On Simulating Liouvillian Flow From Quantum Mechanics Via Wigner Functions

A.N.Mitra *

244 Tagore Park, Delhi-110009, India.

R.Ramanathan

Dept. of Physics, Univ. of Delhi, Delhi-110007, India.

13th November 1997

Abstract

The interconnection between quantum mechanics and probabilistic classical mechanics for a free relativistic particle is derived in terms of Wigner functions (WF) for both Dirac and Klein-Gordon (K-G) equations. Construction of WF is achieved by first defining a bilocal 4-current and then taking its Fourier transform w.r.t. the relative 4-coordinate. The K-G and Proca cases also lend themselves to a closely parallel treatment *provided* the Kemmer-Duffin β-matrix formalism is employed for the former. Calculation of WF is carried out in a Lorentz-covariant fashion by standard 'trace' techniques. The results are compared with a recent derivation due to Bosanac.

PACS: 05.60.+w ; 03.65.Ca ; 11.10.Qr ; 11.40.-q

*e.mail: (1) insa@giasdl01.vsnl.net.in(subj:a.n.mitra); (2) anmitra@csec.ernet.in

1 Introduction: Wigner Function As Link Between QM and CM

Since the advent of Quantum Mechanics (QM), there have been several investigations aimed at understanding Schroedinger equations and their relativistic counterparts from stochastic /statistical premises in classical mechanics, epitomised by Focker-Planck [1,2] and Liouville equations [3-6]. In the process most such investigations have had to resort at some stage to some sort of ansatz designed to introduce the 'square-root' of a probability function [1-4], or make a more direct use of the wave function from the outset [5,6]. This of course does not address the core issue of the foundations of quantum physics, viz., the existence of complex probability amplitudes obeying the "superposition principle", and the need to invoke the "collapse hypothesis" to explain the outcome of a quantum measurement. Even without any commitment to resolve this central mystery of conventional quantum physics, it is both interesting and illuminating to explore the classical roots of quantum mechanics by comparing and relating their respective mathematical structures as in several contemporary investigations [3-6].

Among the several approaches to an alternative 'derivation' of QM, we would like to pick out for explicit scrutiny, a recent formulation due to Bosanac [4] in which the author makes essential use of the probabilistic (instead of 'trajectory-oriented') approach to Classical Mechanics (CM) via the Liouville equation, wherein the Uncertainty Principle is sought to be incorporated at the outset, as a means of capturing a good chunk of QM. Our purpose in this regard is more in the nature of an 'inverse' (to [4]) approach, which stems from a desire to give a more direct role to the QM wave function born out of its 'being already there', thereby according it a more open role in the construction of the Wigner Function [7], as in other orthodox approaches [5,6]. As we shall see, this approach has the advantage of presenting the treatment of different types of relativistic equations (Dirac, Klein-Gordon, Proca, etc) within a unified framework.

The central quantity that satisfies the Liouville equation is the Wigner-function [7] which in itself was conceived within the QM framework, as a bilinear in the 'wave function'. Granting the existence of the wave function in turn implies the existence of its Fourier transform which represents its momentum space counterpart, and the two together guarantee a built-in effect of the Uncertainty Principle, as clearly recognized in [4]. Using such a (hybrid) approach in which the prior existence of the wave function is implicit in the structure of the Wigner function [7], Bosanac [4] claims to derive the Schroedinger equation from the Liouville equation, consistently with current conservation, apparently as an inverse(?) problem. An extension to the corresponding relativistic problem, rather surprisingly, seemed to yield only the Dirac equation (spinor), but not the Klein-Gordon (K-G) equation (scalar), a result whose mystery we seek to clarify in this paper and (hopefully) pave the way to a generalization (see below).

Specifically, we have been actuated by a desire to re-examine the issue of Dirac versus K-G [4], but with a more conscious recognition of an active role of the 'wave function' at the outset, and in the process we find that the Liouville and current conservation equations for both the Dirac and K-G cases follow from identical premises, *provided* one has no qualms about the free use of *multi-component* wave functions in both Dirac and K-G cases, and not (selectively) for the Dirac case only [4]. Indeed it is easily seen from a detailed perusal of the steps in [4] that the author's success in deriving the Dirac

equation is crucial to his *ansatz* of a 4-component spinor, without which the necessary consistency checks would not be forth-coming. On the other hand he did not seem to consider a similar possibility for the K-G case, viz., the existence of the less familiar Kemmer-Duffin equations for integral spins [8], wherein the multicomponent structure of the corresponding wave function satisfying a first order equation as in the Dirac case seems to be the right form for putting the K-G and Proca cases on parallel footing with Dirac. Of course the question of whether, inspite of a multicomponent ansatz for the K-G wave function, a bosanac-like treatment [4] of working backwards from the Liouville-cum-current-conservation equations to the K-G equation would have succeeded, is still open, since the Kemmer-Duffin β-matrices are singular, unlike the Dirac matrices which are not (see Sec.4).

In carrying out this exercise we have been guided solely by the orthodox (Dyson [9]) philosophy that a relativistic quantization of a *single* particle has inherent problems of consistency, whose resolution hinges on the simultaneous existence of an infinite number of particles, i.e., on a *field* concept. Indeed, Dirac's postulation of the infinite sea (of negative energy particles) in the vacuum was enough ground for Pauli-Weisskopf [10] to reinterpret the single particle density from the K-G equation as an average charge density (which could 'locally' have either sign) [9]. Of course from a practical point of view, a single particle interpretation has considerable appeal, but it requires the 'protective cover' of Feynman's positron theory [11] which mandates $\pm$ time directions to be associated with $\pm$ energy propagations respectively.

In what follows, we shall keep in the forefront, the results of [4] for pointwise comparison with the derivation to be presented here under the orthodox 'field' paradigm [9] which is routinely available in any standard text [12]. In Sec.2, we give a 'direct' derivation of the Wigner function for the Dirac equation and thence the corresponding Liouville plus current conservation equations, a procedure whose steps are roughly in 'inverse' order to those of [4]. In Sec.3, we do the corresponding exercise for the K-G and Proca cases via the Kemmer-Duffin [8] formulation [13], and in so doing, bring out the strong similarity of the two systems. Sec.4 concludes with a short discussion, including a comparison with [4].

2 Wigner Function for the Dirac Equation

We begin by writing the Dirac equation for a free field $\psi(x)$ in the 'Euclidean' covariant notation as [13]

$$(\gamma_\mu \partial_\mu + m)\psi(x) = 0; \quad \bar{\psi}(x)(m - \partial_\mu \gamma_\mu) = 0. \tag{2.1}$$

Now define a bilocal current $j_\mu(x_1, x_2)$ as

$$j_\mu(x_1, x_2) = \bar{\psi}(x_1) i\gamma_\mu \psi(x_2) \tag{2.2}$$

Next put $x_1 = x - q$ and $x_2 = x + q$ as in [4], and define a 4-vector $V_\mu(x, p)$ as

$$V_\mu(x, p) = \pi^{-4} \int d^4q e^{2ip.q} \bar{\psi}(x + q) i\gamma_\mu \psi(x - q). \tag{2.3}$$

The fourth component of V_μ may be identified with the standard WF ρ. As a check on this quantity, its integral w.r.t. d^4p yields the usual 4-current $J_\mu(x)$ as

$$J_\mu(x) = \int d^4p V_\mu(x, p) = \bar{\psi}(x) i\gamma_\mu \psi(x) \tag{2.4}$$

from which the standard current conservation equation is recovered as

$$\partial_\mu J_\mu(x) = 0. \tag{2.5}$$

Next we make a Gordon reduction of V_μ (always possible for a free field $\psi(x)$) so as to bring out the separate contributions V^c_μ and V^s_μ arising from the 'convective' and 'spin' currents respectively:

$$V^c_\mu(x,p) = \pi^{-4}\int d^4q e^{2ip.q}\bar{\psi}(x+q)\frac{p_\mu}{m}\psi(x-q); \tag{2.6}$$

$$V^s_\mu(x,p) = \pi^{-4}\partial^{(x)}_\nu \int d^4q e^{2ip.q}\bar{\psi}(x+q)\frac{\sigma_{\mu\nu}}{2m}\psi(x-q). \tag{2.7}$$

We check from these that *two* separate conservation equations follow:

$$\partial_\mu V^c_\mu(x,p) = \partial_\mu V^s_\mu(x,p) = 0, \tag{2.8}$$

the former being a consequence of the K-G equation for each component of $\psi(x)$ and $\bar{\psi}(x)$, and the latter being identically satisfied.

We now seek to use the Gordon reduction as a key element for deriving the Liouville equation a la [4], and in so doing bring out its role in putting our procedure in closer correspondence to [4]. For, the central issue is one of demonstrating that the density function $\rho(x,p)$ and the current function $V_i(x,p)$ are related by the 'classical' velocity factor v_i, so as to ensure the 'correct' structure of the Liouville equation [4]. To that end we note from (2.3,6,7) that the effect of Gordon reduction in (3) may be expressed in operator form sandwiched between ψ and $\bar{\psi}$:

$$i\gamma_\mu = \frac{p_\mu}{m} + \frac{1}{2m}\partial^{(x)}_\nu \sigma_{\mu\nu} \tag{2.9}$$

The spatial part of this equation is seen as the precise counterpart of eq.(27) of [4] (after integration w.r.t. d^4pd^4q), with the following identifications: The convective term p_μ in (2.9) has a space part p_i which corresponds to the j_i term of [4], since it bears the ratio v_i to the time component p_0, as the correct factor connecting the current function $V_i(x,p)$ to the density function $V_0(x,p)$. The space part of the full γ_μ term collectively represents the two terms $a_i d$ and $a_i\epsilon$ in eq.(27) of [4]. Finally the last (derivative) term in (2.9) corresponds to the $\nabla\times\mathbf{\Phi}$ term in eq.(27) of [4] since it has a zero divergence by itself, as seen from the second part of eq.(2.8), and therefore has no separate influence on the equation of continuity. Therefore, following the logic of [4], this term has no effect on the anatomy of the Liouville equation which is thus entirely contained in the first (convective) part of eq.(2.8). To reveal this structure more explicitly, the $\rho(x,p)$ function of [4] reads in our 4×4 Dirac matrix notation, as follows:

$$\rho(x,p) \equiv V_0(x,p) = \pi^{-4}\int d^4q e^{2ip.q}\bar{\psi}(x+q)\gamma_4\psi(x-q). \tag{2.10}$$

which satisfies the Liouville equation

$$\partial_0\rho(x,p) + v_i\partial_i\rho(x,p) = 0 \tag{2.11}$$

that is merely a paraphrase of the first part of eq.(2.8) representing the conservation of the *convective* part of the V_μ function.

To end this section, we outline a construction for the ρ function analogously to eqs.(51-53) of [4], using a fully covariant procedure. To that end, we write down Fourier representations for the Dirac spinors on the lines of [12], but adapted to the Euclidean phase convention [13]:

$$\psi(x_1) = (2\pi)^{-3/2} \int d^3p_1 (m/E_1)^{1/2} [b_{p_1 r_1} u^{r_1}(p_1) e^{ix_1.p_1} + d^*_{p_1 r_1} v^{r_1}(p_1) e^{-ix_1.p_1}]; \tag{2.12}$$

$$\bar{\psi}(x_2) = (2\pi)^{-3/2} \int d^3p_2 (m/E_2)^{1/2} [b^*_{p_2 r_2} \bar{u}^{r_2}(p_2) e^{-ix_2.p_2} + d_{p_2 r_2} \bar{v}^{r_2}(p_2) \times e^{ix_2.p_2}]; \tag{2.13}$$

where b and d are (c-number) electron and positron amplitudes respectively and u and v are the corresponding positive and negative energy spinors. The latter are more conveniently expressed in terms of Lorentz boost operators as follows [12]:

$$u^{r_1}(p_1) = [(m - i\gamma.p_1)/(2\sqrt{m(m+E_1)})] u^{r_1}(0); \tag{2.14}$$

$$v^{r_1}(p_1) = [(m + i\gamma.p_1)/(2\sqrt{m(m+E_1)})] v^{r_1}(0). \tag{2.15}$$

These Lorentz boost factors are proportional to energy projection operators for the corresponding spinors. To specify their spin states, we can further multiply these quantities by the spin projection operators P_n [12]

$$P_n = [1 + i\gamma.\hat{n}\gamma_5]/2, \tag{2.16}$$

where $\hat{n}$ is a unit 4-vector representing the direction of spin [12]. After these manipulations on the spinors through the sequence of equations (2.10-16), and their substitution in the defining eq.(2.3) for V_μ, the spinor dependence can be totally removed by 'tracing'. To simplify these expressions note that only the b^*b and d^*d terms will survive, while the cross terms will drop out of the traces, since the $\pm$ energy spinors at a given momentum (zero) are orthogonal to each other. The rest of the simplification is routine and the result can be compactly expressed in terms of two quantities $Tr^{\pm}_{\mu}$ defined as follows:

$$Tr^{\pm}_{\mu} = \frac{\pm m(p_1+p_2)_\mu + \epsilon_{\mu\lambda\rho\sigma} p_{1\lambda} p_{2\rho} \hat{n}_\sigma}{2\sqrt{E_1(m+E_1)E_2(m+E_2)}} \tag{2.17}$$

Before substituting these quantities in eq.(2.3) for $V_\mu(x,p)$, note that of the two terms on the right, the first (convective) term has the desired structure for the 'Liouville form', eq.(2.11), but the second (spin) term has not. However, in accordance with the discussion immediately following eq.(2.9), in line with [4], one need not insist on the 'Liouville connection' separately for both the terms since the 'spin' term (proportional to $\hat{n}$) can be traced back to the 'derivative' term of (2.9) which has no separate effect on the equation of continuity. Therefore we need only the $\mu = 4$ component in eq.(2.17) above to extract the ρ-function fully, whence the corresponding 'current' function V_i can be generated by multiplying with $v_i = p_i/p_0$ only.

To proceed further, integration over d^3p_1 and d^3p_2 may be simplified via the transformations $p_{1,2} = p' \pm k/2$, remembering that $x_{2,1} = x \pm q$. Further, the 3D integrals may be converted to 4D integrations over p' and k via the identity

$$d^3p_1 d^3p_2 = 2E_1 2E_2 d^4p' d^4k \delta(p'^2 + m^2 + k^2/4) \delta(2p'.k) \tag{2.18}$$

making use of the result $\delta(a)\delta(b) = \delta((a+b)/2)\delta(a-b)$. Integration over d^4q now gives $\delta^4(p' \mp p)$ for the terms b^*b and d^*d respectively; subsequent integration over d^4p' gives $p' = \pm p$. Collecting all these results and substituting in (2.10), gives

$$\begin{aligned}\rho(x,p) &= \pi^{-3}\int d^4k\delta(p^2+m^2+k^2/4)\delta(2p.k) \\ &\times[b^*(\mathbf{p}+\mathbf{k/2})b(\mathbf{p}-\mathbf{k/2})e^{ik.x} + d^*(\mathbf{p}+\mathbf{k/2})d(\mathbf{p}-\mathbf{k/2})e^{-ik.x}] \\ &\times(2mp_0 + 2i\hat{n}.(\mathbf{p}\times\mathbf{k}))\sqrt{\frac{E_1E_2}{(m+E_1)(m+E_2)}} \end{aligned} \tag{2.19}$$

This expression corresponds to eqs.(51-53) of [4], after taking due account of the difference in the choice of the respective basis functions (Dirac- vs. Pauli- types), as well as the covariant vs. non-covariant notations. Note that the integration in (2.19) is basically angular $d\Omega_k$ in content, like in [4], since the 4D integration d^4k is also multiplied by the two δ-functions which take care of the magnitudes of both $\mathbf{k}$ and k_0. Our 3-vector $\mathbf{k}$ is $2\mathbf{k}$ of [4], and $\hat{n}$ is $\hat{\omega}$ of [4]. Finally, our expression for ρ compactly covers both $\pm$ energy components.

We next turn to the K-G and Proca cases via Kemmer-Duffin equations.

3 Wigner Function for Integral Spin

Because of the relative lack of exposure of the Kemmer-Duffin equations [8] in the contemporary literature, it will first be useful to recall some basic results which are summarized in Roman [13] in the Euclidean phase convention, from which we shall freely draw on several results without much comment. The first order differential equation for integral spin is expressible in β- matrix notation as [13]

$$(\beta_\mu\partial_\mu + m)\psi(x) = 0; \quad \bar{\psi}(m - \partial_\mu\beta_\mu) = 0 \tag{3.1}$$

where $\psi(x)$ is a multicomponent field, each of whose components ψ_i satisfies the K-G equation, and $\bar{\psi} = \psi^\dagger\eta_4$ and $\eta_4 = 2\beta_4^2 - 1$. The spin-0 and spin-1 cases correspond to 5- and 10- component irreducible representations respectively. These components are

$$\begin{aligned} Spin-0 : \psi_i(i=1-4) &= -\partial_\mu\phi(x)/m; \quad \psi_5 = \phi \\ Spin-1 : \psi_i(i=1-6) &= -F_{\mu\nu}/m; \quad \psi_i(i=1-4) = \phi_\mu \end{aligned} \tag{3.2}$$

The various quantities like bilocal 4-current $j_\mu(x_1,x_2)$, phase space 4-current $V_\mu(x,p)$, and local 4-current $J_\mu(x)$ follow very closely the corresponding constructions (1.2-5) for the Dirac case, with the replacements: γ_μ by β_μ ; γ_4 by η_4. Thus the phase space 4-current is given by

$$V_\mu(x,p) = \pi^{-4}\int d^4q e^{2ip.q}\bar{\psi}(x+q)i\beta_\mu\psi(x-q) \tag{3.3}$$

and its break-up into convective and spin components is achieved via 'Gordon reduction' analogously to the Dirac case. However the algebra of the β matrices is a bit more involved than that of the γ matrices because the former are singular. Thus [8,13]

$$\beta_\lambda\beta_\mu\beta_\nu + \beta_\nu\beta_\mu\beta_\lambda = \beta_\lambda\delta_{\mu\nu} + \beta_\nu\delta_{\mu\lambda}; \quad \eta_4^2 = 1 \tag{3.4}$$

The 'spin' operator $S_{\mu\nu}$ is given by

$$iS_{\mu\nu} = \beta_\mu\beta_\nu - \beta_\nu\beta_\mu \tag{3.5}$$

and, except for a factor of half, has exactly the same role as that of $\sigma_{\mu\nu}$, including its antisymmetry property. For writing down the counterpart of Gordon reduction, the following equations are needed:

$$(\partial_\mu - \beta_\nu\beta_\mu\partial_\nu)\psi(x) = 0; \quad \bar{\psi}(x)(\partial_\mu - \beta_\nu\beta_\mu\partial_\nu) = 0 \tag{3.6}$$

With the help of (3.1) and (3.6), and making use of the definition (3.5) for the 'spin', the matrix β_μ in (3.3) may be written in a form analogously to (2.9) as

$$i\beta_\mu = \frac{p_\mu}{m} + \partial_\nu^{(x)}\frac{S_{\mu\nu}}{m} \tag{3.7}$$

which shows a clear division between the convective and spin parts, with the latter having a 'derivative' structure whose divergence vanishes by itself, as in the second part of eq.(2.8). Therefore this term has no direct role in the Liouville equation like in the Dirac case, exactly as in [4].

So far this β-formalism for integral spin covers both spin-0 (K-G) and spin-1 (Proca) cases, and one may wonder why for the former there should be a spin term $S_{\mu\nu}$ at all, in the Gordon reduction (3.7). Actually this term is effectively *zero* for the spin-0 case since, as has been explained in Roman [13], for the 5×5 representation the eigenvalues of $S_3 = S_{12}$ are non-measurable. Indeed the probability density [13]

$$\rho = -i(\psi_4^*\psi_5 + \psi_5^*\psi_4) = -i(\partial_0\phi^*\phi - \phi^*\partial_0\phi)/m \tag{3.8}$$

vanishes in the eigenstates that correspond to $S_3 = \pm 1$, so that the presence of the $S_{\mu\nu}$-term in the spin-0 case is purely symbolic. Substituting this information in (3.3), and noting from (3.8) that only the components $\psi_{4,5}$ survive for the phase space density function $\rho(x,p)$ for the K-G case, the final result works out as

$$\rho(x,p) = \frac{1}{m}\pi^{-4}i\int d^4q e^{2ip.q}[\partial_0\phi^*(x+q)\phi(x-q) - \phi^*(x+q)\partial_0\phi(x-q)] \tag{3.9}$$

while eq.(3.7) ensures that the phase space current function $\mathbf{V}(x,p)$ is related to $\rho(x,p)$ by the factor $\mathbf{p}/p_0 = \mathbf{v}$, so that the structure of the Liouville equation is preserved. As a further check, the integration of (3.9) over d^4p reproduces the K-G density function $\rho(x)$ of (3.8) exactly. The last quantity is of course not positive definite, as expected [13] for the K-G case. Again as in the Dirac case, it is possible to write down an expression for $\rho(x,p)$ closely analogous to (2.19), except for the absence of the 'spin' term, in the third line. We omit this routine step for brevity.

For the Proca case, the 'spin' term in the Gordon reduction (3.7) will now contribute to the $\rho(x,p)$ function, while the Liouville connection is again maintained via the convective p_μ term only. From the structure of $\rho(x,p)$, which corresponds to $\mu = 4$, it is seen quite clearly that the 'spin' contribution arising from the second term of eq.(3.7) in the Proca case is proportional to $\mathbf{S}.\mathbf{E} \times \mathbf{H}$, where $\mathbf{E}$ and $\mathbf{H}$ are the 'electric' and 'magnetic' components of $F_{\mu\nu}$ respectively. Again we skip the Fourier representation (2.19) for the full $\rho(x,p)$ function, as this step will not illuminate the structure any further beyond the above demonstration of a close parallelism between the spin- 1/2 and integral spin (0,1) cases.

4 Discussion and Conclusion

This study was motivated by a desire to understand some results due to Bosanac [4] in which he claims to derive the Dirac equation, starting from the (classical) premises of the Liouville equation. What intrigued us was the assertion [4] that the 'derivation' leads to the Dirac equation, instead of to the Klein-Gordon equation. To examine this question more closely, we were led to start from the *opposite* direction to [4], viz., a 'direct' derivation of the Liouville equation through an explicit construction of the Wigner Functions [7] themselves, as in more orthodox approaches [5,6], with a view to throwing some light on this apparent asymmetry between the Dirac and the K-G cases. Rather surprisingly, our result indicates a close parallelism between both spin-1/2 and spin-(0,1) cases, brought about by the Kemmer-Duffin [8] formalism [13] for integral spin as a natural counterpart to the Dirac equation for spin-1/2. In both cases, we have made essential use of the Gordon reduction into 'convective' and 'spin' contributions to the 4-current, wherein the convective part gives the desired 'Liouville ratio' $p_0 : \mathbf{p}$ between the density and current functions in phase space, but the spin part does not. However the latter has a derivative structure by virtue of which its 4-divergence vanishes identically, so as to yield a 'spin' current conservation by itself [eq.(2.8)], and hence has no direct effect on the dynamics of the Liouville equation as such, which is entirely governed by the convective term only. This result seems to accord rather well with the corresponding result of Bosanac [4], viz., his eq.(27), despite a vast difference in the respective procedures. In particular, since Gordon reduction in our derivation has proved crucial for a neat identification of the origin of these two terms as convective and spin contributions respectively, it may be safely assumed that the same mechanism underlies the derivation in [4] as well, irrespective of the manner of the derivation.

Without going into the philosophical aspects of the classical foundations of Quantum Mechanics, it is pertinent to ask why only the Dirac equation has been amenable to the Bosanac derivation [4]. For, it will be rather naive to argue that no substantial quantum mechanical input has gone into the latter approach, since the very structure of the Wigner Function as a bilinear in the (quantum mechanical) wave function is a tacit acknowledgement of the existence of the latter, with the Uncertainty Principle as a built-in corollary of the same. Further, the structure of the Dirac equation is implicit in the ansatz [4], eq.(16), in terms of Pauli spinors F_1, F_2, with anticommutation relations (23-25) characteristic of Dirac matrices in the Pauli 2-component form, all of which ingredients have been crucial to the (indirect) derivation of the main eq.(27). Indeed the entire exercise [4] is a fine example of how to work backwards from the (bilinear) Liouville equation to the (linear) Dirac equation, albeit in the Pauli 2-component form. Yet it goes without saying that the ansatz of a 4-component spinor (satisfying a Dirac-type equation (16) [4]) has been very much of an input in the derivation.

A comparison with our 'direct' derivation would suggest that a possible reason why the K-G equation was not amenable to the Bosanac derivation was the missing element of a multicomponent wave function in his formulation of the K-G problem, something we have sought to rectify through the Kemmer- Duffin β-matrix formalism [8] which restores the formal analogy to the Dirac case. However, and this may be important, even if the Bosanac formalism had included a multicomponent wave function for integral (including zero) spin field, it is doubtful if working backwards from the Liouville to the K-G equation would have proved successful in this case, since the β- matrices are *singular*, unlike the

Dirac matrices, so that a routine inversion of steps on the lines of [4] might well not have been possible; (we did not succeed in this effort). But there seems to be no problem in the 'forward' steps (from QM equation to Liouville) going through in the K-G and Proca cases with equal ease, as the above derivation indicates.

The foregoing comparison would suggest that the structure of Quantum Mechanics is probably richer and more varied than can be fully captured in one go, through alternative formulations. And yet the basically indeterministic nature of QM poses a continuous challenge to innovative ideas designed to capture its full flavour from 'objective-realist' premises, so as to provide a more 'classical' approach to its uncanny predictions. Ref.[4] may be regarded as one more effort in this direction, as part of an ongoing movement [14] for resolving the foundational paradoxes of orthodox QM which has acquired fresh momentum with heavyweights for and against the Copenhagen Interpretation arrayed in comparable strengths on both sides of the Great Divide [15].

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