a very wide canvas and is fully attuned to Bethe's Second Principle Theory (*BSPT* for short). Its general framework equips it with arms to meet most of the conditions (A)-(F), ranging from flexibility to wide-ranging predictivity, thus lending it some credibility within the broad premises of 'BSPT'. In particular, its natural roots in field theory endow it with standard features like dynamical breaking of chiral symmetry (*DBχS*) via the non-trivial solution of the SDE, thus giving it the powers to subsume the contents of the NJL-model[4]. Indeed, soon after the discovery of the NJL model[4], a field-theoretic understanding of its underlying idea was achieved in the form of a generalized *DBχS* in the QED domain via the non-trivial solution of the SDE[21]. And after the advent of QCD[22], the same feature showed up through the solution of the BSE for $q\bar{q}$ interaction via one-gluon-exchange[3]. This is a typical non-perturbative effect, although it does not cover all its aspects.

Subsequently the concept of *DBχS* was generalized to show that this feature is shared by any extended *vector-type* 4-fermion coupling[23-24] which preserves the chiral symmetry of the Lagrangian but the same gets *broken dynamically* through the non-trivial solution of the SDE, derivable from such a Lagrangian. Indeed the sheer generality of a Lagrangian-based BSE-cum-SDE framework, by virtue of its firm roots in field theory, gives it a strong mandate, in terms of both predictivity and flexibility, to accommodate additional principles like Markov-Yukawa Transversality[15] (MYTP) while staying within a basically Lagrangian framework, so that criteria (A)-(F) are stil satisfied.

In particular, a basic proximity to QCD is ensured through a vector-type interaction (condition (B))[10,23], which while maintaining the correct o.g.e. structure in the perturbative region, may be fine-tuned to give any desired structure in the infrared domain as well. The latter part is admittedly empirical, but captures a good deal of physics in the non-perturbative domain while retaining a broad QCD orientation, and hence does not rule out a deeper understanding of the infrared part of the gluon propagator within the same framework. More importantly, the non-trivial solution of the SDE corresponding to this generalized gluon propagator[11] gives rise to a dynamical mass function $m(p)$[11] as a result of *DBχS*, even while the input Lagrangian has chiral invariance due to the vector-type 4-fermion interaction[23,24] between almost massless *u-d* quarks. These considerations further strengthen the case of a Lagrangian-based BSE-SDE framework for a theme choice.

1.4 3D-4D BSE: From Spectra To Loop Integrals

Now the canvas of a (second-principle) BSE-SDE framework is broad enough to accommodate a whole class of approaches, and facilitates further fine-tuning in response to the needs of major experimental findings such as the observed O(3)-like spectra[13], which essentially amounts to treating the time-like momenta separately from the space-like ones, as has long been known since the classic work of Feynman *et al.*[25]. In this regard, the MYTP constraint[15] seems to fit this bill, by imparting a 3D support to the pairwise BSE kernel[15-17], an ansatz which can be motivated from several different angles[16,18,19]. As to the 'soft' non-perturbative part of the gluonic propagator, it still remains empirical since orthodox QCD theory does not yet provide a closed form representation out of the infinite chain of equations that connect the successively higher order Green's functions in the standard fashion[26], thus necessitating parametric representations[27]. Parametrization is also compatible with MYPT[15] In ref. [11] wherein the key constants are attuned to the hadron spectra of both 2-[28] and 3-body[29] types, within a common framework. The 3D support ansatze of MYTP[15] in turn gives a characteristic 'two-tier'[16] structure to the entire BSE formalism, wherein the first stage (3D BSE) addresses the meson[28] and baryon spectra[29], while the reçonstructed 4D wave (vertex) functions[16] fit in naturally with the Feynman language of 4D quark loop diagrams for various types of transition amplitudes[11,30-33] in a unified fashion.

A BSE-SDE formulation[9] represents a 4D field-theoretic generalization of 'potential models'[12], and is thus equipped to deal with a wider network of processes .(e.g., high energy processes) not accessible to potential models[12]. In this way, the BSE-SDE approach occupies an intermediate position, sharing the off-shell feature with potential models[12]., as well as the high energy flavour of QCD-SR[2], but its dynamical spirit is much nearer to ref. [12] than to ref. [2]. Indeed the role of the 'potential'[12] is played by the generalized 4-fermion kernel[23] (which is a paraphrase for the non-perturbative gluon propagator[11]). The 4D feature of BSE-SDE gives this framwork a ready access to high energy amplitudes, as in QCD-SR[2] as well as in other models[5-8]., while its 'off-shell' feature gives it a natural access to hadronic spectra[13], in company with (potential oriented) quarkonia models[12]. (In contrast, certain models[2,5-8] do not have a basic infrastructure to address spectroscopy). Now withn this twin feature of off-shellness and Lorentz-covariance, the BSE-SDE framework formally overcomes the shortcomings of 'potential' models[23] in obtaining numerically 'correct' values for the various condensates which are employed as inputs in QCD-SR calculations[2]. This was indeed confirmed by a later derivation of similar results[30] in terms of a Lorentz-covariant formulation[11] of the BSE-SDE framework, which showed that vacuum condensates are calculable within a spectroscopy-rooted[28,29] framework.

1.5 Off-Shellness in BSE: Parametric Links with QCD-SR

The calculations in ref. [23, 30] raise the interesting question of the possibility of a basic connection among the input parameters of different models, although conceived within very different premises. Thus in QCD-SR[2], the 'free' parameters of the theory are the condensates themselves as input, while in the BSE-cum-SDE methods[9,11], the corresponding parameters are contained in the input structure of the infrared part of the gluon propagator[11]. Now since the condensate parameters of QCD-SR[2] are explicitly calculable in BSE-SDE models by quark loop techniques[23,30], using the gluonic parameters[11,30] this result at least settles the issue of a one-way connection: from BSE[11] to QCD-SR[2].

To pursue this question a bit further, let us compare the features of potential models (of which BSE is a 4D generalization) with those of QCD-SR: Potential models are characterized by 'off-shell' features, whose parameters (corresponding to given 'potential' forms) are primarily attuned to low energy spectroscopy, so that their predictions tend to work upwards on the energy scale, starting from the low energy end. QCD sum rules on the other hand are attuned to the perturbative QCD regime, so that their predictions tend to work downwards on the energy scale, starting from the high energy end. The 'softness' aspects of QCD-SR are typically simulated via the Wilson OPE expansion in inverse powers of 4-momentum Q^2 where the 'twist' terms of successively higher dimensions are symbolized by the corresponding 'vacuum condensates' which are thus the free parameters of the theory. Therefore prima facie it appears that the two methods are largely complementary to each other. The former,by virtue of its low energy/off –shell emphasis, is particularly successful on the spectroscopic front, but its techniques do not find easy access to transition amplitudes due to inadequate treatment of the high energy front (lack of covariance). The latter (QCD-SR) on the other hand, is ideally suited to the high energy regime, but does not find ready access to areas involving soft QCD physics, especially the spectroscopic regime. This is at least partly attributable to the methodology of QCD-SR[2] which makes use of the quark-hadron duality' for 'matching' the respective amplitudes[2,20]. Because of the relatively 'macroscopic' nature of the 'matching' which is effected with a 'Borelization' technique[2], the predictions are reliable only for the hadronic *ground* states, but do not readily extend to the spectra of excited states.

Due to the complementary nature of the two descriptions, it is not generally easy to relate the parameters of one to those of the other. However, the off-shellness feature of potential models gives it access to information on the interaction of the quark pair with *vacuum* in the form of 'vacuum condensates'. That the vacuum condensates of QCD-SR[2] can be expressed in terms of potential[23] and BSE[30] models is a reflection of their crucial 'off-shellness' property. As to the converse question, there is no published evidence of a corresponding exercise in the *opposite* direction viz., a derivation of the parameters of the BSE kernel/gluon propagator in terms of the various vacuum condensates that characterize QCD-SR[2]. A possible reason may lie in the role of *microcausality* (condition (F)) which is well satisfied by potential models, but perhaps not by QCD-SR[2]. Thus it would appear that 'microcausality' which underlies the 'off-shellness' feature of the 'potential models' enhances their predictive powers *vis-a-vis* those which do not possess this crucial property.

Now the off-shell structures of all 'potential-oriented' models[9-12] have a fairly direct connection with the 'spectral' predictions, unlike other types of confinement models[2,5-8], which do not permit such predictions in an equally natural way. And for 3-quark states[29], the dichotomy seems to be even sharper, inasmuch as there is a strong tendency in the literature to simplify the 3-quark systems as quark-diquark systems[2,5-8], thus partly "freezing" some genuine 3-body d.o.f.'s and causing a loss of information on the spectra of L-excited states.

The off-shell characteristics of the BSE-SDE framework[9-12] are perhaps the most important single feature responsible for extending their predictive powers all the way from 3D spectra to 4D transition amplitudes (of diverse types) via 4D quark loop integrals, under one broad canvas. The key to this capacity lies in the vehicle of the BS wave (vertex) function which has at its command the entire 'off-shell' information noted above. Here it is important to stress that this wave function is a *genuine* solution of the BS dynamics[11], so that it leaves *no* scope for any *free* parametrization beyond what is already contained in the (input) gluon propagator. (Potential models[12] also have this capacity in principle, but their 3D structure does not allow full play to the 'loop' aspect).

1.6 Markov-Yukawa Transversality on the Null Plane

Covariant Instaneity Ansatz (CIA) on the BSE[16] is not the only form of invoking MYTP to achieve an exact interconnection between the 3D and 4D structures of BSE. As will be found later (see Sec. 4 for details), the CIA which makes use of the *local* c.m. frame of the $q\bar{q}$ composite, has a disadvantage: The 4D loop integrals are ill-defined due to the presence of time-like momentum components in the exponential/gaussian factors (associated with the vertex functions) caused by a 'Lorentz-mismatch' among the rest-frames of the participating hadrons. This is especially so for triangle loops and above, such as the pion form factor, while 2-quark loops[32] just escape this pathology. This problem is probably absent if the null-plane ansatz (NPA) is invoked, as found in an earlier study of 4D triangle loop integrals[33], except for possible problems of covariance[34]. The CIA approach[16] which makes use of the TP[15], was an attempt to rectify the Lorentz covariance defect, but the presence of time-like components in the gaussian factors inside triangle loop integrals[31] impeded further progress on CIA lines.

Is it possible to enjoy the best of both the worlds, i.e., ensure a formal covariance without having to encounter the time-like components in the gaussian wave functions inside the 4D loop integrals? Indeed the problem boils down to a covariant formulation of the null-plane approach. Now the null-plane approach (NPA) itself has a long history[35], and it is not in the scope of this article to dwell on this vast subject as such. Instead our concern is limited to the *covariance* aspects of NPA, a subject which is of relatively recent origin[18,36-38]. However in all these approaches[38], the primary concern has been with the NP-dynamics in 3D form only, as in the other familiar 3D BSE approaches[39] over the decades. On the other hand, the aspect of NPA which is of primary concern for this article, is on the possibility of invoking MYTP for achieving a 3D-4D BSE interconnection on the *covariant* Null Plane, on similar lines to Covariant Instantaneity (CIA) for the pairwise interaction[16]. Now it seems that a certain practical form of the null-plane formalism[33] had all along enjoyed *both* 3D-4D interconnection and a sort of 'pedagogical covariance' (albeit implictly)[40]. This basic feature can be given a formal shape by merely extending the Transversality Principle[15] from the covariant rest frame of the (hadron) composite[16], to a *covariantly defined* null-plane (NP)[41]. Because of its obvious relevance, the subject of 3D-4D interlinkage on the covariant Null-Plane[41] will be covered in Sec.4.2, with a parallel CIA treatment in Sec.4.1.

1.7 Scope of the Article: Outline of Contents

We now focus on a BSE-cum-SDE form of dynamics derivable from a chirally invariant Lagrangian with an effective gluon-exchange-like interaction (Pairwise), as the central theme of this study for a reasonably self-contained presentation, under the further constraint of Markov-Yukawa Transversality Principle (MYTP). The emphasis is on a *pedagogical* perspective on the problem of effective color confinement, converging on a vector exchange mediated Lagrangian whose chiral symmetry gets broken *dynamically*, after giving a bird's eye view of the main approaches to effective confinement[1-11]. Indeed the ($DB\chi S$) theme, although originating from the NJL-model[4] for contact pairwise interaction, admits a simple generalization to a (space-time extended) vector exchange $q\bar{q}/qq$ interaction which exhibit chiral symmetry at the input Lagrangian level, but get broken dynamically via the solution of the Schwinger-Dyson Equation (SDE)[23,27]. A more explicit QCD motivation must be achieved by hand, e.g., identification of the pairwise interaction with the entire gluon propagator (perturbative and non-peturbative[28a]) which in turn has several desirable consequences, such as the color effect which ensures that the strength of the qq force is *half* that of $q\bar{q}$, within a common parametrization.

The second item of emphasis concerns the remarkable facility of an *exact* interconnection between the 3D and 4D BSE forms[16], that is provided by MYTP, a facility that other 3D approaches to BSE[39], or (basically 3D) Null-Plane approaches[35,38], do *not* seem to possess. This property allows the exposition of the BSE-cum-SDE techniques in a very simple way, so as to provide the reader with a quick working knowledge of their applications to a wide class of problems which may be broadly classified in a *two-tier* form: A) Mass spectra; B) Quark-Loop diagrams. Such a division is natural since investigations of types (A) and (B) are mainly governed by the 3D and 4D aspects of the BSE respectively. Therefore after an introductory phase on the general BSE-SDE formulation, an early specialization to its MYTP-governed 3D-4D form (from Sec.4 onwards) will form the basis for this (application oriented) article.

A third item of emphasis is on the *second* stage of the 3D-4D BSE framework, viz., techniques of 4D quark-loop amplitudes, with a comparative study of CIA[16] vs CNPA[41] to bring out their relative (strong/weak) features.

This article has been built on the infrastructure of one with a similar theme[40] written about a decade ago; it incorporates major advances through the present decade on the 3D-4D BSE front[42], viz., the Covariant Instaneity Ansatz (CIA) and its more recent Null-Plane counterpart (CNPA)[41], both under the umbrella of MYTP[15]. The background of ref. [40] will be freely used, but the details on (3D) spectra on which CIA[28] and CNP[41] have similar predictions, will now be omitted, except for drawing attention to their structural similarities. Instead more attention will be paid to the structure of 4D quark loop integrals of selected types to bring out the applicational potential of this MYTP-governed formalism[42]. These types include (i) certain hadronic form factors built out of triangle loops; (ii) typical self-energy problems dealing with $SU(2)$-mass splittings among hadrons; (iii) vacuum condensates which are inputs in QCD-SR[2], but calculable in the 3D-4D BSE-SDE formalism[30].

While giving the details of this article, we repeat at the outset that, except for the contents of Sections 1-3, it is *not* intended as a conventional 'review' of the BSE-SDE framework such as ref. [9]. Nor are conventional 3D BSE approaches[39], or the conventional NPA formalisms[35-38] the subjects of our detailed description. Aspects of *contact* NJL-type 4- and 6-fermion couplings (often employed in the 'nuclear' field), are also not of interest here.

As to the actual details, the Table of contents, preceding the Introduction (Section 1) gives a fair cross section of the included items: Sect.2 gives a panoramic view of the NJL-Model[4] and its aftermath. Sect.3 gives a general derivation of BSE and SDE in an intelinked fashion, with a gluon-like (Vector-exchange) propagator whose mass function $m(p)$ stems via $DB\chi S$ from a spatially extended 4-fermion interaction in the input Lagrangian. With this general background of SDE-BSE as well as of $DB\chi S$, the rest (Sect. 4-11) deals with different facets of the 3D-4D BSE-SDE framework under the Markov-Yukawa Transversality Principle[15] at *two* distinct levels of operation, viz., CIA[16,17] which has been around for some time, and CNPA[41] which is formally a new proposal, although in effective (practical) use for quite some time[33,40].

Of the subsequent Sections, Sect.4 collects the background for interlinked 3D-4D BSE techniques for $q\bar{q}$ hadrons. For the fermionic BSE, we have preferred to stick to its 'Gordon-reduced' version[10a-b] *adapted* to the *off-shell* constituents[10a]. This is a conscious departure[10b] from the standard BSE-form[26] to make the BSE more tractable for wider applications, as in other BSE approaches[10c-d], and does not violate the 'Bethe Second Principle' spirit, since the input 4-fermion coupling is an effective description of the pairwise interaction).

Sect.(5-8) deal with some selected applications of triangle loops (form factors), two-loops (self-energy), and one-loop (vacuum condensates) techniques respectively. These include, among other things, a technique to include QED gauge insertions in arbitrary momentum-dependent vertex functions for the e.m. self energy and form factors. Whereever possible, a parallel treatment is provided for CIA and CNPA for a comparative view of the two distinct MYTP-governed BSE formalisms, but some technical problems with CIA[16] often lead to a preference for CNPA. Some calculational details on the form factor plus normalization are given in *Appendix A*.

Sect. 6 gives a general method for triangular quark-loop integrals applicable to a large class of transition amplitudes for 3-hadron coupling[31], to bring out a major simplifying feature of the resulting structure arising out of a 'cancellation' mechanism between the 4D quark propagators and the 3D *D*-functions in the hadron-quark vertices of the two-tier BS formalism[16]. This prevents free propagation of quarks by eliminating the Landau-Cutkowsky (overlapping) singularities[16,31].

Sects. 7,8 give results for self-energy diagrams[32] and of vacuum condensates[11,30], requiring two and one S_F-functions respectively. The self-energy calculations in Sect. 7 are illustrated with SU(2) mass splittings of the pseudoscalar mesons[32b] A general method to deal with QED gauge corrections to the e.m. mass differences is outlined in Appendix B. For the vacuum condensates[11,30], Sect. 8 offers a new gauge invariant technique for loop integrations, on the lines of Schwinger[43]. We reiterate that such predictions are intimately linked with spectroscopy via the infrared structure of the gluon propagator[11,30].

The third part Sects. 9-11 deals with the BSE formalism for a 3-quark baryon, with emphasis on the *qqq* structure taking into view that in most approaches, including other BSE models[9b], the dynamical treatment has often relied heavily on the quark-diquark approximation[5b,8b,9b,11], which amounts to a "freezing" of the 3-body degrees of freedom. It has also been recognized in the literature that with a 3-body BSE treatment to the baryon, there are some technical problems associated with the status of the spectator[44]. In the Two-tier BSE model this problem has been regularly addressed at various stages of its development[10,40,29]. The 3-quark dynamics is described in three Sections as follows:

Sect. 9: A panoramic view of the baryon dynamics as a general 3-body problem with full permutation symmetries[45] in all the relevant d.o.f.'s incorporated; a detailed correspondence with the quark-diquark model; Complex HO techniques for the *qqq* problem[46]; problems of 3D reduction and 4D reconstruction for *qqq* BSE[47]; and fermionic BSE with gluonic interactions in pairs[29].

Sect.10: Green's function techniques for 3D reduction of the BSE, and reconstruction of the 4D *qqq* wave function[47]: see *Table of Contents*.

Sect. 11: A summary of the relativistic fermionic *qqq* BSE with the same gluonic propagator as employed for the $q\bar{q}$ problem: the 3D reduction[29] of the *qqq* BSE is on closely parallel lines to the two-body case[28]. The derivation of an explicit mass formula is greatly facilitated by taking a complex HO basis[46]. However loop techniques for baryonic amplitudes are not included for explicit presentation.

2 NJL Model: Recent Developments (Nambu)

The precursor of the NJL-model[4] was the 'Nambu-Goldstone' picture of the pion as a zero mass particle arising from the chiral non-invariance of the vacuum[48]. This view of the pion received quantitative shape at the hands of Gell-Mann and Levy[49a] who started with a $SU(2) \times SU(2)$ symmetry of the Lagranjian (termed *SU(2)* σ-model) involving an I=1 pseudoscalar π and an I = 0 scalar σ field. Due to spontaneous symmetry breaking of the vacuum, the σ-field shifts to a minimum $<\sigma> = -f_\pi \neq 0$, while the pion field remains unshifted ($<\pi> = 0$) and stays at zero mass.

The Gell-Mann Levy σ-model set the stage for modern chiral theories, stimulated by an important paper due to Skyrme[49b], to describe pseudoscalar mesons and baryons through a solitonic picture wherein baryons are generated as bound states of weakly interacting mesons. These models were developed in the QCD context wherein, in the large N limit, QCD becomes equivalent to a non-linear meson theory. The underlying logic is that although the QCD Lagrangian has chiral symmetry for massless quarks, this symmetry is spontaneously broken, giving rise to massless pions, etc. These methods give rise to effective Lagrangian descriptions at the tree level[5], but will not concern us any further in this article.

The NJL-model[4] on the other hand, which is the very raison d'etre of this article, is characterized by chiral symmetry breaking in a *dynamical* fashion, and alows a formally *composite* structure of the pion, in company with other hadrons. This is a distinct advance over the elementary field picture of the pion[48,49], and facilitates a more natural understanding of many of its observational properties (form factor, L-excited states, etc.). A short summary of the NJL model follows.

2.1 Outline of NJL Model

The NJL Lagrangian[4] may be written in two different ways:

$$L_{NJL} = L_0 + L_i = (L_0 + L_s) + (L_i - L_s) \equiv L_0' + L_i'; \quad \ldots(1)$$

where $L_0 = -\bar{\psi}\gamma.\partial\psi$ and L_i (see below) are chirally invariant, but $L_s = -m\bar{\psi}\psi$ which stands for the observed fermion, is not, and represents the symmetry breaking effect. The interaction term L_i is given by

$$L_i = g_0[(\bar{\psi}\psi)^2 - (\bar{\psi}\gamma_5\psi)^2] = -g_0[(\bar{\psi}i\gamma_\mu\psi)^2 - (\bar{\psi}i\gamma_\mu\gamma_5\psi)^2]/2 \quad \ldots(2)$$

The rearrangement in eq. (1) is meant to diagonalize L_0', and treat L_i' as a perturbation; this implies a redefinition of the vacuum by introducing a complete set of 'quasi-particle' states which are eigen states of L_0'. The L_s is now determined from the requirement that L_i' shall not yield additional self-energy effects. This gives the standard Schwinger-Dyson Equation (SDE) for m in terms of the loop self energy

$$m = \sum\Big|_{(i\gamma.p+m=0)} = -8img_0\int(2\pi)^{-4}d^4p\frac{F(p,\Lambda)}{m^2+p^2-i\varepsilon} \quad \ldots(3)$$

where $F(p, \Lambda)$ is a cut-off factor. The trivial solution m=0 corresponds to the usual chiral invariant vacuum characteristic of perturbation theory. The non-trivial solution $m=m_{NJL}$ is found from

$$(2\pi)^4 = -8ig_0\int d^4p(m^2+p^2-i\varepsilon)^{-1}F(p,\Lambda)\ldots(4)$$

in terms of g_0 and Λ. It is also called the 'gap' equation, and is based on a shifted vacuum Ω_m which is chiral non-invariant. With a fixed Lorentz-invariant cut-off Λ in Euclidean space (and F=1), eq. (4) reduces to

$$2\pi^2/g_0 = \Lambda^2 - m^2\ln(\Lambda^2/m^2+1); \quad 0 < 2\pi^2 g_0^{-1}\Lambda^{-2} < 1 \quad \ldots(5)$$

The two vacuua Ω_0 (chiral invariant) and Ω_m (non-invariant) are fully orthogonal to each other , and correspond to two different worlds. Ω_m , with the lower energy, is the true ground state. The chirality operator defined as $\chi = \int\bar{\psi}\gamma_4\psi\, d^3x$ commutes with the original hamiltonian H_0 with vacuum Ω_0, but not with H_m with vacuum Ω_m. However χ has no matrix elements connecting the two worlds. Ω_0 and Ω_m, a sort of superselection rule. Now the following paradox arises: The χ-conservation in the Ω_0 basis implies the existence of a conserved current

$$j_{\mu 5} = i\bar{\psi}\gamma_\mu\gamma_5\psi;\ \partial_\mu j_{\mu 5} = 0 \quad \ldots(6)$$

On the other hand, for a massive Dirac particle in the Ω_m basis

$$\partial_\mu(\bar{\psi}\gamma_\mu\gamma_5\psi) = 2m\bar{\psi}\gamma_5\psi \neq 0 \quad \ldots(7)$$

To reconcile these two statements, the χ-current operator between *physical* states suffers radiative corrections w.r.t. the simple term $i\gamma_\mu\gamma_5$ so that, on grounds of Lorentz invariance

$$<p'|j_{\mu 5}|p> = F(k^2)\bar{u}(p')\left[i\gamma_\mu\gamma_5 + \frac{2m\gamma_5 k_\mu}{k^2}\right]u(p); \quad (k = p - p') \quad \ldots(8)$$

Thus the real fermion (quark) is *not* a point particle since its χ-current has an anomalous γ_5 term. This in turn implies a pole at k^2=0 for the γ_5 term, corresponding to a *zero mass* pseudoscalar, whose natural identification is the pion.

The pion which arises here as the lowest $q\bar{q}$ bound state, has clearly the nature of a *collective excitation*, thus also implying the existence of higher excitations in the same package, something which the elementary field model could not provide. Indeed[48–49] the BS amplitude Ψ for the bound state composite is

$$\Psi(x,y) = <0|T(\psi(x)\overline{\psi}(y))|B> \quad \ldots (9)$$

which is related to the vertex function Γ in momentum space as

$$\Psi(p_1,p_2) = S_F(q+P/2)\Gamma(q,P)S_F(q-P/2) \quad \ldots (10)$$

where the individual quark momenta $p_{1,2}=P/2 \pm q$ in terms of the total (P) and relative (q) 4-momenta. For the pseudoscalar state in question, the BSE for $\Gamma(p_1, p_2)$, viz.,

$$(2\pi)^4\Gamma(q,P) = 2ig_0\gamma_5\int d^4q' Tr[\gamma_5 S_F(q'+P/2) \Gamma(q',P)S_F(q'-P/2)] \quad \ldots (11)$$

which for P_μ=0, has a self-consistent solution $\Gamma=C\gamma_5$, C being a constant, *provided* g_0 satisfies eq. (4), which is just the 'gap' equation (SDE) for the mass m.

This crucial result of the NJL model, which shows that in the chiral limit P_μ=0, the BSE and the SDE are *identical*, is a direct consequence of the γ_5-invariance of the input Lagrangian. It also tells us that in the P_μ=0 limit, the $q\bar{q}$ vertex function and the quark mass function m have the same (constant) structure. The constancy of each is of course a consequence of the contact interaction, but the basic equality of these two quantities is also valid for an extended 4-fermion chirally invariant Lagrangian such as a vector mediated one[24].

The true significance of NJL was realized in the QCD context[22], through the study of non-perturbative solutions of SDE as a *DBχS* mechanism for more general 4-fermion couplings[3,8,23,24], including reformulations of the bag model[8a], and renormalization group equations[21d]. And it was eventually subsumed in the generalized BSE-SDE formalism[9–11], which is of course the subject of this review.

2.2 *BCS Mechanism, Mass Relations, SYSY, etc.*

During the last decade, Nambu[50] has abstracted the findings of a new symmetry from BCS-type theories of dynamical symmetry breaking (due to short-range attraction), resulting in a new vacuum state. The residual symmetry in question is a remarkably simple relation among the fermion mass m_f and the composite boson (π, σ) masses as low energy modes in the new vacuum, viz., ($M_\pi : m_f : M_\sigma$)=(0 : 1 : 2). In more complex formulations the fermion mass m_f and the (composite) boson mass (M_1, M_2) obey the generalized relation $M_1^2 + M_2^2 = 4m_f^2$. The low energy properties of the system can be represented by an effective Hamiltonian like in the σ-model[49a] where the coupling constants are so related as to yield such mass relations automatically.

Coming to the SUSY aspects, the essential thrust of Nambu's discovery[50b] is a hidden SUSY in the BCS mechanism, manifesting via two physical scenarios: i) a cascading chain of symmetry breaking (tumbling); ii) a bootstrap mechanism in which the symmetry sustains itself among a set of effective fields without the need to refer to a substructure. The main ideas are the following.

A BCS mechanism has two energy scales: i) The high energy scale corresponds to the force responsible for the formation of Cooper pairs; its analogue in particle physics is the pion decay constant f_π. ii) The low energy scale is the pairing energy, and one of its manifestations is the quasi-fermion mass m_f which corresponds to the constituent quark mass $m_q=m_{NJL}$. More explicitly, there are both fermionic and bosonic excitations in the low energy scale: the quasi-fermion (m_f), the Goldstone boson (pion) and the Higgs (sigma) boson. In the simplest BCS (NJL) mechanism, their masses are in the ratio $m_f : m_\pi : m_\sigma$=1 : 0 : 2. This low energy picture can also be articulated by an effective Ginzburg-Landau-Gell Mann-Levy Hamiltonian involving these fermion and boson fields with Yukawa couplings and a Higgs potential. Their characteristic parameters are the 'high-energy' sigma-condensate $c \sim f_\pi$, and the 'low-energy' dimensionless Yukawa coupling constant $G=m_f/c$. To satisfy the mass ratio constraints, the Higgs self-coupling must be equal to G^2. The non-relativistic analogue of the condensate c is $\sqrt{N}/2$ where N is the density of states of the constituents at the fermi surface. The origin of the mass scales is a more dynamical question depending on the SUSY Hamiltonian structure, for a derivation of which the interested reader is referred to[50a-d].

The physical scenario envisaged by Nambu for this broken SUSY structure is two-fold. The first is a cascading hierarchy of symmetry breaking ('tumbling') which in particle physics[51a] means something like the following. Suppose a symmetry breaking at a high energy level gives rise to a σ-boson at the low energy scale. The latter, being a scalar, will induce attraction between the quasi-fermions, which in turn may generate a second generation symmetry-breaking, and so on. According to Nambu, a similar example of tumbling also exists in nuclear physics. Thus the σ-boson, which is a fall-out of chiral symmetry breaking and quark-mass generation in the bulk of nuclear binding, also causes nuclear pairing which can be estimated quite accurately[50c].

The second scenario[50c] is the theoretical possibility of a (Chew-like) bootstrap, not at the hadron, but at the quark-lepton level, on the assumption that the t'Hooft self-consistency condition[51b] is satisfied between these two levels. This leads to the following bounds on the t-quark and Higgs masses: m_t>120 GeV; m_H>200 GeV. This and other details may be found in ref. [50e].

3 Gauge Theoretic Formulation of SDE-BSE

As seen in Sec. 2, the simple NJL-model[4] succinctly articulates the $DB\chi S$ mechanism which gives rise to dynamical quark-mass generation on the one hand, and a Nambu-Goldstone[48] realization of the massless pion on the other. Another result is the formal identity of the mass-gap equation (SDE) with the homogeneous BSE for the vertex function for a *massless* pseudoscalar $q\bar{q}$ composite. We are now in a position to pursue the same logic to give a formal theoretical basis to a gluon-exchange (vector)-like 4-fermion interaction (to simulate QCD effects) in the input Lagrangian by deriving frm it an interlinked BSE-SDE framework[9-12] which is the backbone of this article. In this respect, we shall skip an alternative non-perturbative treatment of the BCS-NJL pairing mechanism by the Bogoliubov-Valatin method[23] which is not easy to adapt to a Lorentz-invariant formulation.

3.1 Minimal Effective Action: SDE & BSE

We outline a treatment due to Munczek[52] on the derivation of the equations of motion for composite fields. Consider an action functional

$$S = \int dx[\bar{\psi}(-\gamma.\partial - M)\psi + \bar{\psi}\lambda(x) + h.c.] - \frac{1}{2}\iint dxdy \Sigma_s G_s(x-y) J_s(x) J_s(y) \quad \dots (12)$$

where $\lambda(x)$ is an external source, G_s is the propagator of the exchanged boson, and $J_s(x) = \bar{\psi}(x)\,\Gamma_s\,\psi(x)$ is the current function. This form is approximately derivable from the standard generating function for non-abelian QCD with $\Gamma_s = i\gamma_\mu\lambda/2$, when G_s becomes the gluon propagator. The NJL-type contact interaction corresponds to $G_s \equiv \delta^4(x-y)$, but the treatment is more generally valid for non-local interactions too. The standard approach is to introduce bilocal boson fields[53] which for several types of spin excitations has the form[52]

$$\eta(x,y) = \Sigma_s \Gamma_s \psi(x) \Gamma_s \bar{\psi}(y)\, G_s(x-y) \quad \dots (13)$$

where η has a 4×4 matrix form. With a second auxiliary field $B(x, y)$[52], one gets the following generating functional

$$Z = N^{-1}\int D\psi D\bar{\psi} D\eta DB\, exp[iS(\psi,\bar{\psi},B,\eta) + i\int dx(\bar{\psi}\lambda + \bar{\lambda}\psi)] \quad \dots (14)$$

$$S = \int dx \bar{\psi}(-\gamma.\partial - M)\psi - Tr\int dxdy\eta(x,y) \times [B(y,x) - \psi(y)\bar{\psi}(x)] + \frac{1}{2}Tr\iint dxdy \sum_s G_s(x-y)B(x-y)\,\Gamma_s\,B(y,x)\Gamma_s \quad \dots (15)$$

When the functional integration is carried out over $\eta(x, y)$, it gives a δ-function $\delta[B(y, x) - \psi(y)\bar{\psi}(x)]$. Subsequent integratioin over B gives eq. (12). After this check, the order of integration may be reversed so as to integrate out over ψ and $\bar{\psi}$, and yield the effective action

$$S = Tr[-iln(-\gamma.\partial - \eta) - \eta B + \bar{B}B/2]; \quad \bar{B}(x,y) = \Sigma_s G_s(x-y)\Gamma_s B(x, y)\Gamma_s \quad \dots (16)$$

Here η, B, $(\gamma.\partial+M)$, are matrices in spinor, internal symmetry, and configuration space indices, so that

$$\eta B=\int dz\eta(x,z)B(z,y)\equiv<x|\eta B|y>; \qquad \ldots (17)$$
$$Tr[\eta B]=Tr\int dx<x|\eta B|x>$$

Varying S w.r.t. B and η gives

$$\eta(x,y)=B(x,y);\ B=i(-\gamma.\partial-M-\eta)^{-1} \qquad \ldots (18)$$
$$=i(-\gamma.\partial-M-B)^{-1}$$

Replacing B in (18) by the vacuum expectation value $<B>=iS_F$, gives the SDE

$$S_F=(-\gamma.\partial-M-iS_F)^{-1} \qquad \ldots (19)$$
$$=\Sigma_s G_s(x-y)\Gamma_s S_F(x-y)\Gamma_s$$

whose detailed form is

$$(-\gamma.\partial-M)S_F(x-y)-i\int dz\Sigma_s G_s(x-z) \qquad \ldots (20)$$
$$\times\Gamma_s S_F(x-z)\Gamma_s S_F(z-y)=\delta^4(x-y)$$

Next, for the quantum corrections to B, write

$$B(x,y)=iS_F(x-y)+\phi(x,y) \qquad \ldots (21)$$

and obtain the homogeneous equation

$$i\sum_1^{\inf} S_F(\bar{\phi}S_F)^n=iS_F\bar{\phi}S_F+S_F\bar{\phi}\phi; \qquad \ldots (22)$$
$$\bar{\phi}(x,y)=\Sigma_s G_s(x-y)\Gamma_s\phi(x,y)\Gamma_s$$

If the non-linear term in ϕ in (22) is neglected, the result is the homogeneous BSE

$$\phi(x,y)=i\iint dzdtS_F(x-z)\Sigma_s G_s(z-t)\Gamma_s\phi(z-t)$$
$$\times\Gamma_s S_F(t-y) \qquad \ldots (23)$$

which must be solved along with the SDE (19) for the propagator. Note that the kernel of the BSE is G_s, i.e., the *same* form factor as appears in the input Lagrangian itself. This is the basic logic of the interplay of the SDE with the BSE.. Next we describe this interplay in momentum space for the case $\Gamma_s=i\gamma_\mu\lambda_a/2$, to bring out the Nambu-Goldstone nature of a pseudoscalar state (ϕ proportional to γ_5), one in which the Ward identity plays a crucial role.

3.2 Self-Energy vs Vertex Fn in Chiral Limit

The formal equivalence of the mass-gap equation (SDE) and the BSE for a pseudoscalar meson in the chiral limit[24] will now be demonstrated for an arbitrary confining form $D(k)$ (not just the perturbative form k^{-2}). Denoting the mass operator by $\Sigma(p)$ and the vertex function by Γ_H, the SDE after replacing the color factor $\lambda_1.\lambda_2/4$ by its Casimir value 4/3, reads as

$$\Sigma(p)=\frac{4}{3}i(2\pi)^{-4}\int d^4kD_{\mu\nu}(k)\gamma_\mu S'_F(p-k)\gamma_\nu;$$
$$D_{\mu\nu}(k)=(\delta_{\mu\nu}-k_\mu k_\nu/k^2)D(k) \qquad \ldots (24)$$

S'_F is the full propagator related to the mass operator $\Sigma(p)$ by

$$\Sigma(p)+i\gamma.p=S_F^{-1}(p)=A(p^2)[i\gamma.p+m(p^2)] \qquad \ldots (25)$$

thus defining the mass function $m(p^2)$ in the chiral limit $m_c=0$. In the same way the vertex function $\Gamma_H(q, P)$ for a $q\bar{q}$ hadron (H) of 4-momentum P_μ made up of quark 4-momenta $p_{1,2}=P/2\pm q$ satisfies the BSE

$$\Gamma_H(q,P)=-\frac{4}{3}i(2\pi)^{-4}\int d^4q'D_{\mu\nu}(q-q')\gamma_\mu S_F$$
$$\times(q'+P/2)\Gamma_H(q',P)S_F(q'-P/2)\gamma_\nu \qquad \ldots (26)$$

The complete equivalence of (24) and (26) for the pion case in the chiral limit $P_\mu\to 0$ is easily established. Indeed, with the self-consistent ansatz $\Gamma_H=\gamma_5\Gamma(q)$, eq. (26) simplifies to

$$\Gamma(q)=\frac{4}{3}i(2\pi)^{-4}\int d^4k\gamma_\mu S'_F(k-q)\Gamma(q-k)$$
$$\times S'_F(q-k)\gamma_\nu \qquad \ldots (27)$$

where the replacement $q'=q-k$ has been made. Substitution for S'_F from (25) in (27) gives

$$\Gamma(p)=-\frac{4}{3}i(2\pi)^{-4}\int d^4k$$
$$\times\frac{D(k)\Gamma(p-k)}{A^2(p-k)(m^2((p-k)^2)+(p-k)^2)} \qquad \ldots (28)$$

where we have relabelled $q\to p$. On the other hand substituting for S'_F (25) in (24) gives for the mass term of $\Sigma(p)$ the result

$$A(p^2)m(p^2) = -\frac{4}{3}i(2\pi)^{-4}\int d^4k$$
$$\times\frac{D(k)A(q')m(q'^2)}{A^2(q')(m^2(q'^2)+q'^2)} \quad \ldots (29)$$

where $q'=p-k$. A comparison of (28) and (29) shows their equivalence with the identification $\Gamma(q)=A(q)m(q^2)$, i.e. the identity of the vertex and mass functions in the chiral limit provided $A=1$. This last is a consequence of the Landau gauge for $D_{\mu\nu}$ in eq. (24), since in this gauge, the function $A(p)$ does not undergo renormalization[54], so that it may be set equal to unity. Note that this result is more general than in the contact type NJL model, since both quantities are now functions of momentum due to the extended nature of the 4-fermion coupling caused by the gluonic propagator $D(k)$.

3.3 $\Sigma(p)$ vs $\Gamma(q, P)$ via Ward Identities

The connection between $\Sigma(p)$ and $\Gamma(q, P)$ away from the chiral limit ($P_\mu=0$) is achieved via a systematic use of the Ward identities for vector and axial vector types. The following derivation due to ref. [24a] may be instructive for applications. Consider some approximation scheme (based on a BSE with a specified kernel) to determine $\Sigma(p)$ via eq. (24), so as to obey the Ward-Takahashi identities. e.g., the quark-gluon vertex function Γ_λ satisfies the *inhomogeneous* equation

$$\Gamma_\lambda = \gamma_\lambda - \frac{4}{3}i(2\pi)^{-4}\int d^4q'\gamma_\nu S'_F(q'+P/2)$$
$$\times\Gamma_\lambda S'_F(q'-P/2)\gamma_\mu D_{\mu\nu}(q-q') \quad \ldots (30)$$

Multiplying (30) by P_λ and using the WT-identity

$$P_\lambda\Gamma_\lambda(q,P) = S_F'^{-1}(q+P/2) - S_F'^{-1}(q-P/2) \quad \ldots (31)$$

gives the result

$$\frac{1}{S'_F(P/2+q)} - \frac{1}{S'_F(q-P/2)} = \gamma.P - \frac{4i}{3}\int\frac{d^4q'}{(2\pi)^4}$$
$$\times D_{\mu\nu}(q-q')\gamma_\nu[S_F(q'-P/2)-S_F(q'+P/2)]\gamma_\mu \quad \ldots (32)$$

which is entirely consistent with (24) when one uses the definition (25) for $\Sigma(p)$. In a similar way, for the axial vector $\Gamma_{\mu5}$, the corresponding BSE obeying chiral symmetry is

$$\Gamma_{\lambda5}(q,P) = i\gamma_\lambda\gamma_5 - \frac{4}{3}i(2\pi)^{-4}\int d^4q' D_{\mu\nu}(q-q')$$
$$\times\gamma_\nu S_F(q'+P/2)\Gamma_{\lambda5}(q',P)S_F(q'-P/2)\gamma_\mu \ldots (33)$$

It is again consistent with eq. (26) and the definition (25) for $\Sigma(p)$ if one uses the axial WT identity

$$-iP_\lambda\Gamma_{\lambda5}(q,P) = S_F'^{-1}(q+P/2)\gamma_5 + \gamma_5 S_F'^{-1}(q-P/2) \quad \ldots (34)$$

The LHS of (34) must now be identified with the pseudoscalar vertex function $\Gamma_5\,(p\,.\,P)$, so that the corresponding RHS gives its full structure that is consistent with gauge invariance, viz.,

$$\Gamma_5(q,P)\gamma_5 = i\gamma.(q+P/2)A(q+P/2) - i\gamma.(q-P/2)$$
$$\times A(q-P/2) + B(q+P/2) + B(q-P/2);$$
$$B(p) = A(p)m(p^2) \quad \ldots (35)$$

This equation checks with (29), in the Landau gauge (A=1), in the chiral limit $P_\mu=0$, but now provides the corrections for $P_\mu \neq 0$ as well. In the Landau gauge (35) simplifies to

$$\Gamma_5\gamma_5 = i\gamma.P + m(q+P/2) + m(q-P/2) \quad \ldots (36)$$

In recent years, the determination of vertex functions via WT identities has become a fairly standard practice, although it is not always the most convenient method in practice for incorporating gauge-invariance within a given (semi-phenomenological) framework. For the present report, we shall have occasion to incorporate QED gauge invariance in arbitrary momentum-dependent form factors, and the method will be explained in Sec. 5, and in more detail in *Appendix B*, in connection with the *P*-meson e.m. self-energy calculations to be given in Sec. 7.

4 3D-4D SDE-BSE Formalism Under MYTP

As per the programme outlined in Sect. 1, we shall from now on specialize to a more practical form of SDE-BSE framework born out of 3D support (defined covariantly) to a vector-exchange mediated 4-fermion coupling at the input Lagrangian level with 'current' (almost massless) quarks. The vector exchange simulates the effect of a gluonic propagator, encompassing both the perturbative and non-perturbative regimes, and thus preserves the chiral character of the input coupling. The derived SDE and BSE, at Sec. 3, automatically incorporates $DB\chi S$ and hence generates the dynamical mass function $m(p)$ whose low momentum limit $m(0)$ gives the bulk contribution to the it constituent mass m_{cons}, while the *current* mass m_{curr} for *uds* quarks (that enter the input Lagrangian) gives a small effect. This last is in keeping with Politzer's Additivity principle[55], viz. $M_{cons}=m_{curr}+m(0)$, providing a rationale for the quark masses usually employed in potential models[12].

Now to implement the covariant 3D constraint of MYTP[15] on the BSE kernel (which stems from one on the input Lagrangian), we shall consider two methods in parallel for a direct comparison: i) Covariant Instantaneity Ansatz (CIA)[16,17], ii) Covariant Null-Plane Ansatz (CNPA)[41]. The latter[41] gives a formal 'covariance structure' to an earlier pragmatic formulation with essentially the same content[40], while the former[16] is already covariant as it is. We shall now outline a connected account of the 3D BSE reduction for both CIA and CNPA types (with scalar followed by fermion quarks), to bring out the structural identity of the resulting BSE's for a $q\bar{q}$ system. This will be followed by a reconstruction of the 4D BS vertex functions for both types[16] which will serve as the basic framework for 4D quark loop calculations in the subsequent chapters.

4.1 3D-4D BSE Under CIA: Spinless Quarks

To keep the contents fairly self-contained, we start with a few definitions for unequal mass kinematics in the notation of ref. [16, 10b]. Let the quark constituents of masses $m_{1,2}$ and 4-momenta $p_{1,2}$ interact to produce a composite hadron of mass M and 4-momentum P_μ. The internal 4-momentum q_μ is related to these by

$$p_{1,2} = \hat{m}_{1,2}P \pm q;\ P^2 = -M^2; \qquad \ldots (37)$$
$$2\hat{m}_{1,2} = 1 \pm (m_1^2 - m_2^2)/M^2$$

These Wightman-Garding definitions[56] of the fractional momenta $\hat{m}_{1,2}$ ensure that $q.P=0$ on the mass shells $m_i^2 + p_i^2 = 0$ of the constituents, though not off-shell. Now define $\hat{q} = q_\mu - q.PP_\mu / P^2$ as the relative momentum *transverse* to the hadron 4-momentum P_μ which automatically gives $\hat{q}.P \equiv 0$, for all values of $\hat{q}_\mu$. If the BSE kernel $K=K(\hat{q},\hat{q}')$, this is called the "Cov. Inst. Ansatz (CIA)"[16] which accords with MYTP[15]. For two scalar quarks with inverse propagators $\Delta_{1,2}$, this ansatz gives rise to the following BSE for the wave fn $\Phi(q, P)$[16,10b].

$$i(2\pi)^4 \Delta_1 \Delta_2 \Phi(q,P) = \int d^4q' K(\hat{q},\hat{q}')\Phi(q', P);$$
$$\Delta_{1,2} = m_{1,2}^2 + p_{1,2}^2 \qquad \ldots (38)$$

The quantities $m_{1,2}$ are the 'constituent' masses which are strictly momentum dependent since they contain the mass function $m(p)$[55], but may be regarded as almost constant for low energy phenomena $m(p) \cong m(0)$. Further, under CIA, $m(p)=m(\hat{p})$, a momentum-dependence which is governed by the $DB\chi S$ condition[4] (see below).

To make a 3D reduction of eq. (38), define the 3D wave function $\phi(\hat{q})$ in terms of the longitudinal momentum $M\sigma$ as

$$\phi(\hat{q}) = \int M d\sigma \Phi(q, P);\ M\sigma = Mq.P/P^2 \qquad \ldots (39)$$

using which, eq. (38) may be recast as

$$i(2\pi)^4 \Delta_1\Delta_2\Phi(q,P) = \int d^3\hat{q}' K(\hat{q},\hat{q}')\phi(\hat{q}'); \qquad \ldots (40)$$
$$d^4q' \equiv d^3\hat{q}' M d\sigma'$$

Next, divide out by $\Delta_1\Delta_2$ in (40) and use once again (39) to reduce the 4D BSE form (40) to the 3D form

$$(2\pi)^3 D(\hat{q})\phi(\hat{q}) = \int d^3\hat{q}' K(\hat{q},\hat{q}')\phi(\hat{q}');$$
$$\frac{2i\pi}{D(\hat{q})} \equiv \int \frac{Md\sigma}{\Delta_1\Delta_2} \qquad \ldots (41)$$

Here $D(\hat{q})$ is the 3D denominator function associated with the like wave function $\phi(\hat{q})$. The integration over $d\sigma$ is carried out by noting pole positions of $\Delta_{1,2}$ in the σ-plane, where

$$\Delta_{1,2} = \omega_{1,2}^2 - M^2(\hat{m}_{1,2} \pm \sigma)^2;\ \omega_{1,2}^2 = m_{1,2}^2 + \hat{q}^2 \ldots (42)$$

The pole positions are given for $\Delta_{1,2}=0$ respectively by

$$M(\sigma+\hat{m}_1)=\pm\omega_1\mp i\varepsilon;\ M(\sigma-\hat{m}_2)=\pm\omega_2\mp i\varepsilon \quad \ldots (43)$$

where the (±) indices refer to the lower/upper halves of the σ-plane. The final result for $D(\hat{q})$ is expressible symmetrically[16].

$$D(\hat{q})=M_\omega D_0(\hat{q});\ \frac{2}{M_\omega}=\frac{\hat{m}_1}{\omega_1}+\frac{\hat{m}_2}{\omega_2} \quad \ldots (44)$$

$$\frac{1}{2}D_0(\hat{q})=\hat{q}^2-\frac{\lambda(m_1^2,m_2^2,M^2)}{4M^2};\quad \ldots (45)$$
$$\lambda=M^4-2M^2(m_1^2+m_2^2)+(m_1^2-m_2^2)^2$$

The crucial thing for the MYTP is now to observe the *equality* of the RHS of eqs (40) and (41), thus leading to an *exact interconnection* between the 3D and 4D BS wave functions:

$$\Gamma(\hat{q})\equiv\Delta_1\Delta_2\Phi(q,P)=\frac{D(\hat{q}\phi(\hat{q})}{2i\pi} \quad \ldots (46)$$

Eq. (46) determines the hadron-quark vertex function $\Gamma(\hat{q})$ as a product $D\phi$ of the 3D denominator and wave functions, satisfying a relativistic 3D Schroedinger-like equation (41).

Some comments on the entire BSE structure are now in order. The 'two-tier' character of the formalism is seen from the simultaneous appearance of the 3D form (41) and the 4D form (40), leading to their interconnection (46). The 3D form (41) gives the basis for making contact with the 3D spectra[13], while the reconstructed 4D wave (vertex) function (46) in terms of 3D ingredients D and ϕ enables the evaluation of 4D quark-loop integrals in the standard Feynman fashion[40]. Note that the vertex function $\Gamma=D\phi/(2i\pi)$ has quite a general structure, and independent of the details of the input kernel K. Further, the D-function, eq. (44), is universal and well-defined off the mass shell of either quark. The 3D wave function ϕ is admittedly model-dependent, but together with $D(\hat{q})$, it controls the 3D spectra via (41), so as to offer a direct experimental check on its structure. Both functions depend on the single 3D Lorentz-covariant quantity $\hat{q}^2$ whose most important property is its positive definiteness for time-like hadron momenta ($M^2>0$).

4.2 CNPA for 3D-4D BSE: Spinless Quarks

As a preliminary to defining a 3D support to the BS kernel on the null-plane (NP), on the lines of CIA[16], a covariant NP orientation[41] may be represented by the 4-vector n_μ, as well as its dual $\tilde{n}_\mu$, obeying the normalizations $n^2=\tilde{n}^2=0$ and $n.\tilde{n}=1$. In the standard NP scheme (in euclidean notation), these quantities are $n=(001;-i)/\sqrt{2}$ and $\tilde{n}=(001;i)/\sqrt{2}$, while the two other perpendicular directions are collectively denoted by the subscript $\perp$ on the concerned momenta. We shall try to maintain the n-dependence of various momenta to ensure explicit covariance; and to keep track of the old NP notation $p_\pm=p_0\pm p_3$, our covariant notation is normalized to the latter as $p_+=n.p\ \sqrt{2}$; $p_-=-\tilde{n}.p\sqrt{2}$, while the perpendicular components continue to be denoted by $p_\perp$ in both notations.

In the same notation as for CIA[16], the 4th component of the relative momentum $q=\hat{m}_2p_1-\hat{m}_1p_2$, that should be eliminated for obtaining a 3D equation, is now proportional to $q_n\equiv\tilde{n}.q$, as the NP analogue[40] of $P.qP/P^2$ in CIA[16], where $P=p_1+p_2$ is the total 4-momentum of the hadron. However the quantity q-q_nn is still only $q_\perp$, since its square is q^2-$2n.q\tilde{n}.q$, as befits $q_\perp^2$ (readily checked against the 'special' NP frame). We still need a third component q_3, for which a first guess is zP, where $z=n,q/n.P$. And for calculational convenience we shall need to (temporarily) invoke the 'collinear frame' which amounts to $P_\perp.q_\perp=0$, a restriction which will be removed later by a simple prescription of 'Lorentz completion'. Unfortunately the definition $\hat{q}_\mu=(q_{\perp\mu},zP_\mu)$ does not quite fit the bill for a covariant 3-vector, since a short calculation shows again that $\hat{q}^2=q_\perp^2$. The correct definition is seen as $q_{3\mu}=zP_nn_\mu$, where $P_n=P.\tilde{n}$, giving $\hat{q}^2=q_\perp^2+z^2M^2$, as required. We now collect the following definitions/results:

$$q_\perp=q-q_nn;\ \hat{q}=q_\perp+xP_nn;\ x=q.n/P.n;P^2=-M^2;$$
$$q_n,P_n=\tilde{n}.(q,P);\hat{q}.n=q.n;\ \hat{q}.\tilde{n}=0;\ P_\perp.q_\perp=0;$$
$$P.q=P_nq.n+P.nq_n;\ P.\hat{q}=P_nq.n;\ \hat{q}^2=q_\perp^2+M^2z^2 \quad \ldots (47)$$

Now in analogy to CIA, the reduced 3D BSE (wave-fn ϕ) may be derived from the 4D BSE (38) for spinless quarks (wave-fn Φ) when its kernel K is *decreed* to be independent of the component q_n, i.e., $K=K(\hat{q},\hat{q}')$, with $\hat{q}=(q_\perp, P_n n)$, in accordance with the TP[15] condition imposed on the null-plane (NP), so that $d^4q = d^2q_\perp dq_3 dq_n$. Now define a 3D wave-fn $\phi(\hat{q}) = \int dq_n \Phi(q)$, as the CNPA counterpart of the CIA definition (39) and use this result on the RHS of (38) to give

$$i(2\pi)^4\Phi(q) = \Delta_1^{-1}\Delta_2^{-1}\int d^3\hat{q}' K(\hat{q},\hat{q}')\phi(\hat{q}') \quad \ldots (48)$$

which is formally the same as eq. (40) for CIA above. Now integrate both sides of eq. (48) w.r.t. dq_n to give a 3D BSE in the variable $\hat{q}$:

$$(2\pi)^3 D_n(\hat{q})\,\phi(\hat{q}) = \int d^2q'_\perp\, dq'_3\, K(\hat{q},\hat{q}')\phi(\hat{q}') \quad \ldots (49)$$

which again corresponds to the CIA eq. (41), except that the function $D_n(\hat{q})$ is now defined by

$$\int dq_n\, \Delta_1^{-1}\Delta_2^{-1} = 2\pi i D_n^{-1}(\hat{q}) \quad \ldots (50)$$

and may be obtained by standard NP techniques[40] (Chaps 5-7) as follows. In the q_n plane, the poles of $\Delta_{1,2}$ lie on opposite sides of the real axis, so that only *one* pole will contribute at a time. Taking the Δ_2-pole, which gives

$$2q_n = -\sqrt{2}q_- = \frac{m_2^2 + (q_\perp - \hat{m}_2 P)^2}{\hat{m}_2 P.n - q.n} \quad \ldots (51)$$

the residue of Δ_1 works out, after a routine simplification, to just $2P.q=2P.nq_n+2P_nq.n$, after using the collinearity condition $P_\perp . q_\perp = 0$ from (47). And when the value (51) of q_n is substituted in (50), one obtains (with $P_nP.n = -M^2/2$):

$$D_n(\hat{q}) = 2P.n\left(\hat{q}^2 - \frac{\lambda(M^2,m_1^2,m_2^2)}{4M^2}\right\}$$

$$\hat{q}^2 = q_\perp^2 + M^2z^2;\; z = q.n/P.n \quad \ldots (52)$$

Now a comparison of (48) with (49) relates the 4D and 3D wave-fns:

$$2\pi i\Phi(q) = D_n(\hat{q})\,\Delta_1^{-1}\Delta_2^{-1}\phi(\hat{q}) \quad \ldots (53)$$

as the CIA counterpart of (46) which is valid near the bound state pole. The BS vertex function now becomes $\Gamma = D_n \times \phi/(2\pi i)$. This result, though dependent on the NP orientation, is nevertheless formally *covariant,* and closely corresponds to the pedagogical result of the old NPA formulation[40], with $D_n \Leftrightarrow D_+$.

A 3D equation similar to the covariant eq. (49) above, also obtains in alternative NP formulations such as in Kadychevsky-Karmanov[38] (see their eq. (3.48)). Both are 'covariantly' dependent on the orientation n_μ of the NP, i.e., have certain n-dependent 3-scalars, in addition to genuine 4-scalars. However, the *independent* 4-vector $\tilde{n}_\mu$ which has a dual interplay with n_μ in the above CNPA formulation, does not seem to have a counterpart in ref. [38]. Secondly this manifestly covariant 4D formulation needs no 3-vector like **n**, or explicit Lorentz transformations, as in such alternative NP formulations[38]. As to the 'angular condition', a question first raised by Leutwyler-Stern[35d], no special effect has been made to satisfy this requirement, since the very appearance of the 'effective' 3-vector $\hat{q}_\mu$ in the 3D BSE in a rotationally invariant manner is an automatic guarantee (in the sense of a 'proof of the pudding') of the satisfaction of this condition[35d] without further assumptions.

A second aspect of the above 3D-4D BSE under CNPA (which allows for off-shell momenta) is that it has no further need for 'spurions'[38] (to make up for energy-momentum balance due to on-shellness of the momenta in such formulations[38], so that normal 4D Feynman techniques suffice, as in the old-fashioned NPA formulation[33,40]. However, to rid the *physical* amplitudes of n_μ-dependent terms in the external (hadron) momenta, after integration over the internal loop momenta, one still needs to employ a simple technique of 'Lorentz-completion' (to be illustrated in Sec. 5 for the pion form factor calculation) as an alternative to other NP prescriptions[37,38] to remove **n**-dependent terms.

A more succint comparison with other null-plane approaches concerns the inverse process of *reconstruction* of the 4D hadron-quark vertex, eq. (53), which has no counterpart in them[37,38], as these are basically 3D oriented. Thus in ref. [38], the nearest analogue is to express the 3D NP wave function in terms of the 4D BS wave function (see eq. (3.58)[38]), but *not vice versa*. This problem of 'loss of Hilbert space information' inherent in such a process of reconstruction, has been discussed recently in the context of the qqq problem[47]; (see Sec. 10 for details).

4.3 Fermion Quarks with QCD-Motivated BSE

We are now in a position to give a corresponding description of the 3D-4D BSE for fermion quarks, for both CIA and CNPA cases taken together, just as for spinless quarks above. The 4D BSE for fermion quarks under a gluonic (vector-type) interaction kernel with 3D support has the form[10a-b]:

$$i(2\pi)^4\Psi(P,q) = S_{F1}(p_1)S_{F2}(p_2)\int d^4q' K(\hat{q},\hat{q}')\Psi(P,q');$$
$$K = F_{12}\, i\gamma_\mu^{(1)}\, i\gamma_\mu^{(2)}\, V(\hat{q},\hat{q}') \quad \ldots (54)$$

where F_{12} is the color factor $\lambda_1 . \lambda_2/4$ and the V-function expresses the scalar structure of the gluon propagator in the perturbative (o.g.e.) plus non-perturbative regimes. The 'hat' notation on the momenta covers both CIA and CNPA cases simultaneously, where the longitudinal component $\hat{q}_3$ is defined for the CNPA case as $q_{3\mu} = zP_n n_\mu$, with $P_n = P.\tilde{n}$. The full structure of V (used in actual calculations[28,40]) is collected as under, using the simplified notations k for $q-q'$, and $V(\hat{k})$ for the V fn:

$$V(\hat{k}) = 4\pi\alpha_s / \hat{k}^2 + \frac{3}{4}\omega_{q\bar{q}}^2 \int d\mathbf{r}[r^2(1+4A_0\hat{m}_1\hat{m}_2 M_>{}^2 r^2)^{-1/2}$$
$$- C_0/\omega_0^2]e^{ik.r}; \quad \ldots (55)$$

$$\omega_{q\bar{q}}^2 = 4M_>\hat{m}_1\hat{m}_2\omega_0^2\alpha_s(M_>^2);$$
$$\alpha_s(Q^2) = \frac{6\pi}{33-2n_f}\ln(M_>/\Lambda)^{-1}; \quad \ldots (56)$$

$$\hat{m}_{1,2} = [1 \pm (m_1^2 - m_2^2)/M^2]/2;$$
$$M_> = Max(M, m_1+m_2);$$
$$C_0=0.27;\ A_0=0.0283 \quad \ldots (57)$$

And the values of the basic constants (all in MeV) are[28,40].

$$\omega_0 = 158;\ m_{ud} = 265;\ m_s = 415;$$
$$m_c = 1530;\ m_b = 4900. \quad \ldots (58)$$

The BSE form (54) is however not the most convenient one for wider applications in practice, since the Dirac matrices entail several coupled integral equations. Indeed, as noted long ago[10a,b], a considerable simplification is effected by expressing them in 'Gordon-reduced' form, (permissible on the quark mass shells, or better on the surface $P.q=0$), a step which may be regarded as a fresh starting point of our dynamics, in the sense of an 'analytic continuation' of the γ-matrices to 'off-shell' regions (i.e., away from the surface $P.q=0$). Admittedly this constitutes a conscious departure from the original BSE structure (54), but such technical modifications are not unknown in the BS literature[10c-d] in the interest of greater manoeuvreability, without giving up the essentials, in view of the "effective" nature of the BS kernel (see Subsection 1.6).

The 'Gordon-reduced' BSE form of (54) is given by[10a,b]

$$\Delta_1\Delta_2\Phi(P,q) = -i(2\pi)^{-4}F_{12}\int d^4q' V_\mu^{(1)}V_\mu^{(2)}$$
$$\times V(\hat{q},\hat{q}')\,\Phi(P,q'); \quad \ldots (59)$$

where the connection between the Ψ- and Φ-functions is

$$\Psi(P,q) = (m_1 - i\gamma^{(1)}.p_1)(m_2 + i\gamma^{(2)}.p_2)\Phi(P,q);$$
$$p_{1,2} = \hat{m}_{1,2}P \pm q \quad \ldots (60)$$

$$V_\mu^{(1,2)} = \pm 2m_{1,2}\gamma_\mu^{(1,2)};$$
$$V_\mu^{(i)} = p_{i\mu} + p'_{i\mu} + i\sigma_{\mu\nu}^{(i)}(p_{i\nu} - p'_{i\nu}) \quad \ldots (61)$$

Now to implement the Transversality Condition[15] for the entire kernel of eq. (59), all time-like components σ, σ' in the product $V^{(1)}.V^{(2)}$ must first be replaced by their on-shell values. Substituting from (61) and simplifying gives

$$(p_1+p'_1).(p_2+p'_2) = 4\hat{m}_1\hat{m}_2P^2 - (\hat{q}+\hat{q}')^2$$
$$-2(\hat{m}_1-\hat{m}_2)P.(q+q') + \text{"}spin-Terms\text{"}; \quad \ldots (62)$$

$$\text{"}Spin\ Terms\text{"} = -i(2\hat{m}_1P + \hat{q} + \hat{q}')_\mu\sigma_{\mu\nu}^{(2)}\hat{k}_\nu$$
$$+ i(2\hat{m}_2P - \hat{q} - \hat{q}')_\mu\sigma_{\mu\nu}^{(1)}\hat{k}_\nu + \sigma_{\mu\nu}^{(1)}\sigma_{\mu\nu}^{(2)} \quad \ldots (62a)$$

This is identical to eq. (7.1.9) of ref. [40], via the correspondence $\hat{q} \Leftrightarrow q_x, q_y, q_3 (= q_+/P_+)$, so that both CIA[16] and CNPA[41] have formally the same structures as the 'old-fashioned' NPA[40], and hence give identical predictions on the 2-body spectra[28].

The 3D reduction of eq. (59) now goes through exactly as in the spin-0 case, eqs. (38-44), so that without further ado, the full structure of the 3D BSE can be literally taken over from ref. [40]-Chap 7 (derived under old-fashioned NPA). In particular, for harmonic confinement, obtained by dropping the A_0 term in the 'potential' $U(r)$ of (59) (a very good approximation for *light* (*ud*) quarks), the 3D BSE works out as

$$D(\hat{q})\phi(\hat{q}) = \omega_{q\bar{q}}^2 \tilde{D}(\hat{q})\phi(\hat{q}); \quad \dots (63)$$

$$D_n(\hat{q})\phi(\hat{q}) = \frac{P_n}{M}\omega_{q\bar{q}}^2 \tilde{D}(\hat{q})\ \phi(\hat{q}); \quad \dots (64)$$

for the CIA and CNPA cases respectively, where D_n is given by (52) and $D(\hat{q})$ by (44). The other quantities retain the same meaning for both. Thus

$$\tilde{D}(\hat{q}) = 4\hat{m}_1\hat{m}_2 M^2(\nabla^2 + C_0/\omega_0^2) + 4\hat{q}^2\nabla^2 + 8\hat{q}.\nabla + 18 - 8\mathbf{J.S} + (4C_0/\omega_0^2)\hat{q}^2 \quad \dots (65)$$

For the spectroscopic predictions on $q\bar{q}$ hadrons, *vis-a-vis* data, the reader is referred to ref. [28]. For algebraic completeness however, the (gaussian) parameter β of the 3D wave function $\phi(\hat{q}) = exp(-\hat{q}^2/2\beta^2)$, which is the solution of (63-64) for a ground state hadron[40,28] is:

$$\beta^4 = \frac{8\hat{m}_1^2\hat{m}_2^2 M_>^2 \omega_0^2 \alpha(M_>^2)}{[1 - 8C_0\hat{m}_1\hat{m}_2\alpha_s(M_>^2)] < \sigma >};$$

$$< \sigma >^2 = 1 + 24A_0(\hat{m}_1\hat{m}_2 M_>)^2/\beta^2 \quad \dots (66)$$

Note that β is a 4D invariant quantity, independent of n_μ, etc. (For an *L*-excited hadron wave function, see ref. [40]). The full 4D BS wave function $\Psi(P, q)$ in a 4×4 matrix from[40] is then reconstructed from (59)-(60) as in the scalar case, eq. (46), viz.,[40,16,44]

$$\Psi(P,q) = S_F(p_1)\Gamma(\hat{q})\gamma_D S_F(-p_2);$$
$$\Gamma(\hat{q}) = N_H[1; P_n/M]D(\hat{q})\phi(\hat{q})/2i\pi \quad \dots (67)$$

where γ_D is a Dirac matrix which equals γ_5 for a *P*-meson, $i\gamma_\mu$ for a *V*-meson, $i\gamma_\mu\gamma_5$ for an *A*-meson, etc. The factors in square brackets stand for CIA and CNPA values respectively. N_H represents the hadron normalization given by (see *Appendix A*):

$$N_H^{-2} = 2(2\pi)^3 \int d^3\hat{q}[M_\omega; M]\phi^2(\hat{q})[(1 + \delta m^2/M^2) \times (\hat{q}^2 - \lambda/4M^2) + 2\hat{m}_1\hat{m}_2(M^2 - \delta m^2)] \quad \dots (68)$$

where M_ω is given in (44) and $\delta m = m_1 - m_2$, and again the factors in square brackets represent CIA/CNPA values.

4.4 Dynamical Mass As DBχS Solution of SDE

We end this Section with the definition of the 'dynamical' mass function of the quark in the chiral limit (M_π=0) of the pion-quark vertex function $\Gamma(\hat{q})$, in the 3D-4D BSE framework[11]. The logic of this follows from the BSE-SDE formalism outlined in Sec. 3, eqs. (23)-(28), for the connection between eq. (29) for $m(p)$ and eq. (28) for $\Gamma(q)$ in the limit of zero mass of the pseudoscalar. So, setting M=0 in (63)-(64), the scalar part of the (unnormalized) vertex function may be identified with the mass function $m(\hat{p})$, in the limit P_μ=0, where p_μ is the 4-momentum of either quark; (note the appearance of the 'hatted' momentum). The result is[11]

$$m(\hat{p}) = [\omega(\hat{p}); \sqrt{2}p.n]\frac{m_q^2 + \hat{p}^2}{m_q^2}\phi(\hat{p}) \quad \dots (69)$$

under CIA and CNPA respectively. The normalization is such that in the low momentum limit, the constituent *ud* mass m_q is restored under CIA[11], while the corresponding 'mass' under CNPA is p_+[35c].

A more important aspect of the 'dynamical' mass function is its appearance as the non-trivial solution of the SDE under *DB*χ*S*[9,24]. We now give a derivation of the 3D-4D counterpart[11] of this basic result[24]. To that end we start with the non-perturbative part of the gluon propagator $D_{\mu\nu}(k)$=$D(k)[\delta_{\mu\nu} - k_\mu k_\nu/k^2]$ for the (harmonic) interaction of *ud* quarks (forming the 'pion structure), where the scalar factor $D(k)$ has the form[11].

$$D(k) = \frac{3}{4}(2\pi)^3\omega_0^2 2m_q\alpha_s(4m_q^2)[\nabla_{\hat{k}}^2 + C_0/\omega^2]\delta^3(\hat{k}) \quad \dots (70)$$

This form is immediately derivable from the structure of the 'potential' function $V(\hat{k})$ of eq. (55)-(56), with the A_0-term set equal to zero, and taking $M_>$=$2m_q$ for the 'pion' case. Note that $D(\hat{k})$ has a directional dependence n_μ=P_μ/P^2 on the pion 4-momentum P_μ, so that $\hat{k}^2 > 0$ over all 4D space;

it also possesses a well-defined limit for $P_\mu \to 0$. This structure may now be substituted in the 'gap-equation' (28)-(29) for a self-consistent solution in the low momentum limit. This exercise has been carried out[11], wherein the SDE (29) in the Landau gauge $A(p^2)$=1 reduces to the form

$$m(p^2) = \frac{3i}{\pi}\int d^3\hat{k}dk_0 m_q \alpha_s \times [\omega_0^2 \nabla_{\hat{k}}^2 + C_0]\delta^3(\hat{k}) \times \frac{m(p'^2)}{(p'^2 + m^2(p'^2)} \quad \ldots (71)$$

where $p'=p-k$ is 4D, and $(\hat{k}, k_0)$ are (3D, 1D) respectively. The integration is essentially over the time-like k_0, with the 'pole' position at $p'_0=m(p'_0)\equiv m_{NJL}$, leading finally to[11]

$$m_{NJL} = \frac{3m_q\alpha_s}{(m_{NJL})^2}[3\omega_0^2 - C_0 m_{NJL}^2];\ \alpha_s = \frac{6\pi}{29 ln(10 m_q)} \quad \ldots (72)$$

after substituting the value Λ = 200 MeV for the QCD constant. The further identification of m_q with m_{NJL} in this equation, yields an independent self-consistent estimate $m_{NJL} \sim 300$ MeV which may be compared to the input value 265 MeV, eq. (58), employed for spectra[28]. This analysis so far ignores the Politzer relation[55] $m_{ud} = m_c + m_{NJL}$, for the constituent mass m_q away from the chiral limit. The derivation of the pion and σ-meson masses away from the chiral limit, may be found in ref. [11].

5 CNPA Applications: Gauge-Inv Pion FF

The first example of our applications of the 3D-4D BSE structure developed in Sec. 4 is to 4D triangle loop integrals. This example has been chosen to illustrate the difficulties of CIA (as noted in Sec. 1.4) in tackling their ill-defined nature as a result of acquiring time-like momentum components in the exponential/gaussian factors associated with the vertex functions (46) due to a 'Lorentz-mismatch' among the rest-frames of the concerned hadronic composites, for triangle loops and above, such as the pion form factor, while 2-quark loops[32] just escape this pathology. This problem was not explicitly encountered in the old-fashioned NPA treatment[33] of the pion form factor, except for lack of explicit covariance. The CIA approach[16] to MYTP[15] enjoys covariance, but its application to triangle loop integrals causes other problems such as *complexity* of the corresponding amplitudes[57], apparently without good reason. On the other hand, the apparent success of the old-fashioned NPA[33] in circumventing this (complexity) problem[57], gives the hope that with its 'covariant' formulation[41], CNPA, the powers of this method should stand a better chance of testing via the form factor problem.

To recall a short background, the pion form factor has through the ages been a good laboratory for subjecting theoretical models and ideas on strong interactions to observational test. Among the crucial parameters are the squared radius $< r_{exp}^2 >= 0.43 \pm .014\, fm^2$ (ref. 58a), and the scaled form factor at high k^2, viz., $k^2F(k^2) \approx 0.5\pm0.1$ GeV^2 (ref. 58b) that represent important check points for theoretical candidates such as QCD-sum rules[2b], Finite Energy sum rules[59], perturbative QCD[60], covariant null-plane approaches[37,38], Euclidean SDE[61], etc. The issue of interface of perturbative and non-perturbative QCD regimes has been studied in terms of the relative importance of longitudinal vs transverse components[62], but this is the subject of a full-fledged dynamical theory (such as[9-12], and not of some intuitive ansatze[62].

To that end, we outline a calculation of the P-meson form factor for unequal mass kinematics with full gauge invariance, including correction terms arising from QED gauge invariance, and illustrate the techniques of 'Lorentz-completion' to obtain an explicitly Lorentz invariant quantity. As a check on the consistency of the formalism, the expected k^{-2} behaviour of the pion form factor at high k^2 is realized. Some calculational details on the triangle-loop integral for the P-meson form factor are given in *Appendex A* eq. (247).

5.1 P-Meson Form Factor $F(k^2)$ for Unequal Masses

Using the two diagrams (Figs. 1a and 1b) of ref. [33c], and in the same notation, the Feynman amplitude for the $h \to h' + \gamma$ transition contributed by Fig. 1a (quark 2 as spectator is given by[33c]

$$2P_\mu F(k^2) = 4(2\pi)^4 N_n(P)N_n(P')e\hat{m}_1 \int d^4q T_\mu^{(1)} \times \frac{D_n(\hat{q})\phi(\hat{q})D_n(\hat{q}')\phi(\hat{q}')}{\Delta_1\Delta'_1\Delta_2} + [1 \Rightarrow 2]; \quad \ldots (73)$$

$$4T_\mu^{(1)} = Tr[\gamma_5(m_1 - i\gamma\cdot p_1)i\gamma_\mu(m_1 - i\gamma\cdot p_1') \times\gamma_5(m_2 + i\gamma\cdot p_2)]; \quad \Delta_i = m_i^2 + p_i^2; \quad \ldots (74)$$

$$p_{1,2} = \hat{m}_{1,2}P \pm q; \quad p_{1,2}' = \hat{m}_{1,2}P' \pm q'; \quad p_2 = p_2';$$
$$P - P' = p_1 - p_1' = k; \quad 2\overline{P} = P + P'. \quad \ldots (75)$$

After evaluating the traces and simplifying via (74) (75), T_μ becomes

$$T_\mu^{(1)} = (p_{2\mu} - \overline{P}_\mu)[\delta m^2 - M^2 - \Delta_2] - k^2 p_{2\mu}/2 + (\Delta_1 - \Delta_1')k_\mu/4 \quad \ldots (76)$$

The last term in (76) is non-gauge invariant, but it does not survive the integration in (73), since the coefficient of k_μ, viz., $\Delta_1 - \Delta_1'$ is antisymmetric in p_1 and p_1', while the rest of the integrand in (73) is symmetric in these two variables. Next, to bring out the proportionality of the integral (73) to $\overline{P}_\mu$, it is necessary to resolve p_2 into the mutually perpendicular components $p_{2\perp}, (p_2\cdot k/k^2)k$ and $(p_2\cdot\overline{P}/\overline{P}^2)\overline{P}$, of which the first two will again not survive the integration, the first due to the angular integration, and the second due to the antisymmetry of $k=p_1-p_1'$ in p_1 and p_1', just as in the last term of (76). The third term is explicitly proportional to $\overline{P}_\mu$, and is of course gauge invariant since $\overline{P}\cdot k = 0$. (This fact had been anticipated while writing the LHS of (73)). Now with the help of the results

$$p_2\cdot\overline{P} = -\hat{m}_2 M^2 - \Delta_1/4 - \Delta_1'/4;$$
$$2\hat{m}_2 = 1 - (m_1^2 - m_2^2)/M^2; \quad \overline{P}^2 = -M^2 - k^2/4, \quad \ldots (77)$$

it is a simple matter to integrate (73), on the lines of Sec. 4, noting that terms proportional to $\Delta_1\Delta_2$ and $\Delta'_1\Delta_2$ will give zero, while the non-vanishing terms will get contributions only from the residues of the Δ_2-pole, eq. (51). Before collecting the various pieces, note that the 3D gaussian wave functions ϕ, ϕ', as well as the 3D denominator functions D_n, D'_n, do not depend on the time-like components p_{2n}, so that no further pole contributions accrue from these sources. (It is this problem of time-like components of the internal 4-momenta inside the gaussian ϕ-functions under the CIA approach[16], that had plagued a earlier CIA study of triangle diagrams[57]. To proceed further, it is now convenient to define the quantity $\overline{q}\cdot n = p_2\cdot n - \hat{m}_2\overline{P}\cdot n$ to simplify the ϕ- and D_n-functions. To that end define the symbols:

$$(q,q') = \overline{q} \pm \hat{m}_2 k/2; \quad z_2 = \overline{q}\cdot n/\overline{P}\cdot n;$$
$$\hat{k} = k\cdot n/\overline{P}\cdot n; \quad (\theta_k, \eta_k) = 1 \pm \hat{k}^2/4 \quad \ldots (78)$$

and note the following results of pole integration w.r.t. p_{2n} (see ref. [40]):

$$\int dp_{2n}\frac{1}{\Delta_2}[1/\Delta_1; 1/\Delta_1'; 1/(\Delta_1\Delta_1')]$$
$$= [1/D_n; 1/D_n'; 2p_2\cdot n/(D_n D_n')] \quad \ldots (79)$$

The details of further calculation of the form factor are given in *Appendix A*. An essential result is the normalizer $N_n(P)$ of the hadron, obtained by setting $k_\mu=0$, and demanding that $F(0)=1$. The reduced normalizer $N_H = N_n(P)P\cdot n/M$, which is Lorentz-invariant, is given via eq. (251) by:

$$N_H^{-2} = 2M(2\pi)^3\int d^3\hat{q}e^{-\hat{q}^2/\beta^2}[(1+\delta m^2/M^2) \times(\hat{q}^2 - \lambda/4M^2) + 2\hat{m}_1\hat{m}_2(M^2 - \delta m^2)] \quad \ldots (80)$$

where the internal momentum $\hat{q} = (q_\perp, Mz_2)$ is formally a 3-vector, in conformity with the 'angular condition'[35d]. The corresponding expression for the form factor is (see *Appendix A*):

$$F(k^2) = 2MN_H^2(2\pi)^3 exp[-(M\hat{m}_2\hat{k}/\beta)^2/4\theta_k) \times(\pi\beta^2)^{3/2}\frac{\eta_k}{\sqrt{\theta_k}}\hat{m}_1 G(\hat{k}) + [1\Rightarrow 2] \quad \ldots (81)$$

where $G(\hat{k})$ is defined by eqs. (254, 255) of *Appendix A*.

5.2 *'Lorentz Completion' for $F(k^2)$*

The expression (81) for $F(k^2)$ still depends on the null-plane orientation n_μ via the dimensionless quantity $\hat{k} = k\cdot n/P\cdot n$ which while having simple Lorentz transformation properties, is nevertheless not Lorentz invariant by itself. To make it explicitly Lorentz invariant, we shall employ a simple method of 'Lorentz completion' which is merely an extension of the 'collinearity trick' employed at the quark level, viz., $P_\perp \cdot q_\perp = 0$; see eq. (47). Note that this collinearity ansatz has already become reduntant at the level of the Normalizer N_H, eq. (80), which owes its Lorentz invariance to the integrating out of the null-plane dependent quantity z_2 in (80). This is of course because N_H depends only on one 4-momentum (that of a *single hadron*), so that the collinearity assumption is exactly valid. However the form factor $F(k^2)$ depends on *two independent* 4-momenta P, P', for which the collinearity assumption is non-trivial, since the existence of the perpendicular components cannot be wished away! Actually the quark-level assumption $P_\perp \cdot q_\perp = 0$ has, so to say, got transferred, via the $\hat{q}$-integration in eq. (81), to the *hadron level*, as evidenced from the $\hat{k}$-dependence of $F(k^2)$: therefore an obvious logical inference is to suppose this $\hat{k}$-dependence to be the result of the collinearity ansatz $P_\perp \cdot P'_\perp = 0$ at the hadron level. Now, under the collinearity condition, one has

$$
\begin{aligned}
P \cdot P' &= P_\perp \cdot P'_\perp + P \cdot nP' \cdot \tilde{n} + P' \cdot nP \cdot \tilde{n} \\
&= P \cdot nP'_n + P' \cdot nP_n \,; \quad P \cdot \tilde{n} \equiv P_n \,. \qquad \ldots (82)
\end{aligned}
$$

Therefore 'Lorentz completion' (the opposite of the collinearity ansatz) merely amounts to reversing the direction of the above equation by supplying the (zero term) $P_\perp \cdot P'_\perp$ to a 3-scalar product to render it a 4-scalar! Indeed the process is quite unique for 3-point functions such as the form factor under study, although for more involved cases (e.g., 4-point functions), further assumptions may be needed.

In the present case, the prescription of Lorentz completion is relatively simple, being already contained in eq. (82). Thus since $P, P' = \overline{P} \pm k/2$, a simple application of (82) gives

$$
k \cdot n\, k_n = +k^2 \,; \; \overline{P} \cdot n\overline{P}_n = -M^2 - k^2/4 \,;
$$

$$
\hat{k}^2 = \frac{4k^2}{4M^2 + k^2} = 4\theta_k - 4 = 4\eta_k \qquad \ldots (83)
$$

This simple prescription for $\hat{k}$ automatically ensures the 4D (Lorentz) invariance of $F(k^2)$ at the hadron level. (It may be instructive to compare this to the Cov. LF prescription[38] of 'recognizing' the n-dependent terms (unphysical) of $F(k^2)$ and then dropping them). For more involved amplitudes (e.g., 4-point functions) too, this prescription works fairly unambiguously, if their diagrams can be analyzed in terms of more elementary 3-point vertices (which is often possible). We hasten to add however that strictly speaking, a 'Lorentz completion' gopes beyond the original premises of restricting the (pairwise qq) interaction to the covariant null-plane (in accordance with MYTP[15]), but such 'analytic continuations' are not unwarranted, since in Cov. LF theories too[38], implementing the angular condition[35d] involves the introduction of 'derivative' terms, implying a tacit enlargement of the Hilbert space beyond the null-plane (see Chap 2 of ref. [38]).

5.3 *QED Gauge Corrections to $F(k^2)$*

While the 'kinematic' gauge invariance of $F(k^2)$ has already been ensured in Sec. 5.1 above, there are additional contributions to the triangle loops—Figs. 1a and 1b of ref. [33c]—obtained by inserting the photon lines at each of the two vertex blobs instead of on the quark lines themseleves. These terms arise from the demands of QED gauge invariance, as pointed out by Kisslinger and Li (KL)[63] in the context of two-point functions, and are simulated by inserting exponential phase integrals with the e.m. currents. However, this method (which works ideally for *point* interactions) is not amenable to *extended* (mementum-dependent) vertex functions, and an alternative strategy is needed, which is described below.

The way to an effective QED gauge invariance lies in the simple-minded substitution $p_I - e_I A(x_I)$ for each 4-momentum p_I (in a mixed p, x representation) occurring in the structure of the vertex function. This amounts to replacing each $\hat{q}_\mu$ occurring in $\Gamma(\hat{q}) = D(\hat{q})\phi(\hat{q})$, by $\hat{q}_\mu - e_q \hat{A}_\mu$, where $e_q = \hat{m}_2 e_1 - \hat{m}_1 e_2$, and keeping only first order terms in A_μ after due expansion. Now the first order correction to $\hat{q}^2$ is $-e_q \hat{q} \cdot \hat{A} - e_q \hat{A} \cdot \hat{q}$, which simplifies on substitution from eq. (83) to

$$
-2e_q \tilde{q} \cdot A \equiv -2e_q A_\mu [\hat{q}_\mu - \hat{q} \cdot n\tilde{n}_\mu + P \cdot \tilde{n}\hat{q} \cdot nn_\mu / P \cdot n] \qquad \ldots (84)
$$

The net result is a first order correction to $\Gamma(\hat{q})$ of amount $e_q j(\hat{q})\cdot A$ where

$$j(\hat{q})_\mu = -4M_> \tilde{q}_\mu \phi(\hat{q})(1-(\hat{q}^2-\lambda/4M^2)/2\beta^2) \quad \dots (85)$$

The contribution to the P-meson form factor from this hadron-quark-photon vertex (4-point) now gives the QED gauge correction to the triangle loops, Figs. (1a, 1b) of ref. [33c], to the main term $F(k^2)$, eq. (73), of an amount which, after a simple trace evaluation (and anticipating the vanishing of all Δ-terms remaining in the trace, as a result of contour integration over q_n) simplifies to ($\phi=\phi(\hat{q})$, etc.)

$$F_1(k^2) = 4(2\pi)^4 N_H^2 e_q \hat{m}_1 M_>^2 \int d^4q (M^2-\delta m^2)\phi\phi' \times\left[\frac{D'_n\tilde{q}\cdot\overline{P}}{\Delta'_1\Delta_2 P'\cdot n}+\frac{D_n\tilde{q}'\cdot\overline{P}}{\Delta_1\Delta_2 P\cdot n}\right]+[1\Rightarrow 2]; \quad \dots (86)$$

In writing down this term, the proportionality of the current of $2\overline{P}_\mu$ has been incorporated on both sides, on identical lines to that of (73), using results from (74)-(79) as well as from *Appendix A*. Note that e_q is *antisymmetric* in '1' and '2', signifying a change of sign when the second term $[1\Rightarrow 2]$ is added to the first. The term $\tilde{q}\cdot\overline{P}/\overline{P}^2$ simplifies to $2q\cdot n(1-\hat{k}/2)/P\cdot n$, after extracting the proportionality to $\overline{P}_\mu$. Next, after the pole integrations over q_n, q'_n in accordance with (79), it is useful to club together the results of photon insertions on *both* blobs for either index ('1' or '2'); this step generates two independent combinations for the '1' terms (and similarly for '2' terms):

$$A_n = q\cdot n(1-\hat{k}/2);$$
$$B_n = q\cdot n(1-\hat{k}/2)(\hat{q}^2-\lambda/4M_>^2)/2\beta^2, \quad \dots (87)$$

Collecting all these contributions the result of q_n-integration is

$$F_1(k^2) = 8(2\pi)^3 N_H^2 e_q \hat{m}_1 M_>^2 \int d^3\hat{q}(M^2-\delta m^2)\phi\phi' \times\left[\frac{A_n+A'_n-B_n-B'_n}{\eta_k\times(\overline{P}\cdot n)^2}\right]+[1\Rightarrow 2] \quad \dots (88)$$

The rest of the calculation is routine and follows closely the steps of Appendix A for the (main) $F(k^2)$ term, including the translation $z_2 \to z_2 + \hat{m}_2\hat{k}^2/2\theta_k$, and is omitted for brevity. The final result for $F_1(k^2)$ is

$$F_1(k^2) = -e_q\hat{m}_1\hat{m}_2(3\eta_k+\hat{k}^2) \times\left[\frac{(M_>^2-\delta m^2)\eta_k\hat{k}^4(M_>\hat{m}_2)^2}{8G(0)\theta_k^{7/2}\beta^2}\right]+[1\Rightarrow 2] \quad \dots (89)$$

where we have dropped some terms which vanish on including the $[1\Rightarrow 2]$ terms, noting the (1,2) antisymmetry of e_q.

5.4 *Large and Small k^2 Limits of $F(k^2)$*

We close this section with the large and small k^2 limits of the form factors $F(k^2)$ and $F_1(k^2)$. For large k^2, eq. (83) gives $\hat{k}=2$, $\theta_k=2$, and $\eta_k=4M^2/k^2$, so that

$$F(k^2) = 2MN_H^2(2\pi)^3\hat{m}_1\frac{4M^2}{k^2}(\pi\beta^2/2)^{3/2}G(inf) \times exp[-(M\hat{m}_2/\beta)^2/2]+[1\Rightarrow 2] \quad \dots (90)$$

where, from eqs. (252)-(253).

$$G(inf) = (1+\delta m^2/M^2)(\beta^2-\lambda/4M^2+M^2\hat{m}_2^2) + (M^2-\delta m^2)\hat{m}_2 - 2\hat{m}_2^2M^2 \quad \dots (91)$$

Similarly from eq. (86), the large k^2 limit of $F_1(k^2)$ is

$$F_1(k^2) = 2\sqrt{2}M^2k^{-2}e_q\hat{m}_1\hat{m}_2(M_>^2-\delta m^2) \times\left[\frac{M_>^2(\hat{m}_1-\hat{m}_2)}{\beta^2G(0)}\right] \quad \dots (92)$$

where we have taken account of the (1,2) antisymmetry of e_q in simplifying the effect of the $[1\Rightarrow 2]$ term on the RHS. As a check, both $F(k^2)$ and $F_1(k^2)$ are seen to satisfy the 'scaling' requirement of a k^{-2} variation for large k^2. This result can be traced to the input dynamics of the (non-perturbative) gluonic interaction, eq. (55), on the structure of the vertex function, eq. (67). Perturbative QCD of course gives a k^{-2} behaviour[61]. The covariant NP(LF) approach[38] also gives a similar behaviour, but extracted in a somewhat different way from the present 'Lorentz completion' treatment. Note that for the pion case the QED gauge correction term $F_1(k^2)$ gives zero contribution in the large k^2 limit.

For small k^2, on the other hand, we have from eq. (83)

$$\hat{k}^2 = k^2/M^2; \quad [\theta_k, \eta_k] = 1 \pm k^2/4M^2 \qquad \dots (93)$$

In this limit, the form factor, after substituting for N_H from (80), and summing over the '1' and '2' terms, works out as

$$F(k^2) = (1 - 3k^2/8M^2) \times \left[1 - \hat{m}_1\hat{m}_2\left(\frac{k^2}{4\beta^2} - \frac{k^2\delta m^2}{M^2G(0)}\right) - \frac{3k^2\beta^2(1+\delta m^2/M^2)}{8M^2G(0)}\right] \qquad \dots (94)$$

where $G(0)$ is formally given by eq. (252), except for the replacement of $\hat{q}^2$ by $3\beta^2/2$. As a check, $F(k^2)$ is symmetrical in '1' '2', as well as satisfies the consistency condition $F(0)=1$. Similarly the small k^2 value of $F_1(k^2)$, after taking account of the (1,2) antisymmetry of e_q, is of minimum order k^4, so that it contributes neither to the normalization ($F_1(0)=0$), nor to the P-meson radius.

For completeness we record some numerical results for large and small k^2 limits. For the pion case, in the large k^2 limit, eqs. (80)-(81) yield after a little simplification the simple result

$$F(k^2) = C/k^2;$$

$$C \equiv 2\sqrt{2}\frac{M_>^2}{G(0)}(\beta^2 + m_q^2)e^{-M_>^2/8\beta^2} \qquad \dots (95)$$

where m_q=265 MeV stands for $m_1=m_2$; and $M_>$ stands for the bigger of m_1+m_2 and M. Substituting for $\beta^2=0.0602GeV^2$ (ref. 32) and $G(0)=0.166GeV^2$, yields the result $C=0.35GeV^2$, vs the expt value of 0.50 ± 0.10[58b]. For comparison, we also list the perturbative QCD value[60] of $8\pi\alpha_s f\pi^2=0.296GeV^2$, with $f_\pi=133MeV$, and the argument Q^2 of α_s taken as $M_>^2$.

For low k^2, eqs. (80)-(81) yield values of the pion and kaon radii, in accordance with the relation $<R^2>=-\nabla_k^2 F(k^2)$ in the $k^2=0$ limit. Substitution of numerical values from (57)-(58) yields

$$R_K = 0.629 fm(vs: 0.5 - expt^{58a});$$
$$R_\pi = 0.661 fm(vs: 0.656 - expt^{58a}) \qquad \dots (96)$$

We end this Section with the remark that a simple-minded, conventional NP approach[38,40] to BS dynamics had already produced most of the results of this form factor calculation, but had been criticized[34] on grounds of 'non-covariance'. The CNPA with an explicit formulation of the Transversality Principle (TP) on a *covariant* null[10-8] plane (NP), hopefully, keeps both the advantages, since the 4D loop integrals are now not only perfectly well-defined, but a major part of the n_μ dependence has got eliminated in the process of $\hat{q}$ integration, while the remaining NP orientation dependence has been transferred to the external (hadron) 4-momenta. In this regard the present approach is already in the company of a wider NP (LF) community[37-38] which has also to contend with some n_μ dependence. The solution offered here to overcome this problem is a simple prescription of 'Lorentz completion' wherein a 'collinear frame' ansatz $P_\perp \cdot q_\perp = 0$ is lifted on the external hadron momenta P, P' etc, *after* doing the internal $\hat{q}$ integration, so as to yield an explicitly Lorentz-invariant result. The prescription, though different from other LF approaches[38], is nevertheless self-consistent, at least for 3-point hadron vertices, (and amenable to extension to higher-point vertices provided the latter can be expressed as a combination of simpler 3-point vertices). (It may be added parenthetically that the old-fashioned NP treatment[33c] had yielded a slightly better curve for the pion form factor, but this was due to the use of the "half-off-shell" form of the NP wave function[40], which however did not come out naturally from the present 'covariant' treatment).

6 Three-Hadron Couplings Via Triangle-Loops

For a large class of hadronic processes like $H \rightarrow H' + H''$ and $H \rightarrow H' + \gamma$, the quark triangle loop[31] represents the lowest order "tree" diagram for their evaluation. Criss-cross gluonic exchanges inside the triangle (see Fig. 1of ref. [31]) are not important for this kind of description in which the hadron-quark vertices, as well as the quark propagators are *both non-perturbative*, and thus take up a lion's shaie of non-perturbative effects. This is somewhat similar to the "dynamical perturbation theory" of Pagels-Stokar[64] in which such criss-cross diagrams are neglected.

In this Section, we shall give an outline of the calculational techniques for such diagrams for the most general case of unequal mass kinematics $m_1 \neq m_2 \neq m_3$, but with spinless quarks only, since the 'spin' d.o.f. does not introduce any new singlularities over the spin-0 case. In this we shall closely follow the method of ref. [31], which is a 3-hadron generalization of Sec. 5 for the e.m. form factor of a pseudoscalar meson. However, as already noted therein, the CIA form[16] of 3D-4D BSE is fraught with problems of ill-defined integrals (and hence complexity of amplitudes) due to the presence of time-like momentum components[25] in the (gaussian) wave functions of the participating hadrons. So we shall work only with CNPA[41] structures, as derived in Sec. 4.2.

6.1 Kinematical Preliminaries

According to Fig. 1 of ref. [31], and in the same notation, the 3 hadrons with all incoming 4-momenta P_I, with masses M_I, interact via the quark triangle loop wherein P_k dissociates into the quark pair with 4-momenta $(-p_i, p_j)$ and masses (m_i, m_j) respectively, so that $P_k = -p_i + p_j$, and $P_k + P_i + P_j = 0$. Thus[31]:

$$-P_k P_i + P_j \equiv P_{ij}\ ;\ P_k^2 = -M_k^2\ ;\ P_k = -P_{ij} = p_j - p_i \quad \dots (97)$$

where (i, j, k) are cyclic permutations of 1, 2, 3). Similarly the relative 4-momenta q_k between quarks i,j corresponding to the break-up $P_k = p_j - p_I$, and Q_k between hadrons i,j for the break-up $P_k = -P_i - P_j$ are:

$$q_k = \hat{\mu}_{ij} p_j + \hat{\mu}_{ji} p_i\ ;\ Q_k = \hat{m}_{ij} P_j - \hat{m}_{ji} P_i\ ;$$
$$-P_{kj} = P_i + P_j \quad \dots (98)$$

The fractional momenta $\hat{\mu}_{ij}$ at the quark level, and $\hat{m}_{ij}$ at the hadron level, are given by the Wightman-Gaerding[56] definitions

$$2\hat{\mu}_{ij} = 1 + \frac{m_i^2 - m_j^2}{M_k^2};\ 2\hat{m}_{ij} = 1 + \frac{M_i^2 - M_j^2}{M_k^2} \quad \dots (99)$$

The relative sigs are determined by the phase convention of Fig. 1 of ref. [31].

Now to define the 'hatted' relative 3-momenta $\hat{q}_k$ and $\hat{Q}_k$, we must follow the CNPA procedure[41] instead of CIA[31]. Further, since the content of CNPA is for all practical purposes identical with that of the old-fashioned NPA[40], considerable simplification is achieved by adopting the latter notation[40], which is what is already done in ref. [31], albeit with CIA content. Indeed, with the collinear ansatz (Sec. 4) the NPA values of $\hat{q}_i$ are simply $q_{i\perp}, q_{i3}$, where $q_{i3} = M_i q_{i+} / P_{i+}$, etc[40]: and $Q_{i3} = M_i Q_{i+} / P_{i+}$. However, since the q_I's are not all independent, it is necessary to take a basis momentum (say p_2) in terms of which to express others. Now in a fixed p_I basis, we have

$$q_k = p_i + \hat{\mu}_{ij} P_k\ ;\ q_j = p_j - \hat{\mu}_{ik} P_j\ ;$$
$$q_i = p_i - \hat{\mu}_{jk} P_j + \hat{\mu}_{kj} P_k \quad \dots (100)$$

For later purposes we shall consider a p_2 basis, for which

$$q_1 = p_2 + \hat{\mu}_{23} P_1\ ;\ q_3 = p_2 - \hat{\mu}_{21} P_3 \quad \dots (101)$$

We also record some useful results for the kinematics of external particles, if they are on-shell $(Q_i \cdot P_i = 0)$, under the collinearity condition[31]:

$$P_{i\pm} = \pm \frac{M_i}{Q_i} Q_{i\pm}\ ;\quad M_i^2 = P_{i+} P_{i+}\ ;$$

$$Q_i^2 = -Q_{i+} Q_{i-} = \frac{\lambda(M_1^2, M_2^2, M_3^2)}{4M_i^2} \quad \dots (102)$$

which lead to the further symmetry relations

$$Q_1 M_1 = Q_2 M_2 = Q_3 M_3 = \sqrt{\lambda(M_1^2, M_2^2, M_3^2)/4} \quad \dots (1\text{-}3)$$

Further we can define a 3×3 matrix structure n_{ij}, with $n_{ij} \equiv P_{i+} / P_{j+}$, which satisfy the relations

$$n_{ij} n_{ji} = 1;\quad n_{12} n_{23} n_{31} + 1;\quad \Sigma_i\, n_{i1} = 0. \quad \dots (104)$$

showing a definite phase relation among these quantities, which are more explicitly expressed by the matrix structure

$$[n_{ij}]=1;\ -\hat{m}_{13}\pm Q_2/M_2;-\hat{m}_{12}\mp Q_3/M_3 \quad \dots (105)$$
$$-\hat{m}_{23}\mp Q_1/M_1;\ 1;\ -\hat{m}_{21}\pm Q_3/M_3 \dots (106)$$
$$-\hat{m}_{32}\pm Q_1/M_1;\ -\hat{m}_{31}\mp Q_2/M_2;\ 1 \quad \dots (107)$$

with a two-fold sign ambiguity expressed by the statement that only the upper, or only the lower signs, must be taken. It is easily verified that eqs. (104) (hence phase relations) are satisfied by the matrix (105)-(107).

6.2 *Structure of HHH Form Factor*

The full structure of the 3-hadron amplitude due to Fig. 1 of ref. [31] is

$$A(3H)=\frac{2i}{\sqrt{3}}(2\pi)^8\int d^4p_i\Pi_{123}\frac{\Gamma_i(\hat{q}_i)}{\Delta_i(p_i)} \quad \dots (108)$$

exhibiting cyclic symmetry, where the normalized vertex function Γ_l in CNPA[41] is given from Sec. 4-5 as

$$\Gamma_i(\hat{q}_i)=N_i(2\pi)^{-5/2}D_i(\hat{q}_i)\phi_i(\hat{q}_i);$$
$$D_i=2M_i\left(\hat{q}_i^2-\frac{\lambda(M_i^2,m_j^2,m_k^2)}{4M_i^2}\right) \quad \dots (109)$$

where we have defined the 'reduced' denominator function D_l as $D_{i+}M_i/P_{i+}$ and written the (invariant) normalizer N_{iH} as N_i. The color factor and the effect of reversing the loop direction are given by $2/\sqrt{3}$, while $(2\pi)^8$ is the overall BS normalizer[40]. $\Delta_i=m_i^2+p_i^2=\omega_{i\perp}^2-p_{i+}p_{i-}$. Spin and flavour d.o.f. will give rise to a standard 'trace' factor[31] $[TR]$ which is skipped here for simplicity.

To evaluate (106), we first write the cyclically invariant measure:

$$d^4p_i=d_\perp^p\frac{1}{2}d(x_i^2)M_i^2dy_i;\ x_i=p_{i+}/P_{i+};$$
$$y_i=p_{i-}/P_{i-} \quad \dots (110)$$

The cyclic invariance of this quantity ensures that it is enough to take any index, say 2, and first do the pole integration w.r.t. the y_2 variable which has a pole at $y_2=\xi_2\equiv\omega_{2\perp}^2/(M_2^2x_2)$. The process can be repeated, by turn, over all the indices and the results added. Note that the ϕ-functions do *not* include the time-like y_i variables under CNPA[41], so that the residues from the poles arise from only the propagators. The crucial thing to note is that the denominator functions D_1 and D_3 sitting at the opposite ends of the p_2-line in Fig. 1 of ref. [31] will *cancel out* the residues from the complementary (inverse) propagators Δ_3 and Δ_1 respectively. Indeed by substituting the pole value $y_2=\xi_2$, in $\Delta_{1,3}$, the corresponding residues in an obvious notation work out as:

$$\Delta_{1;2}=\xi_2n_{32}M_2^2+x_2n_{23}M_3^2-2\hat{\mu}_{21}M_3^2;$$
$$\Delta_{3;2}=-\xi_2n_{12}M_2^2-x_2n_{21}M_1^2-2\hat{\mu}_{23}M_1^2 \quad \dots (111)$$

It is then found, with a short calculation using (101), that

$$\frac{D_3(\hat{q}_3)}{\Delta_{1;2}}=2M_3x_2n_{23};\ \frac{D_1(\hat{q}_1)}{\Delta_{3;2}}=2M_1x_2n_{21} \quad \dots (112)$$

which shows the precise cancellation mechanism between the D_i-functions and the residues of the propagators Δ_i at the Δ_2 pole. This mechanism thus eliminates[16] the (overlapping) Landau-Cutkowsky poles that would otherwise have caused free propagation of quarks in the loops. The same procedure is then repeated cyclically for the other two terms arising from the $\Delta_{3,1}$ poles. Collecting the factors, the result of all the 3 contributions is compactly expressible as (c.f.[31]):

$$A(3H)=8\sqrt{\frac{2\pi}{3}}\Sigma_{123}\iint M_2n_{23}n_{21}\pi^2dx_2d\xi_2x_2^2$$
$$\times[TR]_2D_2(\hat{q}_2)\Pi_{123}M_iN_i\phi_i \quad \dots (113)$$

where the limits of integration for both variables are $-\inf<(\xi_2,x_2)<+\inf$, since these are governed, not by the on-shell dynamics of standard LF methods ref. [37-38], but by off-shell 3D-4D BSE. The difference from ref. [31] (under CIA[16]) arises from using CNPA[41] here.

6.3 *Discussion on Applicability*

Eq. (113) is the central result of this Section. Its general nature stems from the use of unequal mass kinematics at both the quark and hadron levels, which greatly enhances its applicability to a wide class of problems which involve 3-hadron couplings, either as complete process by themselves (such as in decay processes) or as part of bigger diagrams in which 3-hadron couplings serve as basic building blocks. What makes the formula particularly useful for general applications is its explicit Lorentz invariance which has been achieved through the simple method of 'Lorentz Completion' on the lines of Sec. 5 for the e.m. form factor of P-mesons (pion).

How much of this derivation is model independent, except for the use of the MYTP[15]? The answer lies in the structure $\Gamma_H = D\times\phi$ for the hadron-quark vertex function, which is a direct consequence of the 3D support ansatz which in turn receives support from several angles[17,19], although this specific form[16] does not seem to have been used elsewhere. Its factorable structure in which the denominator function D is quite universal and depends only on kinematics, has helped reduce the 4D loop-integral to a 3D form, and in so doing, has succeeded in eliminating the Landau-Cutkowsky (overlapping) singularities in a very simple and transparent manner, thus preventing the free propagation of quarks in the intermediate (loop) stages. (The only model dependent entity in the 3D wave function ϕ, but it has been related to the (observable) spectra[13].

In the spirit of this generality, this article was not intended for specific applications per se, but some possibilities are readily listed. The simplest class is that of strong decay of a resonance (H_1) into two lighter hadrons (H_1, H_2) under kinematically allowed conditions (whose signature is carried by the external variables Q_I, M_I inside the integral (113). The amplitude $A(3H)$ can also be adapted, via Sec. 5, to include e.m. or semi-leptonic processes, expressed by $H_1 \to H_2+\gamma$, (in which γ is real or virtual), where the signature of virtuality is carried by $M_3^2 \equiv t$. Non-leptonic weak decays (see Fig. 2 of ref. [31]) are also amenable to this treatment. As an example one may cite the experimental discrepancy[65] of the vector form factors in the semi-leptonic process $D \to K^* e\bar{\nu}$ with theoretical models that prefer to represent the intermediate states through effective meson propagators[66]. On the other hand, a crucial role of the appropriate quark-triangle, with a considerable effect of the unequal masses of the participating quarks, seemed to be strongly indicated in resolving the discrepancy[67]. Other applications include the so-called Sullivan process[68], details of which the interested reader may find in ref. [69].

7 Two-Quark Loops: SU(2)-Breaking Problems

To illustrate other applications of the 3D-4D BSE formalism, we now turn to the (simpler) problem of two-quark loops which are useful for estimating SU(2)-breaking effects in phenomena like i) mass-splittings in P-mesons[18a], and ii) $\rho-\omega$ mixing in meson-exchange forces[18b]. Simpler 2-quark loops, such as those involved in the weak and e.m. decay constants of hadrons, are already available in previous studies of this formalism[40], and will not be the subject of this semi-review. Further, its scope does not include detailed analysis of these 2-loop phenomena[18], but only their essential physics, and a quick derivation of their core structures, leaving the reader to ref [18] for numerical results plus more references.

7.1 Strong SU(2)-Breaking in P*-meson Multiplets*

To recaptulate the essential physics of hadronic mass splittings within $SU(2)$ multiplets (I=0.5, 1.0), these were for long thought to be of e.m. origin, until the advent of QCD[22] when the possibility of strong breaking of $SU(2)$ due to the intrinsic u–d mass difference started being taken seriously. (This was despite the prior existence of theGMOR-mechanism[70] which had sought to relate the pseudoscalar masses to the current quark masses and the vacuum condensates). In this respect the trend was set largely by Weinberg's analysis[71], characterized by the 'Weinberg ratios' $m_u/m_d = 0.55$, and $m_s/m_{ud} = 20.1$, confirmed by a recent analysis[72]. A conservative estimate to the u-d mass difference is believed to be d-u= 3-4MeV[71-72]. On the other hand the absolute values of the current masses are not as well known, but the $SU(2)$ mass splitting[13] among the known pseudoascalar multiplets (π, K, D, B) is a useful mathematical laboratory to determine the d-u mass difference from the corresponding 'constituent mass' difference, via Politzer additivity[55]. The issue is basically a dynamical one (in view of the sensitive nature of this laboratory), necessitating a high degree of parametric control on the strong vertex functions involved in the concerned Feynmam diagrams (Figs. 1(a,b,c) of ref. [18a]). The problem clearly goes beyond mere additivity in the quark masses, as the observed pattern of mass splittings[13] seems to suggest a basically decreasing trend from the lightest (pion) to the heaviest (beauty) flavour, tapering off almost to zero for the $B_0 - B_+$ mass difference.

For the meson self-energy, there are 3 basic contributions, a 1a Figs. 1(a,b,c) of (ref. [18a]): i) In Fig. 1a, a 2-point δm_{ud} vertex inserted at each propagator by turn represents the principal source of strong $SU(2)$ breaking; ii) Fig. 1b simulates the e.m. breaking effect by joining the two quark lines internally by a photon propagator; iii) Finally Fig. 1c simulates the effect of the difference of the quark condensates $<u\bar{u}>$ and $<d\bar{d}>$ on the strong $SU(2)$ breaking of hadron masses. Figs. 1(a,c) are one loop diagrams, while Fig. 1b represents a *two-loop* process, which is moreover sensitive to QED gauge constraints[63], as in the e.m. form factor case (see Sec. 5).

Using the dynamical framework collected in Sec. 4. (2-3), it is fairly straightforward to write down the integrals accruing from these diagrams. Both CIA[16] and CNPA[41] are valid mechanisms for evaluating these diagrams, but in view of a prior exposure[18a] of CIA for this problem, it may be instructive to adopt the CNPA alternative here. Thus Fig. 1a gives in terms of the results of the previous sections,

$$\Pi_a(M^2) = i(2\pi)^{-1} N_H^2 \int d^4q D^2(\hat{q})\phi^2(\hat{q}) Tr \times [\gamma_5 S_F(\hat{m}_1 P + q)\gamma_5 S_F(-\hat{m}_2 P + q)] \quad \ldots (114)$$

where we have used the representation of the normalized vertex function given in eq. (107), and D is the reduced denominator fn in CNPA. After evaluating the traces this expression simplifies to

$$\Pi_a(M^2) = -2i(2\pi)^{-1} N_H^2 \int d^4q D^2(\hat{q})\phi^2(\hat{q}) \times \frac{\Delta_1 + \Delta_2 + M^2 - \delta m^2}{\Delta_1 \Delta_2} \quad \ldots (115)$$

where $\delta m = m_1 - m_2$, and all kinematical quantities are as defined in Sec. 4. The integration over the time-like component q_- is carried out very simply, using the result (79). Note that the vertex function $D\times\phi$ does not involve this variable, and also the D-fn exactly cancels out the residues arising from the propagators, as shown generally in Sec. 6. The resultant 3D integration over $d^3\hat{q}$ is expressible simply as

$$\Pi_a(M^2) = 2N_H^2 \int d^3\hat{q}\phi^2(\hat{q})D(\hat{q}) \times \left[\frac{D(\hat{q})}{4Mx_2} + \frac{D(\hat{q})}{4Mx_1} + M^2 - \delta m^2\right] \quad \ldots (116)$$

where $x_i = p_{i+}/P_+ = \hat{m}_i \pm x$ for i=1,2 respectively. The third component q_3 of CNPA[40-41] is simply Mx, so that $\hat{q} = q_\perp, q_3$. The normalizer N_H is given by eq. (80). The parallel CIA result is (ref. [18a])

$$\Pi_a(M^2) = 2N_H^2 \int d^3\hat{q}\left[D^2(\hat{q})\left(\frac{1}{2\omega_1} + \frac{1}{2\omega_2}\right) + D(\hat{q})(M^2 - \delta m^2)\right] \quad \ldots (117)$$

A comparison between the CNPA and CIA forms of Π_a is now in order. In CIA, eq. (117), the D^2 term is well defined and is amenable to simple quardature. On the other hand, the CNPA form, eq. (116), encounters singularities at $x_{1,2}=0$, on integration w.r.t. x, taking account of the relations $x_{1,2} = \hat{m}_{1,2} - x$, and $\hat{q}^2 = q_\perp^2 + M^2x^2$. The final results are quite similar for both cases.

The formulae (116)-(117) for Π_a and (80) for N_H, show explicit dependence on the masses $m_{1,2}$, and facilitate the evaluation of mass splittings within the SU(2) isospin multiplets as follows: For the K, D, B mesons, take m_2 as the mass of the ud-quark with $m_1 > m_2$, and while differentiating w.r.t. m_2, consider the increment δ_c. A little reflection then shows (by virtue of the Politzer[55] additivity relation) that this quantity may be directly identified with the difference $m_{dc} - m_{uc}$ between the current d- and u-masses provided the hadron mass with the u-quark gets subtracted from that with the d-quark (e.g., $K_0 - K_-$, etc). Of course the normal rules of differentiation apply, viz., $\delta f(m_2) = f'(m_2)\delta_c$, where the argument of $f'(m_2)$ must use the average 'constituent mass' of ud-quark, viz.,265 McV, eq. (58).

For the pion case, some extra care is necessary since both the constituents are now *u/d*-quarks so that both m_1 and m_2 must be subjected to differentiation in turn. On the other hand these two contributions come with just equal but opposite signs, so that they cancel out exactly, giving a net vanishing contribution, as seen more directly from the fact that the ΔI=1 and $u\bar{u}-d\bar{d}$ in the Lagrangian cannot contribute to π_+-π_0 anyway. For the details of numerical results on $\delta\Pi(M^2)$, (see ref. [18a]).

7.2 E M. Contribution to Self-Energy

The e.m. contribution to the hadron self-energy is given by 2-loop diagram (Fig. 1b) of ref.[18a] in a slightly simplified notation as follows:

$$\Pi_b(M^2) = N_H^2 e^2 e_1 e_2 \int\int d^4q d^4q' \frac{D\phi D'\phi'}{(2\pi)^5 k^2}$$
$$\times Tr[\gamma_5 (S_F(p_1) i\gamma_\mu S_F(p_1')\gamma_5 S_F(-p_2')$$
$$\times i\gamma_\mu S_F(-p_2)] \qquad \ldots (118)$$

where $k=q-q'=p_1-p'_1=p'_2-p_2$ is the exchanged quantum; e_1 and e_2 are the charges of the quarks involved (in units of e) and $q.q'$ are the internal 4-momenta of the LHS and RHS hadrons respectively. This integral involves simultaneous (pole) integrations over the time-like components of q and q' which do not figure in the respective vertex functions and therefore can be carried out exactly. However the rest of the 3D integrations (twosets) do not quite factor out, so they need strategy before they can be carried out without much tears. To that end a simple device that suggests itself naturally is based on the following observation: By the very topology of the diagram it is fairly clear that the time-like components of both the 4-vectors q and q' are quantitatively similar, so that their effects largely "cancel out" in the factor k^{-2} in eq. (118). As a results the quantity $k=p-q'$ effectively reduces to the space-like quantity $(\hat{q}-\hat{q}')^2$ which can be manipulated to desired numerical accuracy in the 3D integrations over ($\hat{q}$ and $\hat{q}'$). We list both CIA[18a] and CNPA (new) results in the form of 3D integrals in q, q' jointly as follows.

$$\Pi_b(M^2) = 4N_H^2 e^2 e_2 (2\pi)^{-3} \int\int d^3\hat{q} d^3\hat{q}' [\ldots]$$
$$\times \frac{\phi(\hat{q})\phi(\hat{q}')}{(\hat{q}-\hat{q}')^2}; \qquad \ldots (119)$$

the quantity [...] is first listed for CIA as follows[18a]:

$$[\ldots]_{CIA} = (M^2 - \delta m^2)^2 - 2m_1 m_2 (M^2 - m_{12}^2)$$
$$- \delta m^2 (\hat{q}-\hat{q}')^2 + \left(\frac{1}{2m_1} + \frac{1}{2m_2}\right)$$
$$\times \left(\frac{1}{2m_1'} + \frac{1}{2m_2'}\right) D(\hat{q}) D(\hat{q}')$$
$$+ (M^2 - m_1^2 - 2m_2^2 + m_1 m_2) D(\hat{q}) / \omega_2$$
$$+ (M^2 - m_2^2 - 2m_1^2 + m_1 m_2) D(\hat{q}) / \omega_1; \qquad \ldots (120)$$

The dual quantity $[\ldots]_{CNPA}$ may be simple read from the above merely by the replacements $\omega_{1,2} \to 2(Mx_1, Mx_2)$ respectively, where $x_{1,2} = \hat{m}_{1,2} \pm x$.

To convert the mass shifts from quadratic to linear, it is of course necessary to divide both $\Pi_{a,b}$ in the above equations by $2M$. In the e.m. case, no further 'differentiation' w.r.t. m_2 is necessary, since ref. [119] is already of second order in e. As regards the factor $e_1 e_2$, its differential $\delta(e_1 e_2)$ is easily found as +(1/2), −(1/3), +(2/3) and −(1/3) for the differences $\pi_+-\pi_0$, K^0, D^+-D^0, B^0-B^-) in this order. It turns out[18a] that this alternating sign pattern is of great help in reinforcing and reducing the net differences within the *K, D, B* multiplets (after taking account of the strong breaking effects, Figs 1a and 1c), towards a good pattern of accord[18a] with the data[13]. (For results, see ref. [18a]).

We now consider QED gauge corrections[63] to the e.m. value, eq. (119), arising from Fig. 1b of ref. [32a], on the lines of corresponding corrections to the e.m. form factor derived in Sec. 5. This correction is sketched in *Appendix B* for P-mesons, using diagrams listed in ref. [63], and in their notation for the contributing figures. The resulting QED correction for the kaon e.m. mass difference turns out to be nearly a 60 per cent increase over the CIA result 1.032 MeV[18a] arising from the main term (119). We omit the corresponding CNPA treatment for brevity.

7.3 Effect of Quark Condensates

Another source of mass splittings arises from the difference between *u/d*-quark condensates, in accordance with Fig. (1c) of ref. [18a]. Indeed some recent calculations via QCD-sum rules have used this as the principal mechanism[73] for the mass splittings, with much less contribution from Fig. (1a). Indeed the value of $\delta < q\bar{q} >$ in itself has been subject of separate investigations in chiral perturbation theory[74] as well as in QCD-sum rules[75]. On the other hand, the BSE-SDE formation provides a 'direct' ab initio estimate[11,23] of this condensate (as well as others[30]).

To recapitulate the logic of the condensate calculation by the 'direct' method[11] in terms of the quark's *non-perturbative* mass function, $m(p)$, note that the latter is the chiral (M_π=0) limit of the pion-quark vertex function $\Gamma(\hat{q})$, given by eq. (67), and must be used in the expression of the full propagator, $S_F(p)$, Sec. 7, which appears in the formal definition of the condensate as follows[11]:

$$< \bar{q}q >= \frac{iN_c N_f}{(2\pi)^4} Tr\left[\int d^4 p S_F(p)\right]$$

$$= -\frac{3}{4\pi^3}\int d^3\hat{p}\frac{m(\hat{p})}{\sqrt{\hat{p}^2 + m^2(\hat{p})}} \qquad \ldots (121)$$

after doing the pole-integration over the time-like component of p_μ. Here N_c=3, and N_f=1 (since each separate flavour (u/d) is counted). Now to evaluate the 3D integral (121) substitute the CIA structure (70) for $m(\hat{p})$, and $\phi(\hat{p}) = \exp(-\hat{p}^2/2\beta^2)$. This integral formula has an analytic structure in terms of the constituent mass m_q of the u/d-quark, so that it is now a matter of simple differentiation to give an explicit form of its increment w.r.t. δm_q which equals δ_c. The final formula is[18a]:

$$\delta < \bar{q}q >= \frac{-3\delta_c}{\pi^2 m_q}\int dk k^4 \phi_\pi(k)\omega(k)\left[1 - \frac{2k^2}{m_q^2}\right]$$
$$\times(m^2(k)+k^2)^{-3/2} \qquad \ldots (122)$$

Using the inputs from eqs. (55-58) gives β^2=0.0603, and the final results under CIA are

$$< \bar{q}q >= -(266MeV)^3;\ \delta < \bar{q}q >= +0.0664\delta_c \qquad \ldots (123)$$

These values are fully rooted in spectroscopy but are otherwise free from adjustable parameters, except for the quantity δ_c. They have a fair overlap with QCD-SR determinations[76].

For completeness we now give the condensate results under CNPA, substituting the CNPA mass function (69) in (121). This gives

$$< \bar{q}q >= \frac{12i\sqrt{2}}{(2\pi)^4}\int d^3\hat{p}dp_n \frac{p.n\left[1 + \frac{\hat{p}^2}{m_q^2}\right]\phi(\hat{p})}{m_q^2 + p_\perp^2 - 2p.np_n} \qquad \ldots (124)$$

The integration over p_n is trivial and yields

$$< \bar{q}q >= \frac{-3\sqrt{2}}{(2\pi)^3}\int d^3\hat{p}\left[1 + \frac{\hat{p}^2}{m_q^2}\right]\phi(\hat{p}) = -(242McV)^3 \qquad \ldots (125)$$

Substituting the gaussian form (as above) for ϕ and integrating, yields an analytic structure useful for calculating $\delta < \bar{q}q >$:

$$< \bar{q}q >= -3\sqrt{2}(\beta^2/2\pi)^{3/2}[1 + 3\beta^2/m_q^2]$$
$$=-(242M\,\mathrm{eV})^3 \qquad \ldots (126)$$

a value which seems to be even closer to the estimate $-(240)^3$ of QCD-SR[2] than the CIA result $-(266)^3$ of (ref. [18a]

As to the contribution of $\delta < \bar{q}q >$ to the strong SU(2) mass splittings, a 1a Fig. 1c of ref. [18a], we skip the detailed derivation in favour of ref. [18a], since it turns out to be rather small within this BSE framework. This in sharp contrast to the QCD-SR findings[73] wherein the condensate contribution seems to dominate. This is not too surprising since within a BSE-cum-SDE framework, most of the non-perturbative effects are already contained in the hadron-quark vertex function, with a correspondingly smaller role for the condensates. On the other hand in QCD-SR[2] these represent major non-perturbative effects when seen from the high energy perturbative QCD end.

A few comments on the main results of this exercise are in order. The e.m. contributions alternate in sign in the mass splittings between the charged and neutral components in the sequence π, K, D, B. The condensate contribution to strong SU(2)-breaking being small, the sensitivity to the d-u mass difference comes almost entirely from Fig. 1(a) of ref. [18a] Next, the feature of unequal mass kinematics has played a big role in the formalism, being mainly responsible for a sustematic decrease in mass splittings as one goes up on the mass scale. This aspect has come about mainly from the properties of the *D*-functions (mostly model independent). The numerical values show a good overall pattern of agreement with data[13], (within less than half *MeV*), for the parameter δ_c appears well within the phenomenological limits of acceptability[72]. However, as the results of *Appendix B* on QED gauge corrections indicate, inclusion of these tends of decrease the effective value of δ_c. Finally the calculational technique seems to conform to the spirit of "Dynamical Perturbation Theory' of Pagesls-Stoker[64] (neglect of "criss-cross' diagrams) which must be carefully distinguished from a naive interpration of perturbative QCD.

7.4 Off-Shell ρ-ω Mixing

Before concluding this Section, we shall briefly draw attention to a similar SU(2)-breaking phenomenon which has proved to be of considerable interest for the understanding of certain anomalies in nuclear forces[77]: off-shell ρ-ω mixing. Although nuclear topics are not of direct concern for his article, the basic logic of charge-symmetry-breaking (CSB) to explain the Nolen-Schiffer anomally[77] via ρ-ω mixing[78], stimulated by new experiment[79] on polarized *n-p* scattering, comes directly under the theme of this Section. Indeed, the sensitivity of ρ-ω mixing to the *d-u* mass difference δ_c especially off-shell[78a,18b], is as strong as that of *P*-meson masses[18a].

To recall the basic logic, the small difference between the proton vs neutron analyzing powers at an angle θ_0 corresponding to the vanishing of the average analyzing power[79], is proportional to the CSB potential V_{CSB} whose contribution from ρ-ω mixing may be schematically expressed as[78a]

$$V_{CSB}^{\rho-\omega} = < NN|H_{\text{int}}|NN\omega > G_0 < \omega|H_{CSB}|\rho^0 > \times G_0 < \rho^0 NN|H_{\text{int}}|NN > + (\rho^0 \Leftrightarrow \omega) \quad \ldots (127)$$

Here G_0 is the appropriate *V*-meson propagator, and $< \omega|H_{CSB}|\rho^0 >$ gets its dominant theoretical contribution from the *d-u* mass difference δ_c, with $H_{CSB} = \rho.\omega\delta_c^2$, and a partial contribution from the e.m. chain $\rho \Rightarrow \gamma \Rightarrow \omega$ via vector dominance and/or 2-quark loops. Alternatively, the matrix element can be estimated from the experimental $e^+e^- \Rightarrow \pi^+\pi^-$ amolitude at the ω-pole, which gives the on-shell value $\theta(M^2)$ of the ρ-ω mixing amplitude[80]. On the other hand, it is its off-shell value $\theta(q^2)$ which is relevant to the CSB potential, eq. (127), for the *V*-meson exchange in a space-like region where its effect on V_{CSB} has been claimed to be greatly suppressed[78a]. This question in turn requires a theoretical model for the necessary extrapolation which can be defined in terms of a dimensionless parameter λ as[18e]

$$\theta(q^2) = \theta(M^2)[1-(1+q^2/M^2)\lambda] \quad \ldots (128)$$

A calculation of this parameter λ is the central issue of any investigation of the CSB effect, wherein its value has been variously estimated to be within the (0-1) range[78]. In particular, the function of $\theta(q^2)$ is also amenable to the 3D-4D formalism[16], using the self-energy techniques[18b] outlined in this Section. Its on-shell value $\theta(M^2)$[18b] agrees with the data[80], while the off-shell parameter λ comes close to unity, signifiying a change of sign for $\theta(q^2)$ in the transition region between the space-like and time-like momenta, in agreement with a 'nucleonic' self-energy calculation[78d].

8 QCD Parameters from Hadron Spectroscopy

In this Section, we outline a simple method of calculation[30] of QCD condensates in terms of the (spectroscopy-oriented) parameters of the 3D-4D BSE framework. These parameters of QCD simulate non-perturbative effects as coefficients in Wilson's, operator product expansions (*OPE*)[81,55]. The method of QCD sum rules represented the first practical attempt[2a] to relate these quark-gluon quantities to hadronic amplitudes by employing a duality principle[20] between the quark-gluon and meson-baryon pictures. Basically the idea is to find a Q^2 region (≈ 1 GeV^2) where one may incorporate non-perturbative physics, generated via *OPE*[81], into the perturbative QCD treatment of physical processes involving hadrons. The QCD-SR ansatz[2] for the evaluation of a certain correlation function

$\Pi(p)$, is to replace the free quark (or gluon) propagator by one more suitable for the nontrivial vacuum, and on the other hand to express, via dispersion relations, the same correlation function in terms of the variables of the meson-baryon picture. The two results are then equated to yield sum rules connecting the variables of the two physical descriptions.

8.1 Field-Theoretic Definition of Condensates

While QCD-SR per se[2,82] is not the subject of this review, its basic building blocks the condensates, are the main concern of this Section. These may be defined in terms of quark- and gluon-fields[82,30].

$$<\bar{q}O_i q> = \sum_{q,j} <0|:\bar{q}_j^a(0)O_i q_j^a(0):|0>$$

$$= -\int \frac{d^4 p}{(2\pi)^4} Tr\tilde{S}_F^A(p)O_i\,, \qquad \ldots (129)$$

where O_i is an operator representing the nature of condensate, the index A represents the effect of a background field, and $\tilde{S}_F^A(p)$ is the quark propagator with the perturbative part suitably subtracted. At this stage, we must distinguish between the gluonic background field and other external ones (electromagnetic, axial, etc.): The latter can be taken perturbatively, but the former, with its characteristic problem of color gauge invariance, must be addressed more fully, a subject on which there exists a vast literature[83]. However it is possible to incorporate in practice a major fraction of this effect through the simple device of changing the variable of integration in eq (129) from p_μ to $\Pi_\mu = p_\mu - \frac{1}{2}g_s\lambda^a G_\mu^a$, where G_μ^a is the gluon field. This would in general not be possible if one were to evaluate complicated integrals involving more propagators and vertex functions, but since the integral in (129) "sees" only one such quantity, the trick should work, especially since the are mainly interested in a constant background $G_{\mu\nu}$–field, i.e. $G_\mu^a(x) = -\frac{1}{2}x_\mu G_{\mu\nu}^a$. This is basically a non-abelian adaptation of the famous Schwinger method[43] to the present situation but the details of the available method[83] are not necessary for justifying this step. With this understanding, we shall not use any additional subscript or superscript in eq. (129) to specify the gluonic background, but rather take the integration variable p_μ to represent $\Pi_\mu = p_\mu - \frac{1}{2}g_s\lambda^a G_\lambda^a$.

The principal quark condensate $<\bar{q}q>_0$ corresponds to O_i=1 and A=0. The corresponding gluon condensate is defined as

$$<g_s^2G^2> = Tr(\nabla_\mu\nabla_\nu - \delta_{\mu\nu}\nabla^2)g_s^2 D_{\mu\nu}(0), \quad \ldots (130)$$

where ∇_μ is the gauge covariant derivative and $D_{\mu\nu}(x)$ is the non-perturbative part[11] of the gluon propagator. These quantities which are free parameters in QCD-SR, provide access to the non-perturbative domain of QCD, but except for the two principal condensates $<\bar{q}q>_0$ and $<g_s^2G^2>_0$, which are amenable to cross checks against many data, the determination of the higher order ones often leave ambiguities. A partial list is[82]

$$<\bar{q}i\gamma_\mu\gamma_5 q>_A,\ <\bar{q}\sigma_{\mu\nu}q>_F,\ <\bar{q}\tfrac{1}{2}\lambda\sigma.Gq>_0,$$

$$\bar{q}\tfrac{1}{2}\lambda G_{\mu\nu}q>_F\,. \qquad \ldots (131)$$

In the method of QCD-SR[2,82], there is no intrinsic mechanism to evaluate them from first principles but only an extrinsic 'matching' between the two sides of the duality relation with the help of suitable parameters. And for condensates of still higher dimensions, additional assumptions, such as factorization, are needed. The BSE-SDE framework[9-12] on the other hand, has a more microscopic structure which gives it simultaneous access to both high and low energy phenomena under one umbrella. Thus the condensates (131) as well as others, are calculable within such a framework with as much ease as the (low energy) spectroscopy is accessible to it (See Sec. 1.4 for discussions thereof). The same facility also holds for its 3D-4D adaptation which provides a two-tier structure, with the 3D sector specifically attuned to spectroscopy, while the 4D structure is good for loop integrals, thus naturally giving rise to a spectroscopic linkage between the high and low energy descriptions of hadrons via QCD. To that end eqs (23)-(29) of the BSE-SDE interplay[82], adapted to its 3D-4D form[11], are collected in Sec. 4.4: i) an explicit expression[69] for the mass function $m(p)$ derived from the condition that it is the pion-quark vertex function in the chiral limit of M_π=0; ii) the non-perturbative gluon propagator $D(k)$, eq. (70); iii) its more general form $V(\hat{k})$, eqs (55)-(56); iv) and the formula (66) for the inverse range β of the 3d wave function ϕ. These are the main ingredients needed for the condensate calculations in this Section.

8.2 The Gluon Condensate in 3D-4D Formalism

We start by rewriting the gluon propagator in a more general form than (70) by making use of the more complete V-function, eqs (55)-(56), as under:

$$D^{ab}_{\mu\nu}(k) = \delta^{ab}\left(\delta_{\mu\nu} - \frac{k_\mu k_\nu}{k^2}\right)D(k), \qquad \ldots (132)$$

where a,b are the color indices in the adjoint representation. The logic of the connection between the $D_{\mu\nu}$-fn, and the V-fn Eqs (55), (56) comes about from its relation with the fermionic kernel K:

$$K(q,q') \Leftrightarrow \gamma_\mu D_{\mu\nu}(q-q')\gamma_\nu. \qquad \ldots (133)$$

where the scalar part $D(\hat{k})$ in the infrared region may be identified with the confining part of the K-function as[11,28a]

$$D(\hat{k}) = \frac{3}{4}(2\pi)^3\omega_0^2 2m_q\alpha_c(4m_q^2) \times\left[\frac{\nabla_k^2}{\sqrt{1-A_0m_q^2\nabla_k^2}} + \frac{C_0}{\omega_0^2}\right]\delta^3(\hat{k}). \qquad \ldots (134)$$

using the full $q-\bar{q}$ potential $V(\hat{k})$ which fits the spectroscopy for all flavours (light and heavy)[28], but specialized to the equal mass (m_q) case. The constants C_0, ω_0, A_0 are given by eqs (57), (58), while the QCD coupling constant α_s is given by[28]:

$$\alpha_s(Q^2) = \frac{4\pi}{11-\frac{2}{3}N_f}\cdot\frac{1}{ln\frac{Q^2}{\Lambda_s^2}}; \quad \Lambda_s = 200MeV. \qquad \ldots (135)$$

The coordinate representation $D(\hat{R})$ of the gluon propagator (134) is

$$D(\hat{R}) = \tfrac{3}{4}\omega_0^2.2m_q\alpha_s(Q^2)\left[\frac{C_0}{\omega_0^2} - \frac{\hat{R}^2}{\sqrt{1+A_0m_q^2\hat{R}^2}}\right]. \qquad \ldots (136)$$

Note the, in the $\hat{R}\to\infty$ limit, $D(\hat{R})$ is linear in $\hat{R}$ as well as flavor independent (the m_q-factor cancels out), except for the $\alpha_s(Q^2)$ effect. Thus the structure (136), despite its empiricity, respects the standard QCD constraint, but only in the strict confining region. On the other hand, the smallness of A_0(=0.0283) ensures that for light flavours its structure is dominated by the harmonic form, which amounts to setting A_0=0. This is an excellent approximation for the pion-vertex function in the chiral limit (M_π=0), and hence for the quark mass function given by (69)[11], and normalized to $m(0)=m_q$ and identified as the constituent mass for the ud-quarks only (ignoring their small 'current' values). The 3D wave function $\phi(\hat{q})$ is a gaussian with inverse range β given by (66), which for equal masses simplifies to

$$\phi(\hat{q}) = \exp\left(-\tfrac{1}{2}\hat{q}^2/\beta^2\right);\ \beta^4 = \frac{2m_q^2\omega_0^2\alpha_s(4m_q^2)}{1-2\alpha_s(4m_q^2)C_0}. \qquad \ldots (137)$$

For the inputs (57)-(58), β^2 works out as (0.060) GeV^2.

We shall use the mass function (69)/(137) in the next subsection for the quark condensates. Here we indicate briefly a derivation of the gluon condensate, eq. (130), by inserting the gluon propagator (136) in its definition. The C_0-term may be dropped as it will not survive the subsequent differentiations in eq. (130). For the main term, the following integral representation is employed:

$$\frac{\hat{R}^2}{\sqrt{1+A_0m_q^2\hat{R}^2}} = \frac{2m_q\sqrt{A_0}}{2\pi i}\oint dR_0\frac{\hat{R}^2}{1+A_0m_q^2R^2}, \qquad \ldots (138)$$

where $R^2 = \hat{R}^2 - R_0^2$ (Lorentz-invariant). The 4D expression $D(R)$ may now be inferred from its definition in terms of the 3D quantity $D(\hat{R})$:

$$D(R) = \frac{\alpha_s(Q^2)}{\pi}\frac{2m_q^2\omega_0^2\sqrt{A_0}\hat{R}^2}{1+A_0m_q^2R^2}. \qquad \ldots (139)$$

This is as far as one goes by adopting the 3D form (136) for $D(\hat{R})$. However, it is sufficiently suggestive of the extrapolation needed to make it fully covariant, viz. $\hat{R}^2 \to R^2$ in the numerator of eq. (138) which we adopt in what follows. (On the other hand, if this replacement it not adopted, then the resulting gluon condensate will be reduced by a factor ¾). The full propagator in the Landau gauge is (132), where k_μ is read as $k_\mu = -i\partial^R_\mu$. To evaluate the gluon condensate we first note the result:

$$\frac{\alpha_s}{\pi} < G^a_{\alpha\mu} G^b_{\beta\nu} > = \Big[-2\partial^R_\alpha \partial^R_\beta D^{ab}_{\mu\nu}(R) + 2\partial^R_\alpha \partial^R_\nu D^{ab}_{\mu\beta}(R)\Big]\Big|_{R=0}, \quad \ldots (140)$$

and obtain by straighforward differentiation

$$< g_s^2 G^2 > = \sqrt{A_0}\, 4\pi\alpha_s(4m_q^2)(6m_q\omega_0)^2/\pi^2. \quad \ldots (141)$$

The remaining question concerns what value of the quark mass m_q, i.e. what flavor, should be employed for evaluating the gluon condensate. The structure 136) does exhibit the desired features of linear confinement and flavor independence, but the extrapolation of these features in the opposite limit ($R \to 0$), as demanded by eq. (139), bring in an "effective flavor dependence" of the final formula (140). The heavier the flavour, the more important is the corresponding mass (m_q), *vis-a-vis* the A_0-term in the $q\bar{q}$ potential (136). Since, on the other hand, the full potential (136) fits all the flavor sectors rather well[28], a simple "weighting" procedure was chosen in[30], involving only the 3 flavor sectors with a nontrivial flavor mass, viz. $s\bar{s}$, $c\bar{c}$ and $b\bar{b}$ with equal weights (in the sense of a geometric mean), taking account of the m_q-dependence $m_q^2\alpha_s(4m_q^2)$ of eq. (139). This give the result

$$< m_q^2\alpha_s(4m_q^2) > 13.91\{m_u^2\alpha_s(4m_u^2)\}, \quad \ldots (142)$$

in units of its value in the (ud)-region, and its substitution in (139) yields the final estimate $< g_s^2 G^2 > = 0.502\, GeV^4$, versus the value of 0.47 GeV^4 adopted in the QCD sum rule literature[82].

8.3 $< \bar{q} Q_i q >$ Condensates

We now substitute the mass function $m(\hat{p}^2)$, eqs (69)/(137) into the general formula (129), to derive the various condensates for different choices of O_i. As already noted in Sec. 8.1 (in light of color gauge invariance), the quantity p_μ in eqs (69)/(137), and everywhere else in the following, must be read as Π_μ[43], with appropriate non-abelian corrections. The formula (129) now reads as

$$\bar{q} O_i p >_0 = Tr \frac{-i}{(2\pi)^4} \int d^4\Pi \, \frac{m(\hat{\Pi}) - i\gamma.\Pi}{m^2(\hat{\Pi}) + (\gamma.\Pi)^2} O_i \quad \ldots (143)$$

in the absence of external fields. Note that the subtracted part with $m(\hat{\Pi}) = 0$ in this equation gives no effect on tracing in the absence of external fields. We first express the denominator in an alternative form:

$$m^2(\hat{\Pi}) + (\gamma.\Pi)^2 = \hat{\omega}^2 - \Sigma_g - \Pi_l^2 \equiv \Delta - \Sigma_g; \quad \Pi^2 = \hat{\Pi}^2 - \Pi_l^2 \quad \ldots (144)$$

$$\hat{\omega}^2 = m^2(\hat{\Pi}) + \hat{\Pi}^2; \quad \Sigma_g = \tfrac{1}{2} g_s \tfrac{1}{2} \lambda^a G^a_{\mu\nu}\sigma_{\mu\nu}, \quad \ldots (145)$$

where Π_l is the longitudinal component of Π_μ, $d^4\Pi = d^3\hat{\Pi} d\Pi_l$,

and the integration must first be carried out over Π_l. Because of the presence of the Σ_g-term in (146), however, a further "rationalization" of eq. (143) is necessary according to the identity

$$\frac{1}{\Delta - \Sigma_g} \equiv \frac{1}{\hat{\omega}^2 - \Pi_l^2 - \Sigma_g} = \frac{\Delta + \Sigma_g}{\Delta^2 - \Sigma_g^2}. \quad \ldots (146)$$

At this stage it is probably adequate to replace Σ_g^2 in the denominator of (146) spin-color averaged value $< \Sigma_g^2 >$:

$$\Sigma_g^2 \to < \Sigma_g^2 > = \frac{1}{12} < g_s^2 GG > \equiv \mu^4 (= 8.48 m_q^4) \quad \ldots (147)$$

after the necessary substitutions have been made from (142) and (57)-(58). Thus $< \Sigma_g^2 >$ contains a strong signature of the gluon condensate whose large value introduces some bad analyticity properties in the denominator of the integrand in (143) or (146), for purposes of Π_l-integration, since the $\hat{\omega}^2$-term is numerically much smaller than μ^2. It has been emphasized[30] that this feature has nothing to do with the 3D-4D BSE treatment, since we have not yet passed the barrier of the orthodox 4D quark propagator in the backgrond of the gluon field. It is rather a very general minifestation of the strong spin-color effect of the quark-quark interaction via the color magnetic field. The problem is not so serious in QED[43] where the smallness of the coupling constant leaves the counterpart of the μ^2 term well below the positivity limit (i.e., $\hat{\omega}^2-\mu^2>0$), but the large value of μ^2 in the present (QCD) case tends to invalidate the standard analyticity structure of (143) for purposes of further integration with respect to $d^3\hat{\Pi}$. This problem could not be solved in ref. [30], but it seems to deserve more serious attention from a wider community. (Taken literally, it would imply the introduction of a *phase* in the condensates!) In the meantime, a conservative view[30] was taken that the maximum allowed value of $<\Sigma_g^2>$ (consistent with the positivity of the denominator after Π_l-integration) should not exceed $\hat{\omega}^4$ for all values of $\hat{\Pi}^2$, *i.e.*

$$<\Sigma_g^2>=\sigma^2\le m_q^4. \qquad \ldots(148)$$

Thus eq. (146) should be understood as

$$\frac{1}{\Delta-\Sigma_g}\Rightarrow\frac{\Delta+\Sigma_g}{\Delta^2-\sigma^2};\ \ \Delta\equiv\hat{\omega}^2-\hat{\Pi}_l^2. \qquad \ldots(149)$$

For the numerator of eq. (149) which still carries the spin-dependent quantity Σ_g, eq. (145), there is no restriction of magnitude for one Σ_g-factor only, since it contributes to condensates only after contracting with another Σ-factor in eq. (143). (However, other factors which come from the rationalization of the denominator with higher powers of Σ_g must be subject to the same restriction). With this precaution, eq. (143) serves to define two condensates simultaneously, viz., these with, $O_i=1$ and $O_i=g_s(\lambda^a/2)G^a_{\mu\nu}\sigma_{\mu\nu}$, where the latter is expressible in the notation of Ref. [82] as

$$<0|\bar{q}2\Sigma_g q|0>\equiv m_0^2<\bar{q}q>0. \qquad \ldots(150)$$

To evaluate the integral over $d\Pi_l$, we have

$$\frac{1}{2\pi i}\int d\Pi_l\frac{\Delta;\sigma}{\Delta^2-\sigma^2}\equiv[I(\sigma);J(\sigma)], \qquad \ldots(151)$$

where

$$I(\sigma);J\sigma)=\frac{1}{2}\left[\frac{1}{\sqrt{\hat{\omega}^2-\sigma}}\pm\frac{1}{\sqrt{\hat{\omega}^2+\sigma}}\right]. \qquad \ldots(152)$$

After collecting the necessary trace factors the final result for the two condensates is expressible as a simple quadrature ($q=u$ or d):

$$<q\bar{q}>_0[1;m_0^2]=\frac{3}{\pi^2}\int_0^\infty\hat{\Pi}^2d\Pi\ m(\hat{\Pi})\left[I(\sigma);\frac{2<\Sigma_g^2>}{\sigma}J(\sigma)\right]. \qquad \ldots(153)$$

On insertion of the structure (39)/(137) for the mass function, and putting the "maximum allowed value" of σ, viz., m_q^2 eq. (148), the results under CIA are in ref. [2a]

$$<q\bar{q}>_0=(266\,MeV)^3;\ \ m_0^2=0.130\,GeV^2; \qquad \ldots(154)$$

these results may be compared with the QCD-SR (input) value[82] of (240 $MeV)^3$ and 0.8 GeV^2 respectively. The corresponding CNPA[41] result, as worked out in Sec. 7.3 with $m(\hat{p})$ obtained from Sec. 4.4 is (242 $MeV)^3$ for the first item.

We next calculate three induced condensates χ, κ, and ζ, due to a constant external *e.m.* field $F_{\mu\nu}$, which are defined as[82]:

$$<\bar{q}\sigma_{\mu\nu}q>_F\equiv ee_q\chi F_{\mu\nu}<\bar{q}q>0; \qquad \ldots(155)$$

$$g_s<\bar{q}(\lambda^a/2)G^a_{\mu\nu}q>_F\equiv ee_q\kappa F_{\mu\nu}<\bar{q}q>_0; \qquad \ldots(156)$$

$$g_s<\bar{q}(\lambda^a/2)e_{\mu\nu\alpha\beta}G^a_{\alpha\beta}q>_F\equiv ee_q\zeta\,F_{\mu\nu}<\bar{q}q>_0. \qquad \ldots(157)$$

In these equations the relative phases of the induced condensates are defined with respect to the main condensate $<\bar{q}q>_0$, in accordance with (143) and this feature must be kept systematic track of. Like the two condensates (153), the quantities χ and κ are in a sense *dual* to each other, and are best described together. The *e.m.* field is introduced through the substitution

$$[\hat{m}+i\gamma.\Pi]\rightarrow[\hat{m}+i\gamma_\mu(\Pi_\mu-eA_\mu)] \quad ...(158)$$

in the propagator, eq. (129), and keeping only the first order term in A_μ. Thus we have to calculate

$$\int TrS_F(\Pi)(ie\gamma.A)S_F(\Pi)\left[\sigma_{\mu\nu};\tfrac{1}{2}\lambda^a G^a_{\mu\nu}\right] \quad ...(159)$$

This is facilitated, for a constant *e.m.* field, by the representation

$$A_\mu=-\tfrac{1}{2}x_\nu F_{\mu\nu};\ x_\mu=i\frac{\partial}{\partial\Pi_\mu}. \quad ...(160)$$

The substitution in eq. (153) and subsequent trace evaluation is routine but lengthy. However certain precautions are necessary in the matter of extraction of two groups of terms, proportional to $\sigma_{\mu\nu}$ and $G_{\mu\nu}$ respectively, before the trace evaluation, which will survive contraction with the external *e.m.* field $F_{\mu\nu}$. Thus,

$$\Pi_\mu\Pi_\nu\Rightarrow\frac{i}{2}g_s\tfrac{1}{2}\lambda_a G^a_{\mu\nu};\ \gamma_\mu\gamma_\nu\Rightarrow i\sigma_{\mu\nu}. \quad ...(161)$$

In terms like $i\sigma_{\mu\lambda}\Pi_\lambda\Pi_\nu$, additional survivors come from the symmetrized product $\{\Pi_\lambda,\Pi_\nu\}$ for which we make the standard isotropy ansatz. In this respect, their association with (space-like) magnetic effects makes it more meaningful to do an effectively 3D averaging, viz. $\Pi_\mu\Pi_\nu\Rightarrow \tfrac{1}{3}\hat{\Pi}^2(\delta_{\mu\nu}-\hat{\eta}_\mu\hat{\eta}_\nu)$ where $\hat{\eta}_\mu$ is a unit vector whose direction need not be specified too precisely. After this step, the tracing process is straightforward, and we omit the details. But a useful formula is

$$Tr\left[\tfrac{1}{2}\lambda^a G^a_{\mu\nu}g_s\Sigma_g\sigma_{\alpha\beta}F_{\alpha\beta}\right]=\tfrac{1}{3}<g_s^2G^2>F_{\mu\nu}. \quad ...(162)$$

The results for the three quantities χ, κ, ζ are[30]:

$$\chi=-3.56GeV^{-2};\ \kappa=-0.11;\ \zeta=+0.06GeV^{-2}; \quad ...(163)$$

where the QCD-SR value for χ is $[(6\pm2)GeV^{-2}]$[82]

8.4 Axial Condensates

So far there has been no explicit need to subtract the perturbative contribution ($\hat{m}=0$) to the condensates calculated above, since their traces are zero. We now consider the axial condensate ($O_i=i\,\gamma_\mu\gamma_5$) in a constant external axial field A_μ, where an explicit subtraction is necessary to ensure convergence of the integral. This condensate is connected with the axial isoscalar coupling which enters the Bjorken sum rule[84] for DIS of polarized electrons on a polarized proton[85]. It is defined through the relation

$$<\bar{q}i\gamma_\mu\gamma_5 q>_A=A_s A_\mu, \quad ...(164)$$

and its value was calculated in[85b] as $f_\eta^2\approx f_\pi^2$, on the assumption that the axial field interacts with the 8[th] component (isoscalar) of the unitary octet current. In the present treatment it does not need any such extra assumption but can be simply calculated from eq. (129) with ($O_i=i\,\gamma_\mu\gamma_5$), and introducing the axial field by the gauge substitution $\Pi_\mu\rightarrow\Pi_\mu-\gamma_5A_\mu$ in the propagator, and keeping only the first order term in the expansion. The result is expressed by

$$A_sA_\mu=\frac{-i}{(2\pi)^4}Tr\int d^4\Pi\left[S_F(\Pi)i\gamma.A\gamma_5S_F(\Pi)i\gamma_\mu\gamma_5\right] -\left["\hat{m}=0"\right] \quad ...(165)$$

where the term under quotes is the value of the main term for $\hat{m}=0$. Evaluating the trace and using the isotropy conditioin $<\Pi_\mu\Pi_\nu>=\delta_{\mu\nu}\Pi^2/4$ we obtain

$$A_s=\frac{-3i}{(2\pi)^4}Tr\int d^4\Pi\left[\frac{\hat{m}^2-\Pi^2/2+\Sigma_g}{(\Delta-\Sigma_g)^2}+\frac{\Pi^2/2-\Sigma_g}{(\Pi^2-\Sigma_g)^2}\right] \quad ...(166)$$

In this case however it is perhaps not as meaningful to keep track of the Σ_g-terms for numerical purposes as for the *e.m.* case; we shall drop them at this stage. Then with a simple rearrangement $\hat{m}^2-\Pi^2/2=3\Pi^2/2-\Delta/2$, the $\Delta/2$ term can be combined with the last term through a Feynman variable $u(0 \le u \le 1)$ and the pole integration carried out. The final result is

$$A_s = \frac{3}{4\pi^2}\int_0^\infty \hat{\pi}^2 d\hat{\pi}\int_0^\infty du\, \hat{m}^2 \left[\frac{3}{(\hat{m}+\hat{\Pi}^2)^{3/2}} + \frac{1}{(\hat{m}^2 u+\hat{\Pi}^2)^{3/2}}\right] \quad \ldots (167)$$

which yields 0.021 GeV^2, to be compared with $f_\pi^2 \approx 0.018$, or perhaps better with f_η^2 which is the relevant isoscalar quantity[83c] having a larger value[13] than f_π^2.

For a discussion of these results vis-s-vis QCD-SR, see ref. [30]. Since the spectroscopic linkage of the QCD condensates has been main theme of this Section, we should like to end it with the remark that the (MYTP-governed[15]) CIA[16] by itself does not carry information on the dynamics of spectroscopy which must be governed by other considerations (non-perturbative QCD simulated by $DB\chi S^{4,24}$, but it certainly offers a broad enough framework to accommodate such dynamics, without having to look elsewhere. Of course, the importance of spectroscopy as an integral part of any 'dynamical equation based' approach merely reiterates a philosophy initiated long ago by Feynman *et al.*[25]

9 *qqq* Dynamics: General Aspects

The dynamics of baryons as *qqq* systems represents the third stage of the three-body problem in its journey from the atomic through nuclear to the hadronic level of compositeness. The first (atomic) stage had been relatively free of theoretical ambiguities due to its strong QED foundations in the domain of non-relativistic quantum mechanics. In contrast, the second (nuclear) stage, although providing the initial stimulus for few-body dynamics, has from the outset remained bogged down in a continual empiricity in the theoretical foundations of strong interaction dynamics. Indeed by the time the meson exchange picture started being taken seriously for a parallel treatment of meson-nucleon system on the lines of electron-photon systems, the carpet got quietly removed from under its feet, through the slow but sure realization of its tenuous character born out of the quark compositeness of the underlying (meson) fields. Indeed the quark-gluon picture which had taken firm shape by the end of the Seventies, told in no uncertain terms the futility of understanding the inter-hadronic forces directly in terms of their it own species, as if they were elementary fields! On the other hand, the emergence of nuclear three-body techniques in the Sixties had an instant impact on the quark-level 3-body problem, thus providing a big boost to its development in a language strongly remainiscient of the nuclear 3-body problem, on the lines of Bethe's Second Principle Theory (see Sec. 1), except for the realization of its relativistic character which demands the input dynamics to be Bethe-Salpeter-like (albeit with wide variations), rather than Schroedinger-like. In this Section we shall give a panoramic view of three general aspects governing the dynamics of *qqq* baryons: i) classification of baryonic state[86]; ii) problem of connectedness in 3-body dynamics[87]; iii) BS-dynamics for fermionic *qqq* systems under $DB\chi S^{29b}$, in parallel with $\bar{q}q$, Sec. 4.3. The details of topics (ii) and (iii) are taken up in Sections 10 are 11 respectively.

Yet another type of approach to the *qqq* problem, as available in the literature, concerns parametric representations attuned to effective Lagrangians for hadronic transitions to "constituent" quarks, with ad hoc assumptions on the hadron-*qqq* form factor[88a], similar (parametric) ansatze for the hadron-quark-dipquark form factor[88b], or more often direct gaussian parametrizations for the *qqq* wave functions as the starting point of the investigation[88c]. Such approaches are often quite effective for the investigation of some well-defined sectors of hadron physics with quark degrees of freedom, but are in general much less predictive than dynamical-equation-based methods like NJL-Faddeev[89] or BSE-SDE framework[9,12], when extended beyond their immediate domains of applicability.

9.1 *SU(6) ⊗ O(3) Classification*

The initial *qqq* formulation was provided by a non-relativistic form of dynamics, and the first systematic classification[86a,b] of *qqq* states proved remarkably successful for the understanding of many details of hadronic spectra. On the other hand, the high degree of degeneracy of the h.o. model[86a] caused problems on the details of observed states, such as the absence of (the relatively low-lying) **20** states, in favour of more restricted types which, in a broad $SU(6)\times O(3)$ classification, are all 'natural parity' states[86c]

$$[\mathbf{56},(even)^+],\ [\mathbf{70},(odd)^-];\ [\mathbf{70},(even)^+],[\mathbf{56},(odd)^-] \quad \ldots (168)$$

while the (complementary) 'unnatural parity' states like $\mathbf{20}^+$ seemed to be missing from the data[13]. The natural parity baryons in turn are amenable to a simple quark-diquark picture[86d], with diquarks of the types 'scalar-isoscalar' D_s and '(axial) vector-isovector' D^a_μ, as well as (complementary) diquarks of the types (pseudo) vector-isoscalar D_μ and scalar-isovector $[D^a_s]$[90a], all of which go to make up the list[168] above. On the other hand the 'unnatural' parity baryons require diquark ingredients of opposite parity to above, viz., pseudoscalar-isoscalar, vector-isovector, vector-isoscalar, and pseudoscalar-isovector, respectively, to make up a complementary list of $SU(6)\times O(3)$ baryons[29b]

$$[\mathbf{20},(even)^+],\ [\mathbf{70},(even)^-];\ [\mathbf{70},(odd)^+],[\mathbf{56},(even)^-] \quad \ldots (169)$$

which have not yet been observed[13,29b].

Despite the compactness and elegance of the quark-diquark description, a certain amount of *dynamical* 3-body information gets lost due to the 'freezing' of a quark d.o.f. in the (rigid) diquark structure. While a good part of the S_3 (permutation) symmetry can be recovered by appropriate $SU(6)$ classification, the dynamical information in the full 3-body structure is not entirely retrievable, showing up, e.g., in the k^2-dependence of the e.m. form factor of the qqq baryon. To see more clearly the interconnection between the two descriptions, let us write down the baryon wave function, with proper S_3-symmetry, in both the qqq and q–d_q notations. To that end, its full wave function ψ with S_3-symmetry for 3 identical quarks, may be analyzed into its orbital ψ^α, spin χ^β and isospin ϕ^γ components, where α, β, γ superscripts stand for the S_3-symmetry types[25,45], which, in the Verde notation[45], are (s; m', m''; a) for symmetric, mixed-symmetric, and antisymmetric respectively. Since only color singlet states are being considered, we suppress the (antisymmetric) color wave function C^a for brevity, so that the 'active' part of the wave function is symmetric[86b]. The full structures of ψ for 56 and 70 states are[90a]

$$N^d_{56} = (\chi'\phi' + \chi'\phi'')\psi^s/\sqrt{2}$$
$$N^q_{70} = (\phi'\psi' + \phi''\psi'')\chi^s/\sqrt{2};$$
$$N^d_{70} = [(\chi'\phi'' + \chi''\phi')\psi' + (\chi'\phi' - \chi''\phi'')\psi'']/2;$$
$$\Delta^q_{56} = \chi^s\phi^s\psi^s;\ \Delta^d_{70} = (\chi'\psi' + \chi''\psi'')\phi^s/\sqrt{2}; \quad \ldots (170)$$

where the superscripts d and q stand for spin-doublet and spin-quartet respectively, and the product of the orbital (ψ) and spin (χ) functions for higher L-states must be read in the standard sense of adding angular momenta in terms of C.G coefficients. For strange baryon (Λ, Σ) states, the symmetry is reduced to S_2, due to the higher mass mass of the s-quark, and the corresponding states have the following representations[90]

$$\Lambda_{56} = \phi'\chi'\psi^s/\sqrt{2};\ \Lambda_{70} = \phi'(\chi'\psi'' \pm \chi''\psi')/2$$
$$\Sigma_{56} = \phi''\chi''\psi^s/\sqrt{2};\ \Sigma_{70} = \phi''(\chi'\psi' \pm \chi''\psi'')/2 \quad \ldots (171)$$
$$\Lambda_{70} = \phi'\psi'\chi^s;\ \Sigma_{70} = \phi''\psi''\chi^s.$$

To relate these structures to the quark (q)-diquark (D) description, the qD contents of these wave functions in a lorentz-invariant form may be read off from the following correspondence[90a]

$$\chi'\phi' \Leftrightarrow D_s;\ \chi''\phi'' \Leftrightarrow i\gamma_5\gamma_\mu D^a_\mu\tau_a$$
$$\chi^s\phi^s \Leftrightarrow D^a_\mu \in^\alpha_\mu;\ \chi'\phi'' \Leftrightarrow D^\alpha_s\tau^\alpha; \quad \ldots (172)$$
$$\chi^s\phi' \Leftrightarrow D_\mu \in_\mu;\ \chi''\phi' \Leftrightarrow i\gamma_5\gamma_\mu D_\mu;$$
$$\chi^s\phi'' \Leftrightarrow D^a_\mu\tau_\alpha \in_\mu$$

Here for simplicity, a basis spinor symbol ψ on the RHS has been suppressed for all (baryon) states. However, the additional (Rarita-Schwinger) spin and isospin symbols needed for several such states to make up the full baryon structure have been supplied via the unit vectors $\in^\alpha_\mu$ and $\in_\mu$ where necessary. (Of course *orbital* functions ψ are needed to make up the spatial overlap for the qD-pair). This correspondence may be faithfully substituted in the set (170) to give the precise qD content in $SU(6)$-form to give the different cases with *correct* normalizations. The *dynamical* effects are now entirely contained in the orbital wave function ψ.

9.2 Connectedness in a 3-Body Amplitude

The problem of *connectedness*[88] in a 3-particle amplitude has been in the forefront of few-body dynamics since Faddeev's classic paper[87a] showed the proper perspective, by emphasizing the role of the 2-body T-matrix as a powerful tool for achieving the goal. The initial stimulus in this regard came from the separable assumption due to Mitra[87b] which provided a very simple realization of such connectedness via Faddeev's T-matrix structure, a result that was given a firmer basis by Lovelace. An alternative strategy[87c] for connectedness in a more general n-body amplitude was provided by Weinberg[87d] through graphical equations which brought out the relative roles of the T- and V-matrices in a more transparent manner. (In particular Weinberg showed that the T-matrix was not the only way to achieve connectedness). It was emphasized by both Weinberg and Lovelace that an important signal for connectedness in the 3-body (or n-body) amplitude is the *absence* of *any δ-function* in its structure, either explicitly or through its defining equation. This signal is valid irrespective of whether or not the V- or the T-matrix is employed for the said dynamical equation.

The above equations were found for a *non-relativistic* n-body problem within a basically 3D framework[87d] whose prototype dynamics is the Schröedinger equation. For the corresponding *relativistic* problem whose typical dynamics may be taken as the Bethe-Salpeter Equation (BSE) with pairwise kernels within a 4D framework, it should be possible in principle to follow a logic similar to Weinberg's, using the language of Green's functions with corresponding diagrammatic representations[87d], leading to equations free from δ-functions. However there are other *physical* issues associated with a 4D support to the BSE kernel of a confining type, such as contradictions of the spectral predictions[14] with data[13]. Indeed, this very issue has been discussed in detail in Section 1, culminating in the 'two-tier' 3D-4D BSE approach as the central theme of this article, under the name of Covariant Instantaneity[16] for 3D support to the BSE kernel[17] which receives formal justification from the MYTP principle[15]. The principal result of this ansatz is the *exact interconnection* between the 3D and 4D forms of the BSE, at least for the 4D two-body problem[16].

One may now ask: Does a similar interconnection exist in the corresponding BS amplitudes for a *three-body* system under the same conditions of 3D support for the pairwise BS kernel? The question is of great practical value since the 3D reduction of the 4D BSE already provides a *fully connected* integral equation[29b], leading to an approximate analytic solution (in gaussian form)[29b] for the corresponding 3D wave function, as a byproduct of its success on the baryon spectra[13]. Therefore a reconstruction of the 4D qqq wave function in terms of the corresponding 3D quantities should open up a vista of applications to various types of *transition amplitudes* involving qqq baryons, just as in the two-body case outlined in Section 4. This exercise is carried out in Section 10, using Green's function techniques for both the 2- and 3-body systems (the former for checking against the known results of Section 4). There is however a big difference between the two systems, born out of the 'truncation' of the Hilbert space due to the 3D support ansatz for the pairwise BSE kernel. Such truncation, while still allowing an unambiguous reduction of the BSE from the 4D to the 3D level, nevertheles leaves an 'undermined element' in the *reverse direction*, viz., from 3D to 4D. This limitation for the reverse direction is quite general for any n-body system with $n>2$; the only exception is the case of $n=2$ where both transitions are reversible without extra assumptions (a sort of degenerate situation). As will be shown in Sec. 10, the extra assumption (in its simplest form) needed to complete the reverse transition is facilitated by some 1D δ-function which however has nothing to do with connectedness[87].

9.3 Fermionic qqq BSE with $DB\dot{\chi}S$

We now outline the essntial logic of a BSE treatment for a fermionic qqq system, for pairwise kernels with covariant 3D support, under conditions of $BB\chi S$, on closely parallel lines to the qq case (Section 4). In Section 3, the derivation[52] of the equation of motion from an input Lagrangian for extended 4-fermion coupling shows that the BSE structure (38) emerges in the *linear* approximation to the ϕ-field. This immediately suggests that the BSE for a qqq system in the same (linear) approximation must be one with a linear sum over all the three pairs of interaction, which for spinless quarks reads as[10b]:

$$i(2\pi)^4\Phi(p_1p_2p_3) = \sum_{123}\Delta_{F_1}\Delta_{F_2}\int d^q_{12}K(q_{12}q'_{12})(p'_1\,p'_2\,p'_3) \quad \dots (173)$$

where $\Delta_{Fi}=-i\Delta_i^{-1}$, etc. Under CIA[16], the relative momenta q_{ij} are 'hatted', i.e., they are orthogonal to the total momenta P_{ij} of the ij pairs, as explained in Sec. 4. It is however more convenient for calculational purposes to take all these relative 'hatted' momenta $\hat{q}_{ij}$ to be perpendicular to a common 4-momentum $P=p_1+p_2+p_3$, instead of the individual pairs. Technically this amounts to the introduction of 3-body forces at the quark level. However the difference turns out to be small[10b] that the 3-body forces are (expectedly) small. Having thus checked this (small) 3-body effect, we shall from now on consider a common 'hat' symbol for all the three pairs, i.e., $\hat{q}_{ij;\mu}=q_{ij}-q_{ij}.\,PP_\mu/P^2$. It is this version for hatted symbols for the qqq problem that we consider in the next two sections.

10 Interlinking 3D and 4D qqq Vertex Fns

In this Section, we outline a fairly detailed (self-contained) method of Green's functions for 2- and 3-particle scattering *near the bound state pole,* for the 3D-4D interconnection between the corresponding wave functions. For simplicity we consider identical *spinless* bosons, with pairwise BS kernels under CIA conditions[16], first for the 2-body case for calibration, (see Sects. 4.1-2), and then for the corresponding 3-body case, on the basis of the Green's Fn counterpart of the general structure, eq. (173).

10.1 Two-Quark Green's Function Under CIA

In the notation and phase convention of Section 4, the 4D qq Green's fn $G(p_1p_2;\ p_1'p_2')$ near a *bound* state satisfies a 4D BSE (no inhomogeneous term):

$$i(2\pi)^4G(p_1p_2;\,p_1'p_2') = \Delta_1^{-1}\Delta_2^{-1}\int dp_1''p_2''K(p_1p_2;\,p_1''p_2'')G(p_1''p'';\,p_1'p') \quad \dots (174)$$

wheres

$$\Delta_1 = p_1^2+m_q^2, \quad \dots (175)$$

and m_q is the mass of each quark. Now using the relative 4-momentum $q=(p_1-p_2)/2$ and total 4-momentum $P=(p_1+p_2)$ and total 4-momentum $P=p_1+p_2$ (similarly for the other sets), and removing a δ-function for overall 4-momentum conservation, from each of the G- and K-functions, eq. (174) reduces to the simpler form

$$i(2\pi)^4G(q.q')=\Delta_1^{-1}\Delta_2^{-1}\int d\hat{q}''Md\sigma''K(\hat{q},\hat{q}'')G(q'',q') \quad \dots (176)$$

where $\hat{q}_\mu = q_\mu - \sigma P_\mu$, . with $\sigma=(q.P)/P^2$, is effectively 3D in content (being orthogonal to P_μ). Here we have incorporated the ansatz of a 3D support for the kernel K (independent of σ and σ'), and broken up the 4D measure dq'' arising from (174) into the product $dq''Md\sigma''$ of a 3D and a 1D measure respectively. We have also suppressed the 4-momentum P_μ label, with ($P^2=-M^2$), in the notation for $G(q.q')$.

Now define the fully 3D Green's function $\hat{G}(\hat{q},\hat{q}')$ as (47)

$$\hat{G}(\hat{q},\hat{q}') = \iint M^2d\sigma d\sigma' G(q,q') \quad \dots (177)$$

and two (hybrid) 3D-4D Green's functions $\tilde{G}(\hat{q},q')$, $\tilde{G}(\hat{q},\hat{q}')$ as

$$\tilde{G}(\hat{q},q') = \int Md\sigma G(q,q');$$
$$\tilde{G}(q,\hat{q}') = \int Md\sigma' G(q,q'); \quad \dots (178)$$

Next, use (178) in (176) to give

$$i(2\pi)^4\tilde{G}(q,\hat{q}') = \Delta_1^{-1}\Delta_2^{-1}\int dq''K(\hat{q},\hat{q}'')\tilde{G}(q'',\hat{q}') \quad \dots (179)$$

Now integrate both sides of (176) w.r.t. $Md\sigma$ and use the result

$$\int Md\sigma\Delta_1^{-1}\Delta_2^{-1} = 2\pi i D^{-1}(\hat{q});\ D(\hat{q}) = 4\hat{\omega}(\hat{\omega}^2-M^2/4);$$
$$\hat{\omega}^2 = m_q^2+\hat{q}^2 \quad \dots (180)$$

to give a 3D BSE w.r.t. the variable $\hat{q}$, while keeping the other variable q' in a 4D form:

$$(2\pi)^3\tilde{G}(\hat{q},q') = D^{-1}\int d\hat{q}''K(\hat{q},\hat{q}'')\tilde{G}(\hat{q}'',q') \quad \dots (181)$$

A comparison of (176) with (181) gives the desired connection between the full 4D G-function and the hybrid $\tilde{G}(\hat{q},q')$-function:

$$2\pi iG(q,q') = D(\hat{q})\Delta_1^{-1}\Delta_2^{-1}\tilde{G}(\hat{q},q') \quad \dots (182)$$

Again, the symmetry of the left hand side of (182) w.r.t. q and q' allows rewriting the right hand side with the roles of q and q' interchanged. This gives the dual form

$$2\pi i G(q,q') = D(\hat{q}')\Delta_1^{-1}\Delta_2^{-1}\tilde{G}(q,\hat{q}') \quad \dots (183)$$

which on integrating both sides w.r.t. $Md\sigma$ gives

$$2\pi i\hat{G}(\hat{q},q') = D(\hat{q}')\Delta_1^{-1}\Delta_2^{-1}\hat{G}(\hat{q},\hat{q}'). \quad \dots (184)$$

Substitution of (184) in (182) then gives the symmetrical form

$$(2\pi i)G(q,q') = D(\hat{q})\Delta_1^{-1}\Delta_2^{-1}\hat{G}(\hat{q},\hat{q}')D(\hat{q}')\Delta_1^{-1}\Delta_2^{-1}. \quad \dots (185)$$

Finally, integrating both sides of (181) w.r.t. $Md\sigma''$, we obtain a fully reduced 3D BSE for the 3D Green's function:

$$(2\pi)^3\hat{G}(\hat{q},\hat{q}') = D^{-1}\hat{q}\int dq''K(\hat{q},\hat{q}'')\hat{G}(\hat{q}'',\hat{q}'). \quad \dots (186)$$

Eq. (185) which is valid near the bound state pole, expresses the desired connection between the 3D and 4D forms of the Green's functions; and eq. (186) is the determining equation for the 3D form. A spectral analysis can now be made for either of the 3D or 4D Green's functions in the standard manner, viz.,

$$G(q,q') = \sum_n \Phi_n(q;P)\Phi_n^*(q';P)/(P^2+M^2) \quad \dots (187)$$

where Φ is the 4D BS wave function. A similar expansion holds for the 3D G-function $\hat{G}$ in terms of $\phi_n(\hat{q})$. Substituting these expansions in (185) one immediately sees the connection between the 3D and 4D wave functions in the form:

$$2\pi i\Phi(q,P) = \Delta_1^{-1}\Delta_2^{-1}D(\hat{q})\phi(\hat{q}) \quad \dots (188)$$

whence the BS vertex function becomes $\Gamma=\Delta\times\phi/(2\pi i)$ as found in[16]. We shall make free use of these results, taken as qq subsystems, for our study of the qqq G-functions in Subsections 2 and 3.

10.2 3D BSE Reduction for qqq G-fn

As in the two-body case, and in an obvious notation for various 4-momenta (without the Greek suffixes), we consider the most general Green's function $G(p_1p_2p_3;\, p_1'p_2'p_3')$ for 3-quark scattering *near the bound state pole* (for simplicity) which allows us to drop the various inhomogeneous terms from the beginning. Again we take out an overall delta function $\delta(p_1+p_2+p_3-P)$ from the G-function and work with *two* internal 4-momenta for each of the initial and final states defined as follows[10b]

$$\sqrt{3}\xi_3 = p_1 - p_2;\ 3\eta_3 = -2p_3 + p_1 + p_2 \quad \dots (189)$$

$$P = p_1 + p_2 + p = p_1' + p_2' + p_3' \quad \dots (190)$$

and two other sets ξ_1, η_1 and ξ_2, η_2 defined by cyclic permutations from (189). Further, as we shall consider pairwise kernels with 3D support, we define the effectively 3D momenta $\hat{p}_i$, as well as three (cyclic) sets of internal momenta $\hat{\xi}_i, \hat{\eta}_i, (i = 1, 2, 3)$ by[10b].

$$\hat{p}_i = p_i - v_iP;\ \hat{\xi}_i = \xi_i - s_iP;\ \hat{\eta}_i - t_iP \quad \dots (191)$$

$$v_i = (P.p_i)/P^2;\ s_i = (P.\xi_i)/P^2;\ t_i = (P.\eta_i)/P^2 \quad \dots (192)$$

$$\sqrt{3}s_3 = v_1 - v_2;\ 3t_3 = -2v_3 + v_1 + v_2 \quad \dots (193)$$

(+cyclicpermutations)

The space-like momenta $\hat{p}_i$ and the time-like ones v_i satisfy[10b]

$$\hat{p}_1 + \hat{p}_2 + \hat{p}_3 = 0;\ v_1 + v_2 + v_3 = 1 \quad \dots (194)$$

Strictly speaking, in the spirit of covariant instantaneity, we should have taken the relative 3D momenta ξ', η' to be in the instantaneous frames of the concerned pairs, i.e. w.r.t. the rest frames of $P_{ij}=p_i+p_j$; however the difference between the rest frames of P and P_{ij} is small and calculable, while the use of a common 3-body rest frame (P=0) lends considerable simplicity and elegance to the formalism.

We may now use the foregoing considerations to write down the BSE for the 6-point Green's function in terms of relative momenta, on closely parallel lines to the 2-body case. To that end note that the 2-body relative momenta are $q_{ij}=(p_i-p_j)/2=\sqrt{3}\xi_k/2$, where (ijk) are cyclic permutations of (123). Then for the reduced qqq Green's function, when the *last* interaction was in the (ij) pair, we many use the notation $G(\xi_k\eta_k;\xi'_k\eta'_k)$, together with "hat' notations on these 4-momenta when the corresponding time-like components are integrated out. Further, since the pair $\xi_k\eta_k$ is *permutation invariant* as a whole, we may choose to drop the index notation from the complete G-function to emphasize this symmetry as and when needed. The G-function for the qqq system satisfies, in the neighbourhood of the bound state pole, the following (homogeneous) 4D BSE for pairwise qq kernels with 3D support:

$$i(2\pi)^4 G(\xi\eta;\xi'\eta') = \sum_{123} \Delta_1^{-1}\Delta_2^{-1}\int d\hat{q}''_{12} M d\sigma''_{12} K(\hat{q}_{12}, q''_{12}) G(\xi''_3\eta''_3;\xi'_3\eta'_3) \quad \ldots (195)$$

where we have employed a mixed notation (q_{12} versus ξ_3) to stress the two-body nature of the interaction with one spectator at a time, in a normalization directly comparable with eq. (176) for the corresponding two-body problem. Note also the connections

$$\sigma_{12} = \sqrt{3}s_3/2; \hat{q}_{12} = \sqrt{3}\hat{\xi}_3/2; \hat{\eta}_3 = -\hat{p}_3, etc. \quad \ldots (196)$$

the next task is to reduce the 4D BSE (195) to a fully 3D form through a sequence of integrations w.r.t. the time-like momenta s_i, t_i applied to the different terms on the right hand side, *provided both* variables are simultaneously permuted. We now define the following fully 3D as well as mixed (hybrid) 3D-4D G-functions according as one or more of the time-like ξ, η variables are integrated out:

$$\hat{G}(\hat{\xi}\hat{\eta};\hat{\xi}'\hat{\eta}') = \int\int\int\int(ds dt ds' dt' G(\xi\eta;\xi'\eta') \quad \ldots (197)$$

which is S_3-symmetric.

$$\tilde{G}_{3\eta}(\xi\hat{\eta};\xi'\hat{\eta}') = \int\int(dt_3 dt'_3 G(\xi\eta;\xi'\eta'); \quad \ldots (198)$$

$$\tilde{G}_{3\xi}(\hat{\xi}\eta;\hat{\xi}'\eta') = \int\int(ds_3 ds'_3 G(\xi\eta;\xi'\eta'); \quad \ldots (199)$$

The last two equations are however *not* symmetric w.r.t. the permutation group S_3, since both the variables ξ, η are not simultaneously transformed; this fact has been indicated in eqs. (198)-(199) by the suffix "3" on the corresponding (hybrid) $\hat{G}$-functions, to emphasize that the "asymmetry' is w.r.t. the index "3". We shall term such quantities "S_3-indexed", to distinguish them from S_3-symmetric quantities as in eq. (197). The full 3D BSE for the G-function is obtained by integrating out both sides of (195) w.r.t. the st-pair variables $ds_i ds'_j dt_i dt'_j$ (giving rise to an S_3-symmetrtic quantity), and using (197) together with (196) as follows:

$$(2\pi)^3\hat{G}(\hat{\xi}\hat{\eta};\hat{\xi}'\hat{\eta}') = \sum_{123} D^{-1}(\hat{q}_{12})\int d''\hat{q}_{12} K(\hat{q}_{12}, \hat{q}''_{12})\hat{G}(\hat{\xi}''\hat{\eta}'';\hat{\xi}'\hat{\eta}') \quad \ldots (200)$$

This integral equation for $\hat{G}$ which is the 3-body counterpart of (186) for a qq system in the neighbourhood of the bound state pole, is the desired 3D BSE for the qqq system in a *fully connected* form, i.e., free from delta functions. Now using a spectral decomposition for $\hat{G}$

$$\hat{G}(\hat{\xi}\hat{\eta};\hat{\xi}'\hat{\eta}') = \sum_n \phi_n(\hat{\xi}\hat{\eta}; P)\phi_n^*(\hat{\xi}'\hat{\eta}'; P)/(P^2+M^2) \quad \ldots (201)$$

on both sides of(200) and equating the residues near a given pole $P^2 = -M^2$, gives the desired equation for the 3D wave function ϕ for the bound state in the connected form:

$$(2\pi)^3\phi(\hat{\xi}\hat{\eta}; P) = \sum_{123} D^{-1}(\hat{q}_{12})\int d\hat{q}'_{12} K(\hat{q}_{12}, \hat{q}''_{12})\phi(\hat{\xi}''\hat{\eta}''; P) \quad \ldots (202)$$

Now the S_3-symmetry of ϕ in the $((\hat{\xi}_i, \hat{\eta}_i)$ pair is a very useful result for both the solution of (202) *and* for the reconstruction of the 4D BS wave function in terms of the 3D wave (202), as is done in the subsection below.

10.3 Reconstruction of 4D qqq Wave Function

We now attempt to *re-express* the 4D G-function given by (195) in terms of the 3D $\hat{G}$-function given by (200), as the qqq counterpart of the qq results (185)-(186). To that end we adapt the result (185) to the hybrid Green's function of the (12) sub-system given by $\tilde{G}_{3\eta}$, eq. (198) in which the 3-momenta $\hat{\eta}_3, \hat{\eta}'_3$ play a parametric role reflecting the spectator status of quark # 3, while the *active* roles are played by $q_{12}q'_{12} = \sqrt{3}(\xi_3, \xi'_3)/2$, for which the analysis of subsec. 10.1 applies directly. This gives

$$(2\pi i)^2\tilde{G}_{3\eta}(\xi_3\hat{\eta}_3;\xi'_3\hat{\eta}'_3) = D(\hat{q}_{12})\Delta_1^{-1}\Delta_2^{-1}\hat{G}(\hat{\xi}_3\hat{\eta}_3;\hat{\xi}'_3\hat{\eta}'_3)D(\hat{q}'_{12})\Delta_1'^{-1}\Delta_2'^{-1} \quad \ldots (203)$$

where on the right hand side, the 'hatted' G-function has full S_3-symmetry, although for purposes of book-keeping) we have not shown this fact explicitly by deleting the suffix '3' from its arguments. A second relation of this kind maybe obtained from (195) by noting that the 3 terms of the right hand side maybe expressed in terms of the hybrid $\tilde{G}_{3\xi}$ functions vide their definitions (197), together with the 2-body interconnection between $((\xi_3, \xi'_3)$ and $(\hat{\xi}_3, \hat{\xi}'_3)$ expressed once again via (203) but without the 'hats' on η_3 and η_3^T. This gives

$$(\sqrt{3}\pi i)^2 G(\xi_3\eta_3;\xi_3'\eta_3')$$
$$= (\sqrt{3}\pi i)^2 G(\xi\eta;\xi'\eta')$$
$$= \sum_{123} \Delta_1^{-1}\Delta_2^{-1}(\pi i\sqrt{3})\int d\hat{q}_{12}'' M d\sigma_{12}'' K(\hat{q}_{12},\hat{q}_{12}'')$$
$$\times G(\xi_3''\eta_3'';\xi_3'\eta_3')$$
$$= \sum_{123} D(\hat{q}_{12})\Delta_1^{-1}\Delta_2^{-1}\tilde{G}_{3\xi}(\hat{\xi}_3\eta_3;\xi_3'\eta_3')\Delta_1'^{-1}\Delta_2'^{-1} \quad \dots (204)$$

where the second form exploits the symmetry between ξ,, η and $\xi'\eta'$.

At this stage, unlike the 2-body case, the reconstruction of the 4D Green's function is *not yet* complete for the 3-body case, as eq. (204) clearly shows. This is due to the *truncation* of Hilbert space implied in the ansatz of 3D support to the pairwise BSE kernel K which, while facilitating a 4D to 3D BSE reduction without extra charge, does *not* have the *complete* information to permit the *reverse* transition (3D to 4D) without additional assumptions. The physical reasons for the 3D ansatz for the BSE kernel have been discussed in detail elsewhere[47], vis-à-vis contemporary approaches. Here we look upon this 'inverse' problem as a purely *mathematical* one.

We must now look for a suitable ansatz for $\tilde{G}_{3\xi}$ on the right hand side of eq. (204) in terms of *known* quantities, so that the reconstructed 4D G-function satisfies the 3D equation (200) exactly, as a 'check-point' for the entire exercise. We therefore seek a structure of the form

$$\tilde{G}_{3\xi}(\hat{\xi}_3\eta_3;\hat{\xi}_3'\eta_3') = \hat{G}(\hat{\xi}_3\hat{\eta}_3;\hat{\xi}_3'\hat{\eta}_3')\times F(p_3,p_3') \quad \dots (205)$$

where the unknown function F must involve only the momentum of the spectator quark # 3. A part of the η_3,η_3' dependence has been absorbed in the $\hat{G}$ function on the right, so as to satisfy the requirements of S_3-symmetry for this 3D quantity[47].

As to the remaining factor F, it is necessary to choose its form in a careful manner so as to conform to the conservation of 4-momentum for the *free* propagation of the spectator between two neighbouring vertices, consistently with the symmetry between p_3 and p_3'. A possible choice consistent with these conditions is the form:

$$F(p_3,p_3') = C_3\Delta_3^{-1}\delta(\nu_3-\nu_3') \quad \dots (206)$$

Here Δ_3^{-1} represents the "free" propagation of quark # 3 between successive vertices, while C_3 represents some residual effects which may at most depend on the 3-momentum $\hat{p}_3$, but must satisfy the main constraint that the 3D BSE, (200), be *explicitly* satisfied.

To check the self-consistency of the ansatz (206), integrate both sides of eq. (204) w.r.t. $ds_3ds_3'dt_3dt_3'$ to recover the 3D S_3-invariant $\hat{G}$-function on the left hand side. Next, in the first form on the right hand side, integrate w.r.t. ds_3ds_3' on the G-function which alone involves these variables. This yields the quantity $\tilde{G}_{3\xi}$. At this stage, employ the ansatz (206) to integrate over dt_3dt_3'. Consistency with the 3D BSE, eq. (200), now demands.

$$C_3\iint d\nu_3 d\nu_3'\Delta_3^{-1}\delta(\nu_3-\nu_3') = 1;\ (since{:}\ dt{=}d\nu) \quad \dots (207)$$

The 1D integration w.r.t. $d\nu_3$ may be evaluated as a contour integral over the propagator Δ^{-1}, which gives the pole at $\nu_3 = \hat{\omega}_3/M$, (see below for its definition). Evaluating the residue then gives

$$C_3 = i\pi/(M\hat{\omega}_3);\ \hat{\omega}_3^2 = m_q^2 + \hat{p}_3^2 \quad \dots (208)$$

which will reproduce 3D BSE, eq. (200), *exactly*! Substitution of eq. (206) in the second form of eq. (204) finally gives the desired 3-body generalization of eq (185) in the form

$$3G(\xi\eta;\hat{\xi}'\eta') = \sum_{123} D(\hat{q}_{12})\Delta_{1F}\Delta_{2F}D(q_{12}')\Delta_{1F}'\Delta_{2F}'$$
$$\hat{G}(\hat{\xi}_3\eta_3;\hat{\xi}_3'\eta_3')[\Delta_{3F}/(M\pi\hat{\omega}_3)] \quad \dots (209)$$

where for each index, $\Delta_F = -i\Delta^{-1}$ is the Feynman propagator.

To find the effect of the ansatz (206) on the 4D BS *wave function* $\Phi(\xi\eta;P)$, we do a spectral reduction like (201) for the 4D Green's function G on the left hand side of (204). Equating the residues on both sides gives the desired 4D-3D connection between Φ and ϕ:

$$\Phi(\xi\eta;P) = \sum_{123} D(\hat{q}_{12})\Delta_1^{-1}\Delta_2^{-1}\phi(\hat{\xi}\hat{\eta};P)$$
$$\times\sqrt{\frac{\delta(\nu_3-\omega_3/M)}{M\hat{\omega}_3\Delta_3}} \quad \dots (210)$$

defines the 4D wave fn in terms of piecewise vertex fns V_i, as

$$\Phi(p_1p_2p_3) \equiv \frac{V_1+V_2+V_3}{\Delta_1\Delta_2\Delta_3} \quad \dots (211)$$

From (210)-(211), we infer the baryon-qqq vertex function V_3 corresponding to the 'last' interaction in the 12-pair as

$$V_3 = D(\hat{q}_{12}\phi(\hat{\xi},\hat{\eta}) \times \sqrt{2\Delta_3\delta(v_3^2M^2 - \hat{\omega}_3^2)} \quad \dots (212)$$

and so on cyclically. (The argument of the δ-function inside the radical for V_3 simplifies to $(p_3^2+m_q^2)$. This expression is essentially the same as eq. (5.15) of ref. [10b], which had been obtained from largely intuitive considerations.

To account for the appearance of the 1D δ-fn under radical in (212), it is explained elsewhere[47] that it has nothing to do with connectedness[88] as such, but merely reflects a 'dimensional mismatch' due to the 3D nature of the pairwise kernel K[16] imbedded in a 4D Hilbert space. This in turn is the result of the 'contact' nature (in time dimension) of the pairwise interaction, somewhat analogous to Fermi δ-fn potential to simulate the effect of the (short range) *nuclear n–p* interaction in the "molecular' problem of (specular) neutron scattering by a hydrogen molecule[91]. As a further self-consistency check, it is instructive to compare (212) with one obtained by taking the limit of a point interaction, which amounts to setting *K=Constant* in the entire derivation above. This sturcture[47] which is worked out in *Appendix C*, is free from radicals, and explicitly 4D-invariant, in agreement with the so-called NJL-Faddeev (contact[4]) model[92] of 3-particle scattering.

11 Fermion Quarks: QCD-Motivated *qqq* BSE

We now turn to the more realistic case of fermion quarks for which we shall draw freely from a relatively recent analysis[29b] of a qqq baryon, which is basically a 3-body generalization of Subsection 4.3 for the two-body case. For simplicity of description, without sacrificing the essential physics, we shall specialize to equal mass kinematics (mass=m_q).

11.1 3D Reduction of 4D qqq BSE

The starting 4D BSE has the form (c.f. (173)):

$$(2\pi)^4\psi(p_1p_2p_3) = iS_F(p_1)S_F(p_2)\sum_{123}\int d^4q'_{12}K(\hat{q}_{12},\hat{q}'_{12})\psi(p_1p_2p'_3) \quad \dots (213)$$

where the kernels K_{ij} are given by eqs (54)-(55) for each ij pair, except for the Casimir value of the color factor $F_{12}\equiv\lambda_1.\lambda_2/4$ for the $\bar{3}$ state of a qq pair (to produce a color-singlet baryon), which is just half its 'singlet' value for a $q\bar{q}$ pair. And of course the $DB\chi S$ mechanism is built-in as in the two-body case of Sec. 4.3. The Gordon reduction of the product of two γ_μ-matices[10] also goes through as in Sec. 4.3, leading to[29b]:

$$\Phi(p_1p_2p_3) = \sum_{123}\frac{-iF_{12}}{(2\pi)^4\Delta_1\Delta_2}\int d^4q'_{12}V_\mu^{(1)}V_\mu^{(2)}V(\hat{q}_{12},\hat{q}'_{12})\Phi(p'p'_2p_3) \quad \dots (214)$$

where, following the steps of Sec. 4.3, the 'bosonic' Φ-fn is related to the fermionic Ψ-fn, as in eq. (60), by[10b,29b]

$$\psi(p_i) = \Pi_1^3 S_F^{-1}(-p_i)\,\Phi(p_i); \quad \Delta_i = m_q^2 + p_i^2 \quad \dots (215)$$

while the 4-vectors $V_\mu^{(i)}$ are given by eq. (61)

Next, for the 3D reduction of eq. (214), we need to define the transverse $\hat{p}_i$ and longitudinal v_i components of the 4-momenta p_i, as in Sec. 10.2, eqs. (191)-(194), and multiply the pairs of V_μ-fns, as in Sec. 4.3, replacing in the processes the longitudinal components, v_i by their *on shell* values $\hat{\omega}_i/M$, where $\hat{\omega}_i^2 = m_q^2+\hat{p}_i^2$ (ref. 29a), uniformly from such products[10b]. Now define the 3D wave function ψ in the as in Sec. 10.2, viz.[10b,29b],

$$\psi(\hat{p}_1\hat{p}_2\hat{p}_3) = \int ds_i dt_i \Phi(p_1p_2p_3); \quad \sqrt{3}s_3 = v_1 - v_2\,; \quad 3t_3 = -2v_3 + v_1 + v_2 \quad \dots (216)$$

The product ds_idt_i is cyclically invariant, so that the definition (216) can be taken over for all the three terms on the RHS of (214), with proper indexing. The rest of the procedure is straightforward, and follows closely the pattern laid out in the original formulations. Thus one integrates both sides of (214) w.r.t. ds_3dt_3, making use of (216) as well as the measure $d^4q'_{12} = d^3\hat{q}'_{12}Mds'_3\sqrt{3}/2$ to give on its RHS $\int ds'_3 dt_3\Phi' = \psi(\hat{p}'_1\,\hat{p}'_2\,\hat{p}_3)$. The additional ds_3-integration on the RHS is expressed by the result[10b]:

$$\frac{\sqrt{3}}{2}\int\frac{Mds_3}{\Delta_1\Delta_2}=\frac{2i\pi}{D_{12}};$$

$$D_{12}=-\Omega_{12}\lambda[(M^2(1-v_3)^2,\omega_1^2,\omega_2^2]/2M^2(1-v_3)^2 \quad \ldots (217)$$

$$2/\Omega_{12}=\hat{\mu}_{12}/\omega_1+\hat{\mu}_{21}/\omega_2;$$

$$\hat{\mu}_{12;21}=\frac{1-v_3}{2}\pm\frac{\omega_1^2-\omega_2^2}{2M^2(1-v_3)} \quad \ldots (218)$$

where v_3 has its on-shell value ω_3/M in the foregoing equations, as befits a spectator quark in the first term of (214). The resultant 3D reduction of (214) now takes the form:

$$\psi(\hat{p}_1\hat{p}_2\hat{p}_3)=\sum_{123}\frac{F_{12}}{(2\pi)^3 D_{12}} \times\int d^3\hat{q}'_{12}V^{(1)}\cdot V^{(2)}V(\hat{q}_{12},\hat{q}'_{12})\psi(\hat{p}'_1\,\hat{p}'_2\,\hat{p}_3) \quad \ldots (219)$$

11.2 Reduction of 6D Harmonic Basis

The next task is to reduce eq. (219) to a more transparent form suitable for numerical treatment. To that end we base our procedure[29b] on the expected smallness of the S_3-invariant quantity $\delta=M-\omega_1-\omega_2-\omega_3$ compared to ω_i and/or M. This gives the crucial result:

$$D_{12}=-4\omega_1\omega_2\delta+O(\delta^2);\ -\delta=\omega_1+\omega_2+\omega_3-M \quad \ldots (220)$$

which ensures that in (219), all the three terms on its RHS have a *common denominator* δ which, when transferred to the LHS, serves as a natural 'energy denominator' for the entire qqq equation. [Since the terms of $O(\delta^2)$ in (220) are fully calculable, any effect on their omission can be estimated perturbatively if necessary]. Next, from Sec. 4.3, the confining part of $V(\hat{q}_{12},\hat{q}_{12})$ is harmonic for ud-quarks (A_0=0), so that a perturbative treatment is possible, based on the (harmonic) confining part of $V(\hat{q},\hat{q}')$:

$$V_{con}=\frac{3}{4}(2\pi)^3\omega_{qq}^2[\nabla_{\hat{q}}^2+C_0/\omega_0^2]\delta^3(\hat{q}-\hat{q}');$$

$$\omega_{q1q2}^2=4M_{12}\hat{\mu}_{12}\hat{\mu}_{21}\omega_0^2\alpha(M_{12}^2) \quad \ldots (221)$$

In this formula, the definitions (218) for the fractional momenta $\hat{\mu}_{12,21}$ conform to their Wightman-Gaerding[56] definitions for unequal mass kinematics, a la eq. (37) of Section 4, since the unequal masses arise from the mass-shifts $m_q\rightarrow\omega_i$ of the quarks (1, 2) in the presence of the spectator # 3. Since such shifts are small, it is fairly accurate to approximate the fractional momenta as $(1-v_3)/2$ each, while $M_{12}\approx M-\omega_3$ only. Now to emphasize the 3D character of the various momenta, define the pairwise items:

$$\boldsymbol{L}_{ij}=-i\boldsymbol{q}_{ij}\times\nabla_{ij};\ \nabla_{12}=\nabla_1-\nabla_2;\ 2\boldsymbol{q}_{12}=\boldsymbol{p}_1-\boldsymbol{p}_2;$$

$$\hat{Q}_{12}=4\boldsymbol{q}_{12}^2\nabla_{12}^2+8\boldsymbol{q}_{12}.\nabla_{12}+6 \quad \ldots (222)$$

Also to take full advantage of the HO form (221), recast the (small) energy denominator δ in the alternative form[29b]:

$$-2M\delta\approx(\omega_1+\omega_2+\omega_3)^2-M^2\leq 3(\omega_1^2+\omega_2^2+\omega_3^2)\equiv$$

$$\Delta=9m_q^2+9(\xi^2+\eta^2)/2-M^2 \quad \ldots (223)$$

The resulting 'Master Equation' (219) is in pairwise notation[29b]:

$$\Delta\psi=(W_{con}+W_{OGE})\psi W_{con}=M\omega_0^2\sum_{123}(1-v_3)^2\alpha_{12}^s M_{12}\times\left[\nabla_{12}^2+\frac{C_0}{\omega_0^2}+\frac{1}{\omega_1\omega_2}NCT\right] \quad \ldots (224)$$

$$NCT=\frac{\hat{Q}_{12}}{4}-\frac{C_0}{\omega_0^2}-\mathbf{L}_{12}(\sigma_1+\sigma_2)+\left[\frac{i}{2}\mathbf{p}_3\times\nabla_{12}(\sigma_1+\sigma_2)-\sigma_1\dot{\sigma}_2\right] \quad \ldots (225)$$

and the OGE term in a 'mixed' ($\boldsymbol{r}$, $\boldsymbol{p}$) representation is [29b]:

$$W_{OGE}=\frac{4M}{3}\sum_{123}\alpha_{12}^s\left[\frac{1}{r_{12}}+\frac{1}{\omega_1\omega_2}\left(q_{12}\frac{1}{r_{12}}\cdot q_{12}+\pi\delta^3(r_{12})\right)(1-2\sigma_1\cdot\dot{\sigma}_2/3)+etc\right] \quad \ldots (226)$$

The OGE-term if calculated perturbatively in a 6D HO basis given by the main confining term in (224)[29b], which in a common (ξ, η) basis[29a], provided by eqs. (189)-(194) of Section 10, reads[29b]:

$$\Delta\psi=M\omega_0^2(1-\bar{v})^2\bar{\alpha}_s(M-\bar{\omega})\left[2\nabla_\eta^2+2\nabla_\xi^2+\frac{C_0(M^2-m_q^2+\Delta)}{w\omega_0^2\bar{\omega}^2}+\frac{\hat{Q}_B-8\boldsymbol{J}\dot{\boldsymbol{S}}+18}{4\bar{\omega}^2}\right]\psi \quad \ldots (227)$$

where the operators $\hat{Q}_B$, etc. are defined in ref. [29a], and the symbols (α, ϖ, υ) with 'bars' represent their 'average' values[29b]. From this H.O. equation, the scale parameter β, analogous to the 2-body quantity (66) may be inferred as[29b]

$$\beta^4 = 4M\omega_0^2\hat{\alpha}_s(1-\hat{v})^2(M-\overline{\omega})/9;$$
$$\hat{\alpha}_s = 1/[\overline{\alpha}_s^{-1} - 2C_0M(1-\overline{v})^2/(M-\overline{\omega})] \quad \ldots (228)$$

so that the basis function ψ_0 in its ground state is $exp[-\frac{1}{2}(\xi^2+\eta^2)/\beta^2]$; similar functions exist for L-excited states[40], providing a basis for perturbative treatment[29a] of the OGE terms (226).

11.3 Complex HO Basis for qqq States

It is however mathematically simpler to convert eq. (227) to a complex basis. To this end we define the complex dimensionless 3-vectors z_i, z^*_i, and their (derivative) conjugate momenta ∂_{z_i}, as

$$\sqrt{2}\beta[z_i; z_i^*] = \xi_i \pm i\eta_i;$$
$$\sqrt{2}\beta^{-1}[\partial_{z_i}; \partial_{z_i^*}] = \partial_{\xi_i} \mp i\partial_{\eta_i} \quad \ldots (229)$$

For the construction of angular momenta in complex basis, see Appendix D.

A more convenient basis for handling the various terms in (227) is provided by the creation/annihilation operator representation[29b, 46b] defined by *two* sets of complex operators

$$\sqrt{2}a_i = z_i + \partial l_{z_i^*}; \; \sqrt{2}a_i^* = z_i^* + \partial_{z_i}; \; \sqrt{2}a_i^{\dagger} = z_i^* - \partial_{z_i};$$
$$\sqrt{2}a_i^{*\dagger} = z_i - \partial_{z_i}^* \quad \ldots (230)$$

which satisfy the commutation relations

$$[a_i, a_j^{\dagger}] = [a_i^*, a^{*\dagger}] = \delta_{ij} \quad \ldots (231)$$

with all other pairs commuting. In the next subsection *11.4*, we define the number operators N_c and N_c^* which now play the role of N_ξ and N_η, but unlike the latter, the former can be simultaneously diagonalized; so their sum n and difference N_a are *both* constants of motion. Together with certain two-step operators, they form several sets of SO (2, 1) algebras (described below), which diagonalize the momentum dependent operators Q_B, etc, in terms of their respective Casimirs[46b], so that the solution of eq. (227) takes a simple algebraic form[29b]:

$$F(M,N) \equiv F_{con}(M,N) + F_{OGE}(M,N) = N+3 \quad \ldots (232)$$

where the first term is given by Eq. (55) of (29b), while the second term lends itself to a simple perturbative treatment (see [29b] for details). *Appendix D* gives a summary of the normalized $SU(6)\times O(3)$ structures of the 3D ψ-fns in the complex basis, which are needed for calculating the F_{OGE} term of eq. (232) above.

11.4 SO(2, 1) Algebras of Bilinear Operators

We start by defining the number operators N_c, N_c^*, and the mixed quantities N_m, N_m^* [46b]

$$N_c = a_i^{\dagger}a_i; \; N_c^* = a_i^{*\dagger}a_i^*; \; N_m = a_i a_i^{*\dagger};$$
$$N_m^{\dagger} = N_m^* = a_i^* a_i^{\dagger} \quad \ldots (233)$$

and their linear combinations

$$N = N_c + N_c^* = N_\xi + N_\eta; \; N_a = N_c - N_c^* \quad \ldots (234)$$

Note that both N and N_a are simultaneously diagonal in this (complex) representation, while in the real (ξ, η) basis, only their sum is diagonal. Next define the two-step operators (and their h.c.'s)[46b]:

$$A = 2a_i a_i^*; \; C = a_i a_i; \; C^* = a_i^* a_i^*$$
$$A^{\dagger} = 2a_i^{\dagger}a_i^{*\dagger}; \; C^{\dagger} = a_i^{\dagger}a_i^{\dagger}; \; C^{*\dagger} = a_i^{*\dagger}a_i^{*\dagger} \quad \ldots (235)$$

Now the trio A, $A^{\dagger}$ and N form an S_3-symmetric set, whose normalized forms

$$Q_+ = A^{\dagger}/2; \; Q_- = -A/2; \; Q_3 = (N+3)/2 \quad \ldots (236)$$

form an SO (2, 1) algebra (bounded from below with the Casimir[93,46b]

$$u(u+1) = \mathbf{Q}^2 \equiv -(AA^{\dagger} + A^{\dagger}A)/8 + (N+3)^2/4 \quad \ldots (237)$$

where $u(u+1)$=+3/4) for even N and +2 for odd N, while the eigenvalues of Q_3 are $-u+k$, (with k=0,1,2, ...). These imply that u=–3/2 and u=–2 for even and odd N respectively. Similarly the mixed symmetric set (C, $C^{\dagger}$, N_c) form an $SO(2, 1)$ algebra in the normalized form[46b]

$$Q_{c+} = C^{\dagger}/2; \; Q_{c-} = -C/2; \; Q_{c3} = \frac{1}{2}(N_c + 3/2) \quad \ldots (238)$$

with the corresponding Gasimir[46]

$$u_c(u_c+1) \equiv \mathbf{Q}_c^2 = -(CC^{\dagger} + C^{\dagger}C)/8 + (N_c + 3/2)^2/4 \quad \ldots (239)$$

This spectrum is again bounded from below[93], with the eigenvalues Q_{c3}=–u_c+k, where u_c=-3/16 for even N_c and u_c=+5/16 for odd N_c. An identical structure holds for the 'starred' operators (C^*, $C^{*\dagger}$, N_c^*), with the same eigenvalues. Finally the trio (N_m, $N_m^\dagger = N_m^*$, N_a) which is S_3-antimmetric, satisfy a 'normal' $SO(3)$ algebra[46b].

$$[N_m, N_a] = 2N_m;\ [N_m^\dagger, N_a] = -2N_m^\dagger; [N_m, N_m^\dagger] = -N_a \qquad \ldots (240)$$

with spectra bounded from both above and below. The corresponding Casimir is

$$s(s+1) = (N_m N_m^\dagger + N_m^\dagger N_m)/2 + N_a^2/4 \qquad \ldots (241)$$

The spectrum is here determined from the condition that both N_c and N_c^* are non-negative integers. The result is[46b]

$$-N \le N_a \le N;\ s = N/2 \qquad \ldots (242)$$

11.5 "Exotic" qqq States

The comparison of eq. (232) with the baryon spectra is described at some length in[29b], and it is not our intention here to go into these details afresh. Instead, we shall end this Section with some qualitative analysis on the capacity of this model to identify some exotic baryonic states which have for long remained elusive. The main reason for such optimism stems from the precise predictions on the 'spectroscopic' locations of the states on the one hand, and the possibility of making more reliable $SU(6) \otimes O(3)$ assignments for such states on the basis of their *decay* characteristics which the model also allows within its broad framework. To see the logic, a good calibration is first provided by the fairly accurate location of several 'known' states in a parameter-free manner; see Table I of[29b] for comparison. With this first check, a more sensitive test is now a comparison of the alternative $SU(6) \otimes O(3)$ assignments for the mass locations of the same states; see Table II of[29b]. Specifically the competition is between the **56**, *odd*⁻ and **70**, *odd*⁻ assignments for Δ-like states for the same total quantum number *N*. The question is clearly of physical interest since in the entire history of baryon spectroscopy **56**, *odd*⁻ states have suffered from popular perceptions of elusiveness, despite occasional attempts to the contrary[91]. The analysis in[29b] suggests that the **56** assignment has a slight edge over **70**, at least for a couple of odd-parity states by virtue of 'location', but a more sensitive test requires a more detailed comparative study of the decay and/or production characteristics that these alternative assignments provide, a vis-à-vis the data (which are still elusive). In this respect it was shown in[29b] that the general mechanism of 'direct' versus 'recoil' modes of single-quark transitions[45b], do *not* inhibit in any way the production of natural parity 56^- states w.r.t. the corresponding $\mathbf{70}^-$ states, (perhaps contrary to popular beliefs).

11.6 CIA vs CNPA for Fermionic qqq Dynamics

In the foregoing we have mostly described the CIA predictions[29b] on the baryon spectra. How about the corresponding *qqq*-scenario with the other MYTP-governed CNPA dynamics whose $q\bar{q}$ counterpart has been employed in Sections 4-6 ? The reason for avoiding this exercise for the *qqq* problem is one of pedagogical necessity. For from the results of Sections 4-6, it has been fairly clear that the earlier NPA treatment[40] based on the old-fashioned NP-language[35] formally provides the same CNPA structure of 3D BSE as well as 4D vertex functions for $q\bar{q}$ systems, so that a similar *qqq* structure should be expected. In this respect, the baryon spectral results[29a] based on the old-fashioned NPA treatment are already available in detail[40], and the comparison with the CIA treatment[29b] shows considerable overlap therewith. As for the reconstruction of the *qqq* vertex function under CNPA, a closely analogous treatment akin to Section 10 formally leads to almost identical results, with the CIA-CNPA correspondence already indicated in Section 4.

How about the reconstruction of the 4D *qqq* vertex function for fermion quarks? Again the treatment, which is analogous for both CIA and CNPA, consists in reducing the fermionic structure to an effectively scalar problem via eq. (215) which relates the fermionic BS wave function Ψ to the 'scalar' function Φ fits in smoothly with the treatment outlined in Section 10 for spin zero quarks, with almost no change, thus rendering unnecessary another fresh formulation for fermions. As for quark loop applications to the *qqq* problem, the general problem of 'Lorentz mismatch' of 3D wave functions in a quark-loop integral, that had led us to abandon the CIA treatment in favour of CNPA for the $q\bar{q}$ problem (see, e.g., Sections 5-6), is also encountered in the

qqq case, so that it is profitable to adopt CNPA[41] for baryonic transition amplitudes as well.

The only exceptions are two-loop integrals, as in the self-energy problem (Sec. 7), or one-loop integrals, as in the vacuum condensate problem (see Sec. 8), where this pathology is just avoided. Full-fledged baryon-loop calculations' are still being developed, so such topics are not intended for a detailed coverage in this article, except for indicating the results of a recent calculation of $SU(2)$ n-p-mass splitting[95] analogous to the treatment of pseudoscalar mass splittings (Sec. 7)[32a] by this method. Thus, using the same value (4 *MeV*) of the 'current' d-u mass difference, the total n-p mass difference works out as 1.28 *MeV*[95], to be compared with the experimental value of 1.29 *MeV*[13], except for possible QED gauge corrections[63]. On this last item, an indication of the expected correction is available from its effect on the Kaon e.m. mass difference, which yields a ~60% upward revision on its (uncorrected) value of about 1 *MeV*[32a]; *see Appendix C* for an estimation of this correction. If this value for the kaon case is taken as rough indication of the same effect expected for the nucleon case, then (on a proportionate basis) the QED gauge corrected value for n-p mass difference comes down to ~1 *MeV*. For details of this methodology, see ref [95].

12 Summary and Conclusions

In this article an attempt has been made to present a somewhat 'less than conventional' BSE-SDE formalism based on the Markov-Yukawa Transversality Principle (MYTP)[15] on the one hand, and a strongly QCD motivated 4-fermion Lagrangian which generates the BSE-SDE framework by breaking its chiral symmetry dynamically ($DB\chi S$)[23-27], on the other. The MYTP mechanism provides an *exact* interconnection between the 3D and 4D forms of the BSE, so that both can be used interchangeably, a facility which does not seem to exist in other alternative 3D BSE formalism[39], or the null-plane formulations-both non-covariant[35] and covariant[36-38]. This twin property of the MYTP-governed BSE formalism[16], termed 3D-4D BSE for short, gives rise to a natural 'two-tier' description[40], the 3D sector (with its relativistic Schroedinger-like BSE) being appropriate for making contact with the hadron spectra[13], while the reconstructed 4D BSE yields a vertex function which allows the direct use of the language of Feynman diagrams for evaluating transition armplitudes as 4D loop integrals. (This contrasts with other 3D formulations[35-39] which require specialized versions of Feynman diagrams[37] for calculating loop integrals).

At a more quantitative dynamical level, both $q\bar{q}$ and qqq hadrons are amenable to a unified treatment, since their respective BSE's emanate from a common (input) chiral 4-fermion Lagrangian with a gluon-like propagator whose 'color-factor' has the right relative strengths for both systems. And while the 3D-4D structure of the $q\bar{q}$ BSE[16,28], as well as the 3D reduction of the qqq BSE[29], have been around for some time, the missing link of a *reconstructed* 4D BS wave function for the qqq system (only conjectured in[106] has now (hopefully) been supplied through a formal derivation in Sec. 10 via Green's Function techniques[47]. Indeed the main emphasis in this Article has been on the 'second stage' of this two-tier formalism, relating to the calculation of 4D quark-loop integrals, of which some selected examples have been given in Secs. 5-7 to bring out the feasibility of its applications to the meson sector. The corresponding applications to baryonic amplitudes via loop integrals are still being developed, and only a 'pilot' example, relating to $SU(2)$ mass splitting[95], is as yet available. However the scope (and feasibility) of such applications is quite substantial[96].

The capacity of this BSE-SDE formalism to relate its parameters to the 'vacuum condensates' of QCD-SR theory[2] has been sought to be brought out in Sec. 8, wherein it has been shown that several types of condensates (both direct and induced) lend themselves with great ease to this simple treatment[30], while the corresponding QCD-SR treatments[82-85] often need additional ansatze for their evaluation. This facility it owes to its (input) gluon propagator on the one hand, and the (derived) mass function $m(\hat{p})$ from the SDE solution[11] on the other. The two fundamental parameters[11] of the infrared gluon propagator are not calculable within this (Bethe's Second Principle oriented) framework, but they are firmly rooted in spectroscopy, as their contact[28-29] with data[13] reveals. Indeed the performance of this spectroscopy-oriented BSE-SDE framework in predicting the vacuum condensates, can be directly attributed to its *off-shell* structure.

An important (new) aspect of this Study has been a demonstration of the powers of MYTP extending from its original mandate[15] of transversality in terms of Çovariant Instantaneity' (CIA)[16-17], to a wider 'transversality' on a Covariant Null Plane (CNPA)[41], thus vastly enhancing the applicability of this important principle. In this article, we have tried to present both CIA and CNPA on very similar lines, but the mathematical viability of the Latter[41] seems to exceed that of the former[16], in as much as a CIA treatment of triangle (and higher) loop integrals is fraught with problems of 'Lorentz mismatch' of different CIA wave functions, leading to ill-defined integrals due to the presence of time-like momentum components in the exponential/gaussian factors inside the integrals concerned[57]. This problem, which has been known since the FKR paper[25], is properly circumvented in CNPA, except for the (less serious) problem of dependence on the 'null-plane orientation' which can be tackled through other means, e.g., a simple device of 'Lorentz completion' which yields an explicitly Lorentz-invariant structure. This has been illustrated in Sec. 5 for the pion form factor which shows the expected high energy behaviour as well as very reasonable results[58-62] in both the high and low energy regimes. For more general three-hadron amplitudes[31] too, similar calculations in Sec. 6 show that the anomalies of ill-defined 4D loop integrals are absent in a CNPA treatment. The osnly exceptions are two-quark loops[32] (Sec. 7), where both methods, CIA and CNPA, work.

Clearly, the MYTP is a very powerful Principle which helps organize a whole spectrum of phenomena under a single umbrella. It has been possible to study only a very few (though illustrative) examples to bring out its powers, but its potential is vast, and warrants many more of such applications. More importantly, the 3D-4D structure of BS dynamics provided by MYTP takes in its stride the spectroscopy sector as an integral part of the dynamics, as envisaged long ago by Feynman *et al*[25].

A good part of the logic behind this Article was evolved during my tenure of an INSA Professorship (1989-94), while the actual contents of this Article include both published and unpublished material developed subsequently, in my capacity as a freelance workers (unattached to any Institution), as part of an ongoing research process.

Appendix A: Derivation of $F(k^2)$ and N_H for P-meson

In this Appendix we outline the main steps to the derivation of the P-meson form factor (81), as well as the Normalizer (80), given in Sec. 5 of Text. Collecting the various pieces after p_{2n}-pole integration, gives for (73)

$$F(k^2) = 2(2\pi)^3 N_n(P) N_n(P')\hat{m}_1 \times \int d^2q_\perp dz_2 P.ng(z_2) e^{-q_\perp^2/\beta^2} - f(z_2)/\beta^2 + [1 \Rightarrow 2]; \quad \ldots(243)$$

$$f(z_2) = M^2\eta_k^{-2}[\theta_k z_2^2 - z_2\hat{k}^2\hat{m}_2 + \theta_k \hat{m}_2^2\hat{k}^2/4]; \quad \ldots(244)$$

$$D_n + D_n' = 4\overline{P}.n[q_\perp^2 + M^2(z_2^2 - z_2\hat{k}^2\hat{m}_2/2 + \hat{m}_2^2\hat{k}^2/4)/\eta_k - \lambda/4M^2]; \quad \ldots(245)$$

$$g(z_2) = \frac{D_n + D_n'}{4}\frac{M^2+\delta m}{M^2+k^2/4} + h(z_2); \quad \ldots(246)$$

$$h(z_2) = 2\overline{P}.n(\hat{m}_2 - z_2)[M^2 - \delta\ m^2 + \hat{m}_2 M^2(\delta m^2 - M^2 - k^2/2)/(M^2+k^2/4)] \quad \ldots(247)$$

The integration over $q_\perp$ and z_2 are both routine, the latter with a translation $z_2 \to z_2 + \frac{1}{2}\hat{m}_2 k^2/\theta_k$, to reduce the gaussian factor to the standard form. Note that, unlike the conventional (Weinberg) form[69a] of light-front dynamics, the present 4D form which permits off-shellness of the internal momenta, does not restrict in principle the limits of z_2 integration. Thus after the translation, the odd-z_2 terms can be dropped, and $f(z_2)$ reduces to

$$f(z_2) = M^2 z_2^2\theta_k/\eta_k^2 + (M\hat{m}_2\hat{k})^2/4\theta_k \quad \ldots(248)$$

while the g-function is a sum of two pieces g_1+g_2:

$$g_1 = \eta_k[q_\perp^2 + M^2 z_2^2/\eta_k + \frac{1}{4}M^2\hat{m}_2^2\hat{k}^2(1+3\hat{k}^2/4)/\theta_k^2 - \lambda/4M^2](1+\delta m^2/M^2); \quad \ldots(249)$$

$$g_2 = 2\eta_k(M^2 - \delta m^2)\hat{m}_2/\theta_k + 2(\delta m^2 - M^2 - k^2/2)\hat{m}_2^2\eta_k^2/\theta_k \quad \ldots(250)$$

Before writing the final result for $F(k^2)$, it is instructive at this stage to infer the normalizer N_H of the hadron, obtained by setting $k_\mu=0$, and demanding that $F(o)=1$. This gives after some routine steps:

$$N_H(P)^{-2} = 2M(2\pi)^3(P.n/M)^2\int d^3\hat{q}e^{-\hat{q}^2/\beta^2}G(0); \quad ...(251)$$

$$G(0) = [(1+\delta m^2/M^2)(\hat{q}^2-\lambda/4M^2) + 2\hat{m}_1\hat{m}_2(M^2-\delta m^2)] \quad ...(252)$$

where $\hat{q}=(q_\perp, Mz_2)$ is effectively a 3-vector, in conformity with the requirements of the angular condition[35d, 38], which gives a formal meaning to its third component $q_3=M.q.n/P.n=Mz_2$. The normalization factor $N_n(P)$ is also seen to vary inversely as $P.n$, while the multiplying integral is clearly independent of the NP-orientation n_μ. To exhibit this P_n indepdendence more explicitly, define a 'reduced normalizer' N_H which equals $N_n(P)\times P.n/M$ and gives for N_H^{-2} the Lorentz-invariant result, eq. (80) of Text.

Now insert the result $N_n(P)=MN_H/P.\ n$ on the RHS of (243), and note, via eq. (75) that

$$M^2/P.n\ \ P'.n = M^2/(\bar{P}.n)^2\eta_k;\ \eta_k = 1-\hat{k}^2/4. \quad ...(253)$$

One now checks that the factors $\bar{P}.n$ cancel out completely, and the evaluation of the gaussian integrals leads after a modest algebra to eq. (81) of Text, where $G(\hat{k})$, after collecting from eqs (248), is given by

$$G(\hat{k}) = (1+\delta m^2/M^2)h(\hat{k}) + 2(M^2-\delta m^2)\hat{m}_2/\theta_k + 2\hat{m}_2^2\eta_k\theta_k^{-1}(\delta m^2-M^2-k^2/2); \quad (254)$$

$$h(\hat{k}) = (1+\eta k^2/2\theta_k)\beta^2-\lambda/4M^2 + (M\hat{m}_2\hat{k}/2\theta_k)^2(1+\tfrac{3}{4}\hat{k}^2); \quad ...(255)$$
$$\delta m = m_1 - m_2.$$

Appendix B: *Gauge Corrections to Kaon E.M. Mass*

We outline here a practical procedure to evaluate the gauge corrections to the e.m. self-energy of a $q\bar{q}$ system, vide Fig. 1b of ref. [18a]. For brevity we shall refer to the figures of KL[63] in their notation without drawing them anew. Thus Fig. 1b of ref. [18a] corresponds to fig. 1a of KL[63], except for the presence of the hadron lines at the two ends. We shall call this simply "1a', with the understanding that the hadron lines are 'attached' to 1a. For the actual mathematical symbols (including phase conventions) we shall draw freely from[18a], without explanation. In[18a], only 1a of KL[63] was calculated, but now one must add 2(a,b,c,d,e) of KL[63], all with hadron lines understood at the two ends of each. There is no need to calculate 1b or 1c of[63] which are mere e.m. self=energies of single quarks (g.i.by themselves), and are routinely absorbed in quark mass renormalization (of little significance in this study which has these masses as inputs).

A new ingredient is a 4-point vertex in each of 2(a,b,c,d), and *two* 4-point vertices in 2e, except that the word 'point' is now understood as an extended structure characterized by the hadron-quark vertex function $D(\hat{q})\phi(\hat{q})$ where one must insert a photon line in each such $Hq\bar{q}$ blob. Since it is *not* a standard point vertex, the method[63] of inserting exponential phase integrals with each current is not technically feasible; instead we may resort to the simple-minded substitution $p_I-e_IA(x_I)$ for each 4-momentum p_I (in a mixed p, x representation) occurring in the structure of the vertex function, which has the same physical content, at least up to first order in the e.m. field, without further comment. This amounts to replacing each $\hat{q}_\mu$ occurring in $\Gamma(\hat{q})=D(\hat{q})\phi(\hat{q})$, by $\hat{q}_\mu - e_qA_\mu$, where $e_q = \hat{m}_2e_1-\hat{m}_1e_2$. The net result in the first order in A_μ is a first order correction to $\Gamma(\hat{q})$ of amount $e_qj(\hat{q})$. A defined by

$$j(\hat{q}).A = -4M\hat{q}.A\phi(\hat{q})(1-D(\hat{q})/(4M\beta^2)) \quad ...(256)$$

(The effect of the hat structure of $\hat{q}$ on the e.m. substitution is ignored in this approximate treatment). This effective 4-point vertex function is operative at one end in each of 2a,2b,2c,2d of KL[63] and at both ends of 2e. For the e.m. vertex at the quark lines of 2(a,b,c,d), we use simply $ie_I\gamma.A$, as in ref. [18a]. The matrix elements can now be written down on exactly the same lines, and the *same* phase convention as in ref. [18a], to keep proper track of the gauge corrections with sign. We need write these down only for 2a and 2e, noting theequalities 2a=2b, as also 2c=2d, and the further substitutions (1) → (2) and vice versa to generate 2c(=2d) from

2a(=2b). The contribution from 2a[63] to the e.m. quadratic self-energy of a kaon is expressible as

$$M_{2a}^2 = N_H^2(2\pi)^{-5}e_1e_q \times \int j(\hat{q})_\mu D(\hat{q}')\phi(\hat{q}')Tr[\gamma_5 D_{F_{\mu\nu}}(k) S_F(p_1-\hat{m}_1k)ie_1\gamma_\nu S_F(p_1')\gamma_5 S_F(-p_2')]d^4qd^4k \quad ...(257)$$

where $p_1' = p_1 + \hat{m}_2k$ and $p_2 = p_2' = p_2 - \hat{m}_2k$ are the 4-momenta of the quarks at the other (right-hand) end, and the photon propagator in the Landau gauge is $-i(\delta_{\mu\nu} - k_\mu k_\nu/k^2)/k^2$. It is now convenient to change the variable from k_μ to q_μ', noting that $q' = q + \hat{m}_2k$, which gives $d^4k = d^4q'/\hat{m}_2^4$, etc. This shows that Fig. 2a(=2b), where the photon line ends on the heavier quark m_1, gives a bigger contribution than does Fig. 2c(=2d) which would give $\hat{m}_1^{-4}$ arising from the d^4k-measure. Evaluating the traces, and integrating over the poles of the two time-like momenta q_0 and q_0' gives for the sum of the contributions from 2a-2d to the quadratic mass difference between K^0 and K as a product of two 3D quadratures after some simplifications with factorable approximations a 1a[18a]:

$$\delta M_{2(a-d)}^2 = \frac{6N_H^2 M\delta(e_1e_q)}{(2\pi)^3\hat{m}_2^3} \times \int d^3\hat{q}\int d^3\hat{q}'\frac{\phi\phi'}{\hat{q}\hat{q}'\omega_{1k}}\left[1-\frac{D(\hat{q})}{4M\beta^2}\right] \times[(\hat{q}^2(2-4/\pi) - \hat{q}\hat{q}'/3)(M^2-\delta m^2) + D(\hat{q}')\omega_1'^{-1}/2 + D(\hat{q}')\omega_2'^{-1}/2) + \tfrac{1}{3}\hat{m}_2\hat{q}\hat{q}'(D(\hat{q}')\omega_2'^{-1}/2 + M^2 - \delta m^2)] + [1\leftrightarrow 2] \quad ...(258)$$

Here $\delta(e_ie_q)$ is the $\overline{K}^0$ minus K^- difference between the indicated charge factors associated with line 'I', while $\omega_{1,2}'^2 = m_{1,2}^2 + \hat{q}'^2$ and $\omega_{1k}^2 = m_1^2 + (\hat{q} - \hat{m}_1\hat{k})^2$.

Next the contribution to δM^2 arising from Fig. 2e of KL[63] which involves the product of two vertex blobs like eq. (256) is given by

$$dM_{2e}^2 = iN_H^2(2\pi)^{-5}e_q^2\int d^4qd^4kD_{F\mu\nu}^{(K)} \times j(\hat{q})_\mu j(\hat{q})_\nu Tr[\gamma_5 S_F(p_1-\hat{m}_1k)\gamma_5 \times S_F(-p_2+\hat{m}_2k)] \quad ...(259)$$

This integral is somewhat different in structure from (257) in as much as k_μ is fully decoupled from either wave function ϕ, ϕ', both of which have the same argument $\hat{q}$. This makes it possible to integrate first over d^4k as well as the time-like component q_0 of q_μ neither of which is involved in the vertex function. The relevant integral after tracing and rearranging has the form

$$F(\hat{q}) = 3(-i)^2\int d^4k\int dq_0k^{-2}(\delta_{\mu\nu} - k_\mu k_\nu/k^2) [\hat{q}^2 - q_0^2 + m_1m_2 - \hat{m}_1\hat{m}_2(P-k)^2]/(\Delta_1\Delta_2) \quad ...(260)$$

where $\Delta_i = m_i^2 + (p_i - \hat{m}_ik)^2$. The integral which is entirely convergent works out after some standard manipulations involving Feynman techniques as well as differentiation under integral signs as

$$F(\hat{q}) = 6\pi^3[m_1m_2 + \hat{q}^2 + \Lambda] \times[\sqrt{\Lambda} - \sqrt{\Lambda - \hat{m}_1\hat{m}_2M^2}]/(\hat{m}_1\hat{m}_2M)^2 \quad ...(261)$$

where $A = \hat{m}_1\hat{m}_2M^2 + D(\hat{q})/2M$. And the final expression for (259) in terms of (261) is

$$\delta M_{2e}^2 = N_H^2(2\pi)^{-5}\delta(e_q^2)\int d^3\hat{q}j(\hat{q})^2F(\hat{q}) \quad ...(262)$$

Further evaluation of (258) and (262) can be made a 1a ref. [18a] in straightforward way. The key ingredients are

$$\delta e_1e_q = 0.236e^2;\ \delta e_2e_q = 0.139e^2;\ \delta e_q^2 = -0.0294e2. \quad ...(263)$$

The break-up of the final results for the diagrams 2(a-e) after dividing the results of (263) and (262) by $2M$, since $\delta M^2 = 2M\delta M$, is (in MeV) :

$$\delta M_{2a=2b} = -0.6996;\ \delta M_{2c+2d} = +0.1358;\ \delta M_{2e} = -0.0481;\ \delta M_{tot} = -0.612MeV. \quad ...(264)$$

All these corrections, which reinforce one another due to a complex interplay of signs, add up to a figure which increases the value -1.032 *MeV* due to Fig. 1(b) of ref. (18a), to −1.644 *MeV*, roughly a 60 percent (negative) increase, which is a rough indication of the type of QED gauge correction to be expected from such diagrams.

Appendix C: A 4D NJL-Faddeev Model

We summarize here the results of a simplified 4D NJL-Faddeev bound state problem[47,89], with 3 scalar-isoscalar quarks interacting pairwise in a contact fashion, a la NJL[4]. It is merely a special case of 3D-4D-BSE when its kernel K becomes a constant λ. For ease of comparison, we employ the same notation and phase convention for the various quantities as in Secs. (4, 9), but in view of the bound state nature of the problem it is enough to work with the 4D BSE for the wave function only, without invoking Green's functions. We start with the qq problem as a prerequisite for the solution of the qqq problem.

C.1 qq Bound State in NJL Model

The BSE for the 4D wave function Φ for a qq system may be written down for the NJL model:

$$i(2\pi)^4\Phi(q_{12}P_{12}) = \Delta_1^{-1}\Delta_2^{-1}\lambda\int d^4q'_{12}\Phi(q'_{12}P_{12}) \quad \text{... (265)}$$

where λ is the strength of the contact NJL interaction for any pair of (scalar) quarks. The solution of this equation simply reads as[87b]

$$\Phi(q_{12}P_{12}) = A\Delta_1^{-1}\Delta_2^{-1} \quad \text{... (266)}$$

when plugged back into (265), one gets an 'eigeŋvalue' equation for the invariant mass $M_d^2 = -P_{12}^2$ of an isolated bound qq pair in the implicit form of a determining equation for λ:

$$\lambda^{-1} = -i(2\pi)^{-4}\int d_q^4\Delta_1^{-1}\Delta_2^{-1} \equiv h(M_d) \quad \text{... (267)}$$

where $\Delta_{1,2} = m_q^2 + q^2 - M_d^2/4 \pm q.P_{12}$, and we have indicated the result of integration by a function $h(M_d)$ of the mass M_d of the composite bound state (diquark). Unfortunately the integral eq. (267) logarithmically divergent, but it can be regularized with a 4D ultraviolet cut-off Λ, together with a Wick rotation, i.e., $q_0 \to iq_0$, which is allowed by the singularities of the two propagators. The exact result is:

$$16\pi^2\lambda^{-1} = 1 + \ln(4\Lambda^2/M_d^2)\ . \\ -2\frac{\sqrt{4m_q^2 - M_d^2}}{M_d}\arcsin(\mathrm{M_d}/2m_q) \quad \text{... (268)}$$

under the condition $M_d < 2m_q$. A slightly less accurate but much simpler form which is also easier to adapt to the qqq problem to follow, may be obtained by the Feynman method of introducing an auxiliary integration variable $u(0<u<1)$ to combine the two propagators, followed by a Wick rotation and a translation to integrate over d^4q (ignoring surface terms which formally arise due to the logarithmic divergence):

$$16\pi^2\lambda^{-1} = \ln\frac{6\Lambda^2}{6m_q^2 - M_d^2} - 1 \equiv 16\pi^2 h(M_d), \quad \text{... (269)}$$

Thus defining a diquark 'self-energy' function $h(M)$ where the 'on-shell' value is $M=M_d$. Eq. (269) also provides a determining equation for the NJL strength parameter λ in terms of the 'diquark' mass M_d and the cut-off paramter Λ, in a clearly 4D invariant form.

C.2 NJL-qqq Bound State Problem

We now set up the corresponding NJL-qqq problem under the same q-q contact interaction strength λ. Using the same notation for the various 4-momenta and propagators as listed in Sec. 10, the 4D wave function $\Phi(\zeta, \eta; P)$ expressed in terms of any of the S_3 invariant pairs(ζ_i, η_i) of internal 4-momenta satisfies the BSE:

$$i(2\pi)^4\Phi(\xi, \eta; P) = \sum_{123}\lambda\Delta_1^{-1}\Delta_2^{-1}\int d^4q'_{12}\Phi(\xi'_3, \eta_3; P) \quad (270)$$

where the arguments of Φ on the LHS are not-indexed since it is S_3-symmetric as a whole, while those on the RHS are indexed in order to indicate which subsystem is in pairwise interaction (see explanation in Sec. 10). The solution of this equation may be read off from the observation the integration w.r.t. $q'_{12} = \sqrt{3}\xi'_3/2$ leaves the respective integrals as functions of η_i only, where i=1, 2, 3. Thus ref. [87b].

$$\Phi(\xi, \eta; P) = \sum_{123}\Delta_1^{-1}\Delta_2^{-1}F(\eta_3) \quad \text{... (271)}$$

where F is a function of a single variable η_i. Next, plugging back the solution (271) into the main equation (270), gives the following integral equation in a single variable η_3, as a routine procedure applicable to separable potentials[87b]

$$(h(M_d) - h(M_{12}))F(\eta_3) = -i(2\pi)^{-4}\Delta_3^{-1}\int d^4q'_{12})\Delta_1^{'-1} + (1 \leftrightarrow 2)] \quad \text{....} (272)$$

Note that the cut-off parameter Λ drops out from the LHS, as checked by substitution for $h(M)$ from (269). This means that the 4D diquark propagator $(h(Md) - h(M_{12}))^{-1}$ is formally independent of the cut-off Λ, in this simple NJL model.

Next, the meaning of the function $F(\eta)$ can be inferred from an inspection of eq. (271), on similar lines to 3D[87b] or 4D[89] studies: $F(\eta_3)$ is the 4D 'quark(3)-diquark (12)' wave function which is generated by an exchange force represented by the propagators $\Delta_1^{'-1}$ and $\Delta_2^{'-1}$ in the first and second terms on the RHS respectively. And the baryon-qqq vertex function V_3 corresponding to a break-up of the baryon into quark (3) and diquark (12) may be identified by multiplying this quantity with the product of the inverse propagators of quark (3) and diquark (12).:

$$V_3 = V(\eta_3) = \Delta_3 f(\eta_3)F(\eta_3) \quad \text{...} (273)$$

where the diquark inverse propagator is reexpressed as

$$f(\eta_3) = h(M_d) - h(M_{12})$$
$$= (4\pi)^{-2}\, ln\, \frac{6m_q^2 + \eta_3^2 - 4M_B^2/9}{6m_q^2 - M_d^2}, \quad \text{...} (274)$$

making use of eq. (269) and the kinematical relation $\Delta_i = m_q^2 + \eta_i^2 - M_B^2/9$, where M_B is the mass of the bound qqq state, and i=1,2,3. The quantity V_3 of eq. (273) may be compared directly (except for normalization) with the corresponding '3D-4D-BSE' eq. (212).

C.3 Solution of the Bound qqq State Eq. (272)

We now turn to the Lorentz structure of the NJL-qqq equation (272), as well as an approximate analytic solution for the energy eigenvalues of the bound qqq states. To that end we substitute (273) in (272) to give an integral equation for $V(\eta_3)$, with $\eta'_2 \equiv \eta$ for short

$$V(\eta_3) = -2i(2\pi)^{-4}\int d^4\eta V(\eta)f^{-1}(\eta)\times(m_q^2 + \eta^2 - M_B^2/9)^{-1}(m_q^2 + (\eta_3 + \eta)^2 - M_B^2/9)^{-1} \quad \text{...} (275)$$

where the factor 2 on the RHS signifies equal effects of the two terms on the RHS of (272). For a bound state solution of this equation, with $M_B < M_d + m_q$, the singularity structures permit a Wick rotation $\eta_0 \to i\eta_0$ which converts η into a Euclidean variable η_E. This shows without further ado that eq. (275) is 4D-invariant just like its qq counterpart eq. (267). This is not quite the same thing as the old result[14] on O(4)-like spectra with harmonic confinement in the limit of infinite quark mass[14], since this NJL-Faddeev model of contact interaction, patterned after similar approachs[89], lacks a confining interaction, so that although in principle eq. (275) predicts a spectrum of bound states at the qqq level (starting with NJL (contact) pairwise interactions), such spectra cannot be a realistic representation of the *actual* hadron spectra[13]. We now show how this comes about via Wick rotation in (275).

For an approximate analytic solution of eq. (275), note that the logarithmic function $f(\eta)$ in the integral appearing on the right is slowly varying, so that not much error is incurred by taking it out of the integral and replacing it with an average value $V(\eta)$, *provided* any further logarithmic dependence on η is also similarly treated for consistency. The integral is now exactly of the type (267), i.e., logarithmically divergent, and can be handled successively by Wick rotation, Feynman auxiliary variable u, and a translation. The result is again of the form (269), and after cancelling out the factors $V(\eta_3)$ and $V(\eta)$ from both sides, the eigenvalue equation reads:

$$\langle f(\eta)\rangle = 2(4\pi)^{-2}\left[\ln\frac{\Lambda^2}{\langle\eta_3^2/6\rangle + m_q^2 - M_B^2/27} - 1\right] \quad \text{...} (276)$$

To simplify this equation, we express all quantities in terms of the $h(M)$ functions given in (269) and (274) and ignore the difference between $\eta = \eta'_2$ and η_3 inside the logarithms, to give

$$h(M_d) - h(M_{12}) = 2h(M_{12});$$
$$\Rightarrow \lambda^{-1} = h(M_d) = 3h(M_{12}) \quad \text{...} (277)$$

The last equation brings out clearly the fact that the baryong binding comes about from *three* pairs of qq interaction, albeit off-shell, since the function $\langle M_{12}^2 \rangle = \langle n_3^2 \rangle - 4M_B^2/9$ still depends on the (average) value of η^2. The qualitative features are thus on expected lines, but this oversimplified model is not intended for a realistic fit to the nucleon/Delta masses (which at minimum require the introduction of spin-isospin d.o.f.), beyond the general feature of a quark-diquark structure that characterizes an NJL-Faddeev approach[89], as expected from any separable potential[87b], of which the NJL model is a special case.

C.4 Comparison of NJL-Faddeev with 3D-4D-BSE

We end this Appendix with a comparison between the vertex functions (212) and (273). The NJL-Faddeev form (273) of V_3 is Lorentz invariant, being derived from a BSE with a constant kernel, viz. the *K=const* limit of 3D-4D BSE[16]. Its quark diquark form merely reflects the 'separable' nature of the NJL model[4]. there is no motivation here for a 3D BSE reduction, or 4D reconstruction, since 4D invariance is in-built throughout.

In contrast, the vertex function (212), obtained from 3D-4D-BSE[47], is merely Lorentz covariant due to the 3D kernel support, but the derivation is otherwise more general than NJL-Faddeev, since it is valid for any spatial form of the kernel as long as it is 3D in content. This leads to an exact 3D reduction of the (4D) BSE whose formal solution is a 3D wave function $\phi(\bar{\xi},\hat{\eta})$, a function of *two* independent 3-momenta[10b], in contrast to its NJL counterpart $F(\eta_3)$ in (273) which is a function of a single 4-momentum η_3 only. The denominator function $(D(\hat{q}_{12})$ of (212) similarly is a 3D counterpart of the corresponding 4D inverse propagator $f(\eta_3)$ in (273). Finally the big radical in (212) corresponds to the inverse propagator Δ_3 in (273), except for its more involved structure, which we now seek to explain.

While the 'zero extention' in the temporal direction is common to both approaches, NJL-Faddeev has also a zero spatial extension, while 3D-4D-BSE has a 'normal' spatial extension. Thus any ambiguity in the reconstruction of the 4D wave function from the 3D form of the 3D-4D BSE, vanishes in the *K=const* limit, so that the same is directly attributable to the (mere Lorentz covariant) 3D form of the BSE kernel. Indeed the 1Dδ-function in (212) fills up an information gap in the reconstruction from a truncated 3D to the full 4D Hilbert space in the simplest possible manner, while satisfying a vital self-consistency check by reproducing the full structure (200) of the 3D BSE. This already lends *sufficienty* to the ansatz (206) which leads to (212). As to its 'necessity', this ansatz has certain desirable properties like on-shell propagation of the spectator in between two successive interactions, as well as an explicit symmetry in the p_3 and p'_3 momenta. There is a fair chance of its uniqueness within some general constraints, but this is still short of a formal 'proof of necessity'.

The other question concerns the compatibility of the 1D δ-function in (212) with the standard requirement of connectedness[87]. Both the δ-function and the Δ_{3F} propagator appear in *rational* forms in the 4D Green's function, eq. (209) reflecting a free on-shell propagation of the spectator between two vertex points. The square root feature in the baryon-qqq vertex function (212) is a technical artefact corresponding to an equal distribution of this singularity between the initial and final state vertex points, and has no deeper significance. Furthermore, as the steps in Sec. 10.2 indicate, the three-body connectedness has already been achieved at the 3D level of reduction, so the 'physics' of this singularity, generated via eq. (206) must be traced to some mechanism other than a lack of connectedness[87] in the 3-body scattering amplitude. A plausible analogy is to a sort of (Fermi-like) 'pseudopotential' of the type employed to simulate the effect of chemical binding in the coherent scattering of neutrons from a hydrogen molecule in connection with the determination of the *singlet n-p* scattering length[92]. Such δ-function potentials have no deeper significance other than depicting the vast mismatch in the frequency scales of nuclear and molecular interactions. In the present case, the instantaniety in time of the pairwise interaction kernel in an otherwise 4D Hilbert space causes a similar mismatch, needing a 1D [*delta*-function to fill the gap. And just as the 'pseudo-potential' in the above example[92] does not have any observable effect, the singularity under radicals in (212) will *not* show up in any physical amplitude for hadronic transitions via quark loops, since the Green's functions (209) involve both the δ-function and the propagator Δ_{3F} in *rational* forms before the relevant quark loop integrations over them are performed.

Appendix D: SU(6)⊗O(3) Wave Fns In Complex Basis

In this Appendix, we outline a general method of expressing the qqq wave functions in a complex basis[29b,46b]. Such a basis gives a compact realization of the doublet representation of the permutation group S_3, with the two complex vectors z, z* substituting for the real pair ξ, η. The action of the permutations P_{ij} on this basis in the order (12); (31); (23) is [94a]

$$P_{ij}z = [1;\, e^{2i\pi/3};\, e^{-2i\pi/3}]_{z^*};$$
$$P_{ij}z^* = [1;\, e^{-2i\pi/3};\, e^{2i\pi/3}]z \qquad \ldots (278)$$

Identical doublet representations hold for the orbital ψ, spin χ and isospin ϕ d.o.f's, in the notation of Sections 9-11. To that end, define the corresponding complex quantities (except for an overall I-factor)

$$\sqrt{2}[\psi_c \chi_c;\, \phi_c] \equiv [\psi''-i\psi';\, \chi''-i\chi';\, \phi'''-i\phi'] \qquad \ldots (279)$$

together with a second set of complex conjugate relations. Using these definitions, the action of permutation group on the full wave function is

$$P_{ij}[\psi_c \otimes \chi_c \otimes \phi_c;\, \psi_c^* \otimes \chi_c^* \otimes \phi_c^*] = [\psi_c^* \otimes \chi_c^* \otimes \phi_c^*;\, \psi_c \otimes \chi_c \otimes \phi_c] \qquad \ldots (280)$$

Another important result concerns the action of P_{ij} on any pair of component wave functions (λ_c, μ_c^*), where (λ_c, μ_c) are any two out of the full set (ψ, χ, ϕ) of three[94a].

$$P_{ij}[\lambda_c \otimes \mu_c^*;\, \lambda_c^* \otimes \mu_c] = [\lambda_c^* \otimes \mu_c;\, \lambda_c \otimes \mu_c^*] \qquad \ldots (281)$$

As to the quantities $(\psi_s; \psi_a)$, they are S_3-singlets by themselves, with eigenvalues ±1 for the P_{ij} operators. The properly symmetrized $SU(6)\times O(3)$ states, eqs. (168)-(172), are now:

$$|56\rangle^q = \psi^s \chi^s \phi^s;\, |56\rangle^d = (\chi_c \phi_c^* + \chi_c^* \phi_c)/\sqrt{2}; \qquad \ldots (282)$$

$$|70\rangle^q = \chi^s(\psi_c \phi_c^* + \psi_c^* \phi_c)/\sqrt{2};$$
$$|70\rangle^d = (\psi_c \chi_c \phi_c + \psi_c^* \chi_c^* \phi_c^*)/\sqrt{2} \qquad \ldots (283)$$

$$|20\rangle^q = \psi_a \chi^s \phi_a;\, |20\rangle^d = \psi_a(\chi_c \phi_c^* - \chi_c^* \phi_c)/\sqrt{2} \qquad \ldots (284)$$

D.1 Construction of ψ-Fns in Complex Basis

We now turn to the construction of the orbital ψ-functions in terms of (z_i, z_i^*), so as to preserve the total angular momentum adapted to the complex language.To that end, the angular momenta (both diagonal and 'mixed') in the complex basis are given by

$$\mathbf{L}_z = -i\mathbf{z}\times\nabla_z^*;\, \mathbf{L}_{z^*} = +i\mathbf{z}^*\times\nabla_z^*;$$
$$\mathbf{L}_c = -iz^*\times\nabla_z;\, \mathbf{L}_c^* = +iz^*\times\nabla_z^* \qquad \ldots (285)$$

which obey the connections

$$\mathbf{L} = \mathbf{L}_z + \mathbf{L}_{z^*} = \mathbf{L}_\xi + \mathbf{L}_\eta;\, \mathbf{L}_z = \mathbf{L}_a = \mathbf{L}_z + \mathbf{L}_{z^*} \qquad \ldots (286)$$

These quantities transform according to eq. (278) under the elements P_{ij} of S_3.

To construct angular momentum states of correct S_3 symmetry, it is useful to take those of highest seniority[94b], now expressed in appropriate powers of z_+ and z_+^*, and to note that $\mathbf{z.z}^*$, $z_+z_+^*$ and z_+^3 are all S_3-invariant. The angular momenta carried by these basic units are easily checked to be in conformity with the above (complex) definitions (285) and (286) of the angular momenta. Using these basic building blocks, the natural parity states of highest seniority for a given angular momentuma are compactly written in an HO basis as[46b]:

$$|56^+;\, 70^-;\, 70^+;56^-\rangle = (2z_+z_+^*)^l[1;z_+;\, z_+^2;\, z_+^3]e^{-2zz^*} \qquad \ldots (287)$$

The superscripts ±on the various states on the LHS *correctly* describe their parity structures, by noting that z_+^n has parity $(-1)^n$, while zz^* is a 3-scalar. The L^P-values of the states (287) in this order are[46b,29b]:

$$L^P = (2\ell)^+;\, (2\ell+1)^{-1};\, (2\ell+2)^+;\, (2\ell+3)^-; \qquad \ldots (288)$$

while ℓ goes through the values 0, 1, 2, 3..., thus bringing out the *naturalness* of the respective parity structures.

In a similar way it is possible to systematically span all the "unnatural" parity states in the same representation[46b], noting that the main carrier of unnatural parity is the axial vector $\zeta = iz \times z^*$, which is a fully antisymmetric S_3-singlet. The L^P-structures of such states of highest seniority, corresponding to the series (287) are[46b,29b]:

$$|20^+; 70^-; 70^+; 20^-\rangle = \zeta_+(2z_+z_+^*)^\ell \times [1; z_+; z_+^2; z_+^3]e^{-2zz^*} \quad \ldots (289)$$

together with the respective L^P-values

$$L^P = (2\ell+1)^+; (2\ell+2)^-; (2\ell+3)^+; (2\ell+4)^-; \quad \ldots (290)$$

thus bringing out the únnaturalness' of their respective parities. For the construction of more involved states on these lines, see[46b].

A similar construction is possible for the 'spin' wave functions in the complex basis; see[46b] for details.

D.2 Normalization of Natural and Unnatural Parity Baryons

We now outline a new method of integration for the normalization of the spatial wave functions (287) and (289) in the 6D (**z, z***) space, which is rather well-suited to the (complex) variables on hand; see ref. [29b, 46b]. The volume measure in this 6D space is expressible in the spherical basis as

$$d^6\tau = d^3zd^3z^* = (dz_+dz_-^*)(dz_-dz_+^*)(dz_3dz_3^*) \ldots (291)$$

where the six elements on the RHS of (291) have been rearranged into 3 sets of *real* 2D volumes, since the three pairs on the RHS, each form complex conjugate pairs. Now put

$$\sqrt{2}(z_+; z_-^*) = R_1 e^{\pm i\theta_1}; \sqrt{2}(z_-; z_=^*) = R_2 e^{\pm i\theta_2};$$
$$\sqrt{2}(z_3; z_3^*) = R_1 e^{\pm i\theta_3} \quad \ldots (292)$$

Then the volume element becomes

$$d^6\tau = R_1dR_1d\theta_1R_2dR_2d\theta_2R_3dR_3d\theta_3 \quad \ldots (293)$$

$$0 \le R_{1,2,3} \le \inf; 0 \le \theta_{1,2,3} \le 2\pi;$$

$$R_1^2 + R_2^2 + R_3^2 = 2\mathbf{z}.\mathbf{z}^* \equiv R^2 \quad \ldots (294)$$

Since the phase angles (not quite Euler angles) will not appear in the squared modulii of the wave functions, these are integrated out to give

$$d\tau = \pi^3 dR_1^2 dR_2^2 dR_3^2 \quad \ldots (295)$$

Now the natural parity sequence (287) is compactly expressed as

$$\psi = N_{\ell n}(2z_+z_+^*)^\ell z_+{}^n e^{-R^2/2} \quad \ldots (296)$$

where n=0,1,2,3 for the states (287) in sequence, and the normalizer is

$$N_{\ell n}^{-2} = \int d^6\tau[R_1^2R_2^2]^\ell(R_1^2/2)^n e^{-R^2}$$
$$= \pi^3\Gamma(\ell+1)\Gamma(\ell+n+1)/2^n \quad \ldots (297)$$

which agrees with the result for the ξ, η-representation[29a]. For the unnatural parity sequence (289), the extra ζ-factor gives

$$(\zeta_+\zeta_-^*) = R^4/4 - R_3^4/4 - R_1^2R_2^2 \quad \ldots (298)$$

Denoting the corresponding normalizers by $\tilde{N}_{\ell n}$, similar integration now leads to the result

$$\tilde{N}_{\ell n}^{-2} = \frac{\pi^3\Gamma(\ell+1)\Gamma(\ell+n+1)}{12(2^n)} \times[(\ell+n+1)(n+2)+(\ell+1)(\ell+4)] \quad \ldots (299)$$

Radial excitations can be similarly handled. E.g.,. one radial excitation gives an extra multiplicative factor in the normalization integral (297) for natural parity states, giving rise to an extra factor $(2\ell+n+4)$ in (297). Further, the reciprocity between the momentum and coordinate spaces implied in an HO description as above, allows the same formulation to be adapted in the dual space, a result which is useful for evaluating the *OGE* corrections to the mass formula (232).

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