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Abstract

Using the method of Green's functions within the framework of a Bethe-Salpeter formalism characterized by a pairwise qq interaction with a 3D support to its kernel (expressed in a Lorentz-covariant manner), the 4D BS wave function for a system of three identical relativistic spinless quarks is reconstructed from the corresponding 3D form which satisfies a *fully connected* 3D BSE. This result is a 3-body generalization of a similar interconnection between the 3D and 4D 2-body wave functions that had been found earlier under identical conditions of a 3D support to the corresponding BS kernel, using the ansatz of Covariant Instaneity for the pairwise $q\bar{q}$ interaction. The generalization from spinless to fermion quarks is straightforward.

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1 Introduction: Statement of Problem and Summary of Results

The problem of connectedness in a three-particle amplitude has been in the forefront of few-body dynamics since Faddeev's classic paper [1] showed the proper perspective by emphasizing the role of the 2-body T-matrix as a powerful tool for achieving this goal. The initial stimulus in this regard came from the separable assumption to the two-body potential [2] which provided a very simple realization of such connectedness via the T-matrix structure envisaged in [1], a result that was given a firmer basis by Lovelace [3]. An alternative strategy for connectedness in an n-body amplitude was provided by Weinberg [4] through graphical equations which brought out the relative roles of T- and V- matrices in a more transparent manner. As was emphasized in both [3] and [4], an important signal for connectedness in the 3-body (or n-body) amplitude is the *absence of any delta function* in its structure, either explicitly or through its defining equation. This signal is valid irrespective of whether or not the V- or the T- matrix is employed for the said dynamical equation. The above results were found for a non-relativistic n-body problem within a basically 3D framework whose prototype dynamics is the Schroedinger equation. For the corresponding *relativistic* problem whose typical dynamics may be taken as the Bethe-Salpeter equation (BSE) with pairwise kernels within a 4D framework, it should in principle be possible to follow a similar logic, using the language of Green's functions with corresponding diagramatic representations [4], leading to equations free from delta functions. However there are other *physical* issues associated with a 4D support to the BS kernel of a 'confining' type, such as O(4)-like spectra [5], while the data [6] exhibit only O(3)-spectra. To handle this issue in a realistic and physically plausible manner, there have been many approaches in the literature (which hardly need to be cited) that are centred on a basically "instantaneous" approximation to the (pairwise) two-particle interaction. In the spirit of this general philosophy, and keeping close to the *observational* features of the hadron mass spectra [6], a concrete 'two-tier' strategy [7] had been proposed by incorporating the physical content of the Instantaneous Approximation, albeit treated *covariantly* [8], wherein the 3D reduction of the original 4d BSE would serve for the dynamics of the spectra, while the reconstructed 4D wave function would be appropriate for applications to various transition amplitudes by standard Feynman techniques for 4D quark loop integrals [7,8]]. This philosophy found [8] a precise mathematical content through the ansatz of Covariant Instantaneity (CIA for short) which gives a *3D support* to the BSE kernel. This ansatz yields a complete equivalence between the 3D and 4D forms of the BSE, viz., not only is the 4D form exactly reducible to the 3D form, but conversely the 4D BS vertex function Γ is fully expressible as a simple product of only 3D quantities, viz., $Dx\phi$, where D and ϕ are both 3D denominator and wave functions respectively, satisfying a relativistic Schroedinger-like equation [8]. (The ansatz of a 3D support to the BS kernel was also advocated by Pervushin and collaborators [9], but under a different philosophy from the two-tier point of view enunciated in [8], so that the feature of 3D-4D interconnection was apparently not on their agenda). A comparative assessment of this method vis-a-vis more conventional ones employing the BSE has been given elsewhere [10]. One may now ask: Does a similar 3D-4D interconnection exist in the corresponding BS amplitudes for a *three-body system* under the same conditions of 3D support to the pairwise BS kernel? This question is of great practical value since the

3D reduction of the 4D BSE (under conditions of a 3D support for the pairwise kernel) already provides a a *fully connected* integral equation [11], leading to an approximate analytic solution (in gaussian form) for the corresponding 3D wave function, as a by-product of the main results on the baryon mass spectra [11]. Therefore a reconstruction of the 4D qqq wave (vertex) function in terms of the corresponding 3D quantities will open up a vista of applications to various types of *transition amplitudes* involving qqq baryons, just as in the $\bar{q}q$ case [8]. This is the main purpose of the present investigation, with three identical spinless particles for simplicity and definiteness, which however need not detract from the generality of the singularity structures. The answer is found to be in the affirmative, except for the recognition that a 3D support to the pairwise BS kernel implies a truncation of the Hilbert space. Such truncation, while still allowing an unambiguous reduction of the BSE from the 4D to the 3D level, nevertheless leaves an element of ambiguity in the *reverse direction*, viz., from 3D to 4D. This limitation for the reverse direction is quite general for any n-body system where $n > 2$; the only exception is the case of $n = 2$ where both transitions are reversible without any extra assumptions (a sort of degenerate situation). The extra assumption needed to complete the reverse transition in its simplest form is facilitated by some 1D delta-functions which however have nothing to do with connectedness [3,4] of an n-body amplitude (see Sec.4 for a formal demonstration). The paper which makes use of Green's function techniques to formally derive the results stated above, is organized as follows. In Sec.2 we first derive the 3D-4D interconnection [8] at the level of the *Green's function* for a $\bar{q}q$ system, under conditions of a 3D support for the BS kernel, whence we reproduce the previously derived result [8] for the corresponding BS wave functions in 3D and 4D forms. In Sec.3, starting with the BSE for the 4D Green's function for *three* identical spinless quarks (q), when the qq subsystems are under pairwise BS interactions with 3D kernel support, this 4D BS integral equation is reduced to the 3D form by integrating w.r.t. *two* internal time-like momenta, and in so doing, introducing 3D Green's functions as double integrals over two time-like internal momenta. The resulting 3D BSE has a fully connected structure, free from delta-functions, as anticipated from an earlier analysis with 3D BS wave functions [11,12]. With this 3D BSE as the check point, Sec.4 is devoted to the task of reconstructing the full 4D Green's function in terms of its (partial) 3D counterparts, so as to satisfy exactly the above 3D BSE, *after* integration w.r.t. the relevant time-like momenta. (In doing these various manipulations, the inhomogeneous parts of the various Green's functions will not be kept track of, since we shall be mainly concerned with their bound state poles). Sec.5 concludes with a discussion of this result, including the technical issues arising from the inclusion of spin, together with a comparison with contemporary approaches to the *relativistic* qqq problem.

2 3D-4D Interconnection For the $\bar{q}q$ System

If the BSE for a spinless $\bar{q}q$ system has a 3D support for its kernel K in the form $K(\hat{q}, \hat{q}')$ where $\hat{q}$ is the component is the component of the relative momentum $q = (p_1 - p_2)/2$ *orthogonal* to the total hadron 4-momentum $P = p_1 + p_2$, then, as was shown in [8], the 4D hadron-quark vertex function Γ is a function of $\hat{q}$ only, and is expressible as $\Gamma(\hat{q}) = D(\hat{q})\phi(\hat{q})$, where D and ϕ are the respective denominator and wave functions of the BSE in 3D form, viz., $D\phi = \int K\phi \, d\hat{q}$. For this 2-body case the 4D and 3D forms of the BSE are exactly reversible without further assumptions. For the 3-body case a corresponding 3D-4D connection was obtained on the basis of semi-intuitive arguments [12], and therefore needed a more formal derivation, which is the central aim of this paper. To that end it is useful to formulate the 4D and 3D BSE's in terms of Green's functions. Therefore in this section we first outline the derivation of the 3D-4D connection for a two-body system in terms of their respective Green's functions, in preparation for the generalization to the three-body case in the next two sections.

Apart from certain obvious notations which should be clear from the context, we shall use the notation and phase conventions of [8,12] for the various quantities (momenta, propagators, etc). The 4D qq Green's function $G(p_1p_2;p_1'p_2')$ near a *bound* state satisfies a 4D BSE without the inhomogeneous term, viz. [8,12],

$$i(2\pi)^4 G(p_1p_2;p_1'p_2') = {\Delta_1}^{-1}{\Delta_2}^{-1}\int dp_1''dp_2''K(p_1p_2;p_1''p_2'')G(p_1''p_2'';p_1'p_2') \qquad (2.1)$$

where

$$\Delta_1 = {p_1}^2 + {m_q}^2; (m_q = mass of each quark) \qquad (2.2)$$

Now using the relative $q = (p_1 - p_2)/2$ and total $P = p_1 + p_2$ 4-momenta (similarly for the other sets), and removing a δ-function for overall 4-momentum conservation, from each of the G- and K- functions, eq.(2.1) reduces to the simpler form

$$i(2\pi)^4 G(q.q') = {\Delta_1}^{-1}{\Delta_2}^{-1}\int d\hat{q}''Md\sigma''K(\hat{q},\hat{q}')G(q,q') \qquad (2.3)$$

where $\hat{q}_\mu = q_\mu - \sigma P_\mu$, with $\sigma = (q.P)/P^2$, is effectively 3D in content (being orthogonal to P_μ). Here we have incorporated the ansatz of a 3D support for the kernel K (independent of σ and σ'), and broken up the 4D measure dq'' arising from (2.1) into the product $d\hat{q}''Md\sigma''$ of a 3D and a 1D measure respectively. We have also suppressed the 4-momentum P_μ label, with $(P^2 = -M^2)$, in the notation for $G(q.q')$. Now define the fully 3D Green's function $\hat{G}(\hat{q},\hat{q}')$ as [8,12]

$$\hat{G}(\hat{q},\hat{q}') = \int\int M^2 d\sigma d\sigma' G(q,q') \qquad (2.4)$$

and two (hybrid) 3D-4D Green's functions $\tilde{G}(\hat{q},q')$, $\tilde{G}(q,\hat{q}')$ as

$$\tilde{G}(\hat{q},q') = \int Md\sigma G(q,q'); \tilde{G}(q,\hat{q}') = \int Md\sigma' G(q,q'); \qquad (2.5)$$

Next, use (2.5) in (2.3) to give

$$i(2\pi)^4\tilde{G}(q,\hat{q}') = {\Delta_1}^{-1}{\Delta_2}^{-1}\int dq''K(\hat{q},\hat{q}'')\tilde{G}(q'',\hat{q}) \qquad (2.6)$$

Now integrate both sides of (2.3) w.r.t. $Md\sigma$ and use the result [8]

$$\int Md\sigma{\Delta_1}^{-1}{\Delta_2}^{-1} = 2\pi i D^{-1}(\hat{q}); D(\hat{q}) = 4\hat{\omega}(\hat{\omega}^2 - M^2/4); \hat{\omega}^2 = {m_q}^2 + \hat{q}^2 \qquad (2.7)$$

to give a 3D BSE w.r.t. the variable $\hat{q}$, while keeping the other variable q' in a 4D form:

$$(2\pi)^3\tilde{G}(\hat{q},q') = D^{-1}\int d\hat{q}''K(\hat{q},\hat{q}'')\tilde{G}(\hat{q}'',q') \qquad (2.8)$$

Now a comparison of (2.6) with (2.8) gives the desired connection between the full 4D G-function:

$$2\pi i G(q, q') = D(\hat{q}){\Delta_1}^{-1}{\Delta_2}^{-1}\tilde{G}(\hat{q}, q') \tag{2.9}$$

which is the Green's function counterpart, *near the bound state*, of the same result [8] connecting the corresponding BS wave functions. Again, the symmetry of the left hand side of (2.9) w.r.t. q and q' allows us to write the right hand side with the roles of q and q' interchanged. This gives the dual form

$$2\pi i G(q, q') = D(\hat{q}'){\Delta_1}'^{-1}{\Delta_2}'^{-1}\tilde{G}(q, \hat{q}') \tag{2.10}$$

which on integrating both sides w.r.t. $Md\sigma$ gives

$$2\pi i\tilde{G}(\hat{q}, q') = D(\hat{q}'){\Delta_1}'^{-1}{\Delta_2}'^{-1}\hat{G}(\hat{q}, \hat{q}'). \tag{2.11}$$

Substitution of (2.11) in (2.9) then gives the symmetrical form

$$(2\pi i)^2 G(q, q') = D(\hat{q}){\Delta_1}^{-1}{\Delta_2}^{-1}\hat{G}(\hat{q}, \hat{q}')D(\hat{q}'){\Delta_1}'^{-1}{\Delta_2}'^{-1} \tag{2.12}$$

Finally, integrating both sides of (2.8) w.r.t. $Md\sigma'$, we obtain a fully reduced 3D BSE for the 3D Green's function:

$$(2\pi)^3\hat{G}(\hat{q}, \hat{q}') = D^{-1}(\hat{q} \int d\hat{q}'' K(\hat{q}, \hat{q}'')\hat{G}(\hat{q}'', \hat{q}') \tag{2.13}$$

Eq.(2.12) which is valid near the bound state pole (since the inhomogeneous term has been dropped for simplicity) expresses the desired connection between the 3D and 4D forms of the Green's functions; and eq(2.13) is the determining equation for the 3D form. A spectral analysis can now be made for either of the 3D or 4D Green's functions in the standard manner, viz.,

$$G(q, q') = \sum_n \Phi_n(q; P)\Phi_n^*(q'; P)/(P^2 + M^2) \tag{2.14}$$

where Φ is the 4D BS wave function. A similar expansion holds for the 3D G-function $\hat{G}$ in terms of $\phi_n(\hat{q})$. Substituting these expansions in (2.12), one immediately sees the connection between the 3D and 4d wave functions in the form:

$$2\pi i\Phi(q, P) = {\Delta_1}^{-1}{\Delta_2}^{-1}D(\hat{q})\phi(\hat{q}) \tag{2.15}$$

whence the BS vertex function becomes $\Gamma = D \times \phi/(2\pi i)$ as found in [8]. We shall make free use of these results, taken as qq subsystems, for our study of the qqq G-functions in Sections 3 and 4.

3 Three-Quark Green's Function: 3D Reduction of the BSE

As in the two-body case, and in an obvious notation for various 4-momenta (without the Greek suffixes), we consider the most general Green's function $G(p_1p_2p_3; {p_1}'{p_2}'{p_3}')$ for 3-quark scattering *near the bound state pole* (for simplicity) which allows us to drop the various inhomogeneous terms from the beginning. Again we take out an overall delta function $\delta(p_1+p_2+p_3-P)$ from the G-function and work with *two* internal 4-momenta for each of the initial and final states defined as follows [12]:

$$\sqrt{3}\xi_3 = p_1 - p_2\ ; \quad 3\eta_3 = -2p_3 + p_1 + p_2 \tag{3.1}$$

$$P = p_1 + p_2 + p_3 = p_1{}' + p_2{}' + p_3{}' \tag{3.2}$$

and two other sets ξ_1, η_1 and ξ_2, η_2 defined by cyclic permutations from (3.1). Further, as we shall be considering pairwise kernels with 3D support, we define the effectively 3D momenta $\hat{p}_i$, as well as the three (cyclic) sets of internal momenta $\hat{\xi}_i, \hat{\eta}_i$, (i = 1,2,3) by [12]:

$$\hat{p}_i = p_i - \nu_i P\ ; \quad \hat{\xi}_i = \xi_i - s_i P\ ; \quad \hat{\eta}_i - t_i P \tag{3.3}$$

$$nu_i = (P.p_i)/P^2\ ; \quad s_i = (P.\xi_i)/P^2\ ; \quad t_i = (P.\eta_i)/P^2 \tag{3.4}$$

$$\sqrt{3}s_3 = \nu_1 - \nu_2\ ; \quad 3t_3 = -2\nu_3 + \nu_1 + \nu_2 \ \ (+\text{cyclicpermutations}) \tag{3.5}$$

The space-like momenta $\hat{p}_i$ and the time-like ones ν_i satisfy [12]

$$\hat{p}_1 + \hat{p}_2 + \hat{p}_3 = 0\ ; \quad \nu_1 + \nu_2 + \nu_3 = 1 \tag{3.6}$$

Strictly speaking, in the spirit of covariant instantaneity, we should have taken the relative 3D momenta $\hat{\xi}, \hat{\eta}$ to be in the instantaneous frames of the concerned pairs, i.e., w.r.t. the rest frames of $P_{ij} = p_i + p_j$; however the difference between the rest frames of P and P_{ij} is small and calculable [12], while the use of a common 3-body rest frame ($P = 0$) lends considerable simplicity and elegance to the formalism. We may now use the foregoing considerations to write down the BSE for the 6-point Green's function i terms of relative momenta, on closely parallel lines to the 2-body case. To that end note that the 2-body relative momenta $q_{ij} = (p_i - p_j)/2 = sqrt3\xi_k/2$, where (ijk) are cyclic permutations of (123). Then for the reduced qqq Green's function, when the *last* interactio was in the (ij) pair, we may use the notation $G(\xi_k\eta_k;\xi_k{}'\eta_k{}')$, together with 'hat' notations on these 4-momenta when the corresponding time-like components are integrated out. Further, since the pair ξ_k, η_k is *permutation invariant* as a whole, we may choose to drop the index notation from the complete G-function to emphasize this symmetry as and when needed. The G-function for the qqq system satisfies, in the neighbourhood of the bound state pole, the following (homogeneous) 4D BSE for pairwise qq kernels with 3D support:

$$i(2\pi)^4 G(\xi\eta;\xi'\eta') = \sum_{123} \Delta_1{}^{-1}\Delta_2{}^{-1} \int d\hat{q}_{12}'' M d\sigma_{12}{}'' K(\hat{q}_{12}, \hat{q}_{12}'') G(\xi_3{}''\eta_3{}''; \xi_3{}'\eta_3{}') \tag{3.7}$$

where we have employed a mixed notation (q_{12} versus ξ_3) to stress the two-body nature of the interaction with one spectator at a time, in a normalization directly comparable with eq.(2.3) for the corresponding two-body problem. Note also the connections

$$\sigma_{12} = \sqrt{3}s_3/2\ ; \quad \hat{q}_{12} = \sqrt{3}\hat{\xi}_3/2\ ; \quad \eta_3 = -p_3, etc \tag{3.8}$$

The next task is to reduce the 4D BSE (3.7) to a fully 3D form through a sequence of integrations w.r.t. the time-like momenta s_i, t_i applied to the different terms on the right hand side, *provived both* variables are simultaneously permuted. We now define the following fully 3D as well as mixed 3D-4D G-functions according as one or more of the time-like ξ, η variables are integrated out:

$$\hat{G}(\hat{\xi}\hat{\eta};\hat{\xi}'\hat{\eta}') = \int\int\int\int ds dt ds' dt' G(\xi\eta;\xi'\eta') \tag{3.9}$$

which is S_3-symmetric.

$$\tilde{G}_{3\eta}(\xi\hat{\eta};\xi'\hat{\eta}') = \int\int dt_3 dt_3{}' G(\xi\eta;\xi'\eta'); \tag{3.10}$$

$$\tilde{G}_{3\xi}(\hat{\xi}\eta;\hat{\xi}'\eta') = \int\int ds_3 ds_3{}' G(\xi\eta;\xi'\eta'); \tag{3.11}$$

The last two equations are however $S-3$-indexed. The full 3D BSE for the $\hat{G}$-function is obtained by integrating out both sides of (3.7) w.r.t. $ds_i ds_j{}' dt_i dt_j{}'$ (S_3-symmetric), and using (3.9) with (3.8) as follows:

$$(2\pi)^3\hat{G}(\hat{\xi}\hat{\eta};\hat{\xi}'\hat{\eta}') = \sum_{123} D^{-1}(\hat{q}_{12}) \int d\hat{q}_{12}'' K(\hat{q}_{12},\hat{q}_{12}'')\hat{G}(\hat{\xi}''\hat{\eta}'';\hat{\xi}'\hat{\eta}') \tag{3.12}$$

This integral equation for $\hat{G}$ which is the 3-body counterpart of (2.13) for a qq system in thev neighbourhood of the bound state pole, is the desired 3D BSE for the qqq system in a *fully connected* form, i.e., free from delta functions. Now using a spectral decomposition for $\hat{G}$

$$\hat{G}(\hat{\xi}\hat{\eta};\hat{\xi}'\hat{\eta}') = \sum_n \phi_n(\hat{\xi}\hat{\eta};P)\phi_n^*(\hat{\xi}'\hat{\eta}';P)/(P^2+M^2) \tag{3.13}$$

on both sides of (3.12) and equating the residues near a given pole $P^2 = -M^2$, gives the desired equation for the 3D wave function ϕ for the bound state in the connected form:

$$(2\pi)^3\phi(\hat{\xi}\hat{\eta};P) = \sum_{123} D^{-1}(\hat{q}_{12}) \int d\hat{q}_{12}'' K(\hat{q}_{12},\hat{q}_{12}'')\phi(\hat{\xi}''\hat{\eta}'';P) \tag{3.14}$$

The solution of this equation for the ground state was found in [11] in a *gaussian* form which implies that $\phi(\hat{\xi}\hat{\eta};P)$ is an S_3-invariant function of $\hat{\xi}_i^2+\hat{\eta}_i^2$, *valid for any index i*. While the gaussian form may prove too restrictive for more general applications, the mere S_3-symmetry of ϕ in the $(\hat{\xi}_i,\hat{\eta}_i)$ pair may prove adequate in practice, and hence useful for both the solution of (3.14) *and* for the reconstruction of the 4D BS wave function in terms of the 3D wave function (3.14), as is done in Sec.4 below.

4 Reconstruction of the 4D BS Wave Function

In this section we shall attempt to *re-express* the 4D G-function given by (3.7) in terms of the 3D $\hat{G}$-function given by (3.12), as the qqq counterpart of the qq results (2.12-13). To that end we first adapt the result (2.12) to the hybrid Green's function of the (12) subsystem given by $\tilde{G}_{3\eta}$, eq.(3.10), in which the 3-momenta $\eta_3,\eta_3{}'$ play a parametric role reflecting the spectator status of quark #3, while the *active* roles are played by $q_{12}, q_{12}{}' = \sqrt{3}(\xi_3,\xi_3{}')/2$, for which the analysis of Sec.2 applies directly. This gives

$$(2\pi i)^2\tilde{G}_{3\eta}(\xi_3\hat{\eta}_3;\xi_3{}'\hat{\eta}_3') = D(\hat{q}_{12})\Delta_1{}^{-1}\Delta_2{}^{-1}\hat{G}(\hat{\xi}_3\hat{\eta}_3;\hat{\xi}_3'\hat{\eta}_3')D(\hat{q}_{12}')\Delta_1{}'^{-1}\Delta_2{}'^{-1} \tag{4.1}$$

where on the right hand side, the 'hatted' G-function has full S_3-symmetry, although (for purposes of book-keeping)we have not shown this fact explicitly by deleting the suffix '3' from its arguments. A second relation of this kind mau be obtained from (3.7) by noting that the 3 terms on its right hand side may be expressed in terms of $\tilde{G}_{3\xi}$ functions vide their definitions (3.11), together with the 2-body interconnection between $(\xi_3, {\xi_3}')$ and $(\hat{\xi}_3, \hat{\xi}_3')$ expressed once again via (4.1), but without the 'hats' on η_3 and ${\eta_3}'$. This gives

$$\begin{aligned}(\sqrt{3}\pi i)^2 G(\xi_3\eta_3; {\xi_3}'{\eta_3}') &= (\sqrt{3}\pi i)^2 G(\xi\eta; \xi'\eta') \\ &= \sum_{123} {\Delta_1}^{-1}{\Delta_2}^{-1}(\pi i\sqrt{3}) \int d\hat{q}_{12}'' M d\sigma_{12}'' K(\hat{q}_{12}, \hat{q}_{12}'') G({\xi_3}''{\eta_3}''; {\xi_3}'{\eta_3}') \\ &= \sum_{123} D(\hat{q}_{12}){\Delta_1}^{-1}{\Delta_2}^{-1}\tilde{G}_{3\xi}(\hat{\xi}_3\eta_3; \hat{\xi}_3'{\eta_3}'){\Delta_1'}^{-1}{\Delta_2'}^{-1} \qquad (4.2)\end{aligned}$$

where the second form exploits the symmetry between ξ, η and ξ', η'. This is as far as we can go with the qqq Green's function, using the 2-body techniques of Sec.2. However,, unlike the 2-body case where the reconstruction of the 4D G-function in terms of the corresponding 3D quantity was complete at this stage, the process is far from complete for the 3-body case, as eq.(4.2) clearly shows. This is due to the *truncation* of Hilbert space implied in the ansatz of 3d support to the pairwise BSE kernel K which, while facilitating a 4d to 3d BSE reduction without extra charge, does *not* have the *complete* information to permit the *reverse* transition (3d to 4D) without additional assumptions. This limitation of the 3D support ansatz for the BSE kernel affects all n-body systems except $n = 2$ (which may be regarded as a sort of degenerate situation. Now it may be argued: Is this 3D ansatz for the BSE kernel really necessary? Since this paper is not the most convenient place to dwell on this *physical* issue in detail (the same has been discussed elsewhere [10], vis-a-vis contemporary approaches), we shall here take the 3D support ansatz for granted, and look upon this "inverse" problem as a purely *mathematical* one. We add in parentheses however that the physical applications of the 3D ansatz are (indeed) quite widespread since it is directly related to the "instantaneous approximation" on which a considerable amount of low and intermediate energy hadron physics (at the quark level) has been (and is still being) studied. As a purely mathematical problem, we must look for a suitable ansatz for the quantity $\tilde{G}_{3\xi}$ on the right hand side of (4.2) in terms of *known* quantities, so that the reconstructed 4D G-function satisfies the 3D equation (3.12) exactly, which may be regarded as a "check-point" for the entire exercise. We therefore seek a structure of the form

$$\tilde{G}_{3\xi}(\hat{\xi}_3\eta_3; \hat{\xi}_3'{\eta_3}') = \hat{G}(\hat{\xi}_3\hat{\eta}_3; \hat{\xi}_3'\hat{\eta}_3') \times F(p_3, {p_3}') \qquad (4.3)$$

where the unknown function F must involve only the momentum of the spectator quark #3. A part of the $\eta_3, {\eta_3}'$ dependence has been absorbed in the $\hat{G}$ function on the right, so as to satisfy the requirements of S_3-symmetry for this 3D quantity, whether it has a gaussian structure [11] (where it is explicit), or a more general one; see the discussion below eq(3.14). As to the remaining factor F, it is necessary to choose its form in a careful manner so as to conform to the conservation of 4-momentum for the *free* propagation of the spectator between two neighbouring vertices, consistently with the symmetry between p_3 and ${p_3}'$. A possible choice consistent with these conditions is the form

$$F(p_3, {p_3}') = C_3{\Delta_3}^{-1}\delta(\nu_3 - {\nu_3}') \qquad (4.4)$$

where Δ_3 could also be written more symmetrically as $\sqrt{\Delta_3\Delta_3'}$. Here we have taken only the time component of the 4-momentum p_3 in the δ-function since the effect of its space component has already been absorbed in the "connected" (3D) Green's function $\hat{G}$. ${\Delta_3}^{-1}$ represents the "free" propagation of quark #3 between successive vertices, while C_3

represents some residual effects which may at most depend on the 3-momentum $\hat{p}_3$, but must satisfy the main constraint that the 3D BSE, eq.(3.12), is *explicitlysatisfied*. To check the self-consistency of the ansatz (4.4), integrate both sides of (4.2) w.r.t. $ds_3 ds_3' dt_3 dt_3'$ to recover the 3D S_3-invariant $\hat{G}$-function on the left hand side; and in the first form on the right hand side, integrate w.r.t. $ds_3 ds_3'$ on the G-function which alone involves these variables. This yields the quantity $\tilde{G}_{3\xi}$. At this stage, employ the ansatz (4.4) to integrate over $dt_3 dt_3'$. Consistency with the 3D BSE, eq.(3.12), now demands

$$C_3 \int\int d\nu_3 d\nu_3' \Delta_3{}^{-1} \delta(\nu_3 - \nu_3') = 1; (since dt = d\nu) \qquad (4.5)$$

The 1D integration w.r.t. $d\nu_3$ may be evaluated as a contour integral over the propagator Δ^{-1} , which gives the pole at $\nu_3 = \hat{\omega}_3/M$, (see below). Evaluating the residue then gives

$$C_3 = i\pi/(M\hat{\omega}_3); \quad \hat{\omega}_3^2 = m_q{}^2 + \hat{p}_3^2 \qquad (4.6)$$

which will reproduce the 3D BSE, eq.(3.12), *exactly*! Substitution of (4.4) in the second form of (4.2) finally gives the desired 3-body generalization of (2.12) in the form

$$3G(\xi\eta; \xi'\eta') = \sum_{123} D(\hat{q}_{12}) \Delta_{1F} \Delta_{2F} D(\hat{q}_{12}') \Delta_{1F}' \Delta_{2F}' \hat{G}(\hat{\xi}_3 \hat{\eta}_3; \hat{\xi}_3' \hat{\eta}_3')[\Delta_{3F}/(M\pi\hat{\omega}_3)] \qquad (4.7)$$

where for each index, $\Delta_F = -i\Delta^{-1}$ is the Feynman propagator. Before commenting on this structure of the 4D Green's function near the bound state pole, let us first find the effect of the ansatz (4.4) on the 4D BS *wave function* $\Phi(\xi\eta; P)$. This is achieved through a spectral representation like (3.13) for the 4D Green's function G on the left hand side of (4.2). Equating the residues on both sides gives the desired 4D-3D connection between Φ and ϕ:

$$\Phi(\xi\eta; P) = \sum_{123} D(\hat{q}_{12}) \Delta_1{}^{-1} \Delta_2{}^{-1} \phi(\hat{\xi}\hat{\eta}; P) \times \sqrt{\frac{\delta(\nu_3 - \hat{\omega}_3/M)}{M\hat{\omega}_3 \Delta_3}} \qquad (4.8)$$

From (4.8) we can infer the structure of the baryon-qqq vertex function by rewriting the it in the alternative form [12]:

$$\frac{\Phi(\xi\eta; P) = (V_1 + V_2 + V_3)}{\Delta_1 \Delta_2 \Delta_3} \qquad (4.9)$$

where

$$V_3 = D(\hat{q}_{12}) \phi(\hat{\xi}\hat{\eta}; P) \times \sqrt{\frac{\Delta_3 \delta(\nu_3 - \hat{\omega}_3/M)}{M\hat{\omega}_3}} \qquad (4.10)$$

The quantity V_3 is the baryon-qqq vertex function corresponding to the "last interaction" in the (12) pair, and so on cyclically. This is precisely the form (apart from a constant factor that does not affect the baryon normalization) that had been anticipated in an earlier study in a semi-intuitive fashion [12].

5 Discussion and Conclusion

Eqs.(4.7-10) which represent the principal results of this investigation, bring out rather directly the significance of the square root of the δ-function in the energy variable of the spectator. Both the δ-funnction and the Δ_{3F} propagator appear in *rational forms in the*

4D Green's function. reflecting a free on-shell propagation of the spectator between two vertex points. The square root feature in the baryon-qqq vertex function is the result of equal distribution of this singulariity between the initial and final state vertex functions. Further, as the steps indicate, the appearace of this singularity has nothing to do with the connectedness [3,4] of the 3-body scattering amplitude, but rather with the 3D support for the pairwise BSE kernel. More importantly, this singularity will *not* show up in any physical amplitude for hadronic transitions via quark loops, since such amplitudes will always involve both the δ-function and the propagator Δ_{3F} in a *rational* form before the relevant momentum integrationns are performed. The next question concerns the possible uniqueness of the structure (4.7-10). It is certainly "sufficient" since the 3D form (3.12) of the BSE for the $\hat{G}$- function is exactly satisfied. Moreover the underlying ansatz (4.4) has certain desirable properties like on-shell propagation of the spectator in between two successive interactions, as well as an explicit symmetry in the p_3 and $p_3{}'$ momenta. There is a fair chance of its uniqueness within certain general constraints, but so far we have not been able to prove this. As regards spin, the extension of the above formalism to fermion quarks is a straightforward process amounting to the replacement of $\Delta_F = -i\Delta^{-1}$ by the corresponding S_F-functions, as has been shown elsewhere [11,12]. In particular, the fermion vertex function has recently been applied to the problem of proton-neutron mass difference [13] via quark loop integrals, to bring out the practicability of its application without parametric uncertainties, since the entire formalism is linked all the way from spectroscopy to hadronic transition amplitudes [7]. At this stage it is interesting to ask, again as a purely mathematical problem, what would have been the possible scenario if the 3D support ansatz for the BSE kernel had not been made. In that case, the entire "hat" formalism would become redundant, and there would be no special roles of equations like (2.12) or (4.7) connecting the 4D to the 3D Green's functions. The "connected" equations (2.13) or (3.12) would simply remain valid *without* the "hats", viz., as 4D integral equations, and with the replacement of $D(\hat{q}_{12})$ by $\Delta_1\Delta_2/(2i\pi)$. And if closed form solutions of these equations were routinely possible in 4D form, there would be no special advantage in going in for more complicated connected equations [3,4]. This kind of 3-body approach in direct 4D form was indeed attempted in a Wick-rotated Euclidean manner [5], but the predicted O(4)-like spectra did not accord with observation [6]. And while 'normal' 4D kernels (i.e., without Wick rotation) have been employed for $\bar{q}q$ systems [14], there is no corresponding evidence of the qqq BSE attempted on similar lines. The 3D kernel support discussed here is just a concrete alternative which, being otherwise rooted in a 4D framework, acts as an effective bridge between traditional 3D methods employed in the literature for few-body problems at the quark level and more formal 4D treatments [14-17]; (see next para for specific comparison with other qqq problem studies). Its strong connection with the 'instantaneous approximation' gives it a natural applicational base for a systematic treatment of hadronic phenomena at the quark level up to moderately high energies, as indicated by its observational successes [8,10-13]. Finally we would like to comment on this formalism vis-a-vis contemporary approaches to the qqq problem. Many such approaches as are available in the literature are parametric representations attuned to QCD-sum rules [15], effective Lagrangians for hadronic transitions to "constituent" quarks, with ad hoc assumptions on the hadron-qqq form factor [16], similar (parametric) ansatze for the hadron- quark-diquark form factor [17]; or more often simply direct gaussian parametrizations for the qqq wave functions as the starting point of the investigation [18]. Such approaches are often quite effective for the investigations of some well-defined sectors of hadron physics with quark degrees of freedom, but are not readily extensible to other sectors without further assumptions (e.g., meson and baryon parametrizations are quite unrelated to each other), since there is no possibility of a deeper understanding of the crucial functions/parameters involved, from more formal dynamical premises. A more unified approach, albeit at the cost of a bigger dynamical investment, should have the capacity to provide a more natural form of integration of the different sectors, perhaps all the way to hadron spectroscopy, without additional assumptions on the way. Such approaches usually need a "dynamical equation" such as the BSE or SDE, as the starting point for the flow of information. It is precisely in respect of such unification that the philosophy underlying the present formalism for the *qqq* problem differs from some others [15-18]. This is hardly a new philosophy, since the perspective in this respect was shown 25 years ago by Feynman [19], but can stand a reiteration. This work arose out of the need for a formal demonstration of a semi-intuitive ansatz [12] on the structure of the baryon-*qqq* vertex function that had been recently applied to the neutron-proton mass difference problem [13], on the demand of some critics. However it took shape as a self-contained "mathematical" problem in its own right, even though its origin is strongly physical. The final version of this paper was prepared at the International Centre for Theoretical Physics during a short time visit of the author in November 1996. He expresses his appreciation to the Director, Prof. M.A.Virasoro, for this hospitality.

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