Bounds on quark mass matrices elements due to measured properties of the mixing matrix and present values of the quark masses

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Abstract

We obtain constraints on possible structures of mass matrices in the quark sector by using as experimental restrictions the determined values of the quark masses at the M_Z energy scale, the magnitudes of the quark mixing matrix elements $V_{\rm ud}$, $V_{\rm us}$, $V_{\rm cd}$, and $V_{\rm cs}$, and the Jarlskog invariant $J(V)$. Different cases of specific mass matrices are examined in detail. The quality of the fits for the Fritzsch and Stech type mass matrices is about the same with $\chi^2/\mathrm{dof} = 4.23/3 = 1.41$ and $\chi^2/\mathrm{dof} = 9.10/4 = 2.28$, respectively. The fit for a simple generalization (one extra parameter) of the Fritzsch type matrices, in the physical basis, is much better with $\chi^2/\mathrm{dof} = 1.89/4 = 0.47$. For comparison we also include the results using the quark masses at the 2 GeV energy scale. The fits obtained at this energy scale are similar to that at M_Z energy scale, implying that our results are unaffected by the evolution of the quark masses from 2 to 91 GeV.

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I. INTRODUCTION

Quark flavor mixing in the Standard Model arises from unitary matrices which diagonalize the corresponding hermitian mass matrices. The CKM quark-flavor mixing matrix V in the physical basis [1, 2, 3], is given by $V = V_{\rm u}V_{\rm d}^{\dagger}$, where the unitary matrices $V_{\rm u}$ and $V_{\rm d}$ diagonalize the up-quark and down-quark Dirac mass matrices $M_{\rm u}$ and $M_{\rm d}$, respectively. Given this circumstance, complete knowledge of the mass matrices fully determines the corresponding mixing matrices. In practice, however, the mass matrices are guessed at, while experiment can only determine the moduli of the CKM matrix elements.

Recently, evolved quark masses have been given at various energy scales [4]. The quark masses at the M_W ($= 80.403\,\mathrm{GeV}$), M_Z ($= 91.1876\,\mathrm{GeV}$), and $m_{\rm t}$ ($= 172.5\,\mathrm{GeV}$) scales are quite similar. For our analysis we use as input the quark masses at the M_Z energy scale, which lies between the M_W and the m_t scales. So, in this paper we shall use as experimental restrictions the reported values of the CKM matrix elements $|V_{\alpha j}|$ [5], the Jarlskog invariant $J(V)$ [5], and the quark masses at the M_Z energy scale [4], to obtain constraints on the elements of the quark mass matrices in five specific cases for three generations. As a check of the stability of our type of analysis, under the evolution of quark masses, we have repeated it with quark masses at 2 GeV scale [4].

Section II gives the notation and the basic formulas or expressions needed and the general procedure adopted for the analysis. In Sec. III, we consider the Fritzsch-type of mass matrices [6, 7, 8] in the physical basis. As pointed out there $M_{\rm u}$ can be chosen to be real with three parameters while $M_{\rm d}$ has five, two of which are phase angles. In Sec. IV, Stech-type matrices [9] are considered in the basis in which $M_{\rm u}$ is diagonal, while $M_{\rm d} = p\,M_{\rm u} + i\,S$. Here p is a real number and S is a non-diagonal matrix, satisfying $S^{\rm T} = -S$, with three real parameters. In Secs. V and VI we consider mass matrices which are a simple generalization of the Fritzsch-type matrix in that $M_{\rm u}$ and $M_{\rm d}$ have an additional parameter. These were considered recently, in the $M_{\rm u}$ ($M_{\rm d}$) diagonal basis [10]. In Sec. V we consider these cases again and fit them to the data. In Sec. VI we consider the case of these type of mass matrices in the physical basis. Results obtained in the above cases are compared and discussed in the final section VII. Here also the results of the two energy scales (M_Z and 2 GeV) are compared and discussed.

II. NOTATION AND BASIC FORMULAS

The 3×3 hermitian quark mass matrix $M_{\rm q}$ is diagonalized by $V_{\rm q}$ so that $M_{\rm q} = V_{\rm q}^{\dagger}\hat{M}_{\rm q}V_{\rm q}$, q=u,d. The eigenvalues are denoted by ($\lambda_{\rm u}$,$\lambda_{\rm c}$,$\lambda_{\rm t}$) and ($\lambda_{\rm d}$,$\lambda_{\rm s}$,$\lambda_{\rm b}$) for the up and down quark mass matrices, respectively. Note that the eigenvalues are real but not necessarily positive. Each mass matrix can be expressed in terms of its projectors. Thus,

$$M_{\rm u} = \sum_{\alpha={\rm u,c,t}} \lambda_\alpha N_\alpha \qquad \text{and} \qquad M_{\rm d} = \sum_{j={\rm d,s,b}} \lambda_j N_j. \tag{1}$$

Since $V = V_{\rm u}V_{\rm d}^{\dagger}$, it follows that [11]

$$|V_{\alpha j}|^2 = \mathrm{Tr}[N_\alpha N_j], \tag{2}$$

where

$$N_\alpha = \frac{(\lambda_\beta - M_{\rm u})(\lambda_\gamma - M_{\rm u})}{(\lambda_\beta - \lambda_\alpha)(\lambda_\gamma - \lambda_\alpha)} \tag{3}$$

and

$$N_j = \frac{(\lambda_k - M_{\rm d})(\lambda_l - M_{\rm d})}{(\lambda_k - \lambda_j)(\lambda_l - \lambda_j)}, \tag{4}$$

with (α,β,γ) and (j,k,l) any permutation of (u,c,t) and (d,s,b), respectively.

The Jarlskog invariant $J(V)$, which is a measure of CP-violation can be directly expressed in terms of $M_{\rm u}$ and $M_{\rm d}$ and their eigenvalues [12], thus

$$\mathrm{Det}([M_{\rm u}, M_{\rm d}]) = 2i\, D(\lambda_\alpha) D(\lambda_j) J(V), \tag{5}$$

where

$$D(\lambda_\alpha) = (\lambda_{\rm c} - \lambda_{\rm u})(\lambda_{\rm t} - \lambda_{\rm u})(\lambda_{\rm t} - \lambda_{\rm c}) \tag{6}$$

and

$$D(\lambda_j) = (\lambda_{\rm s} - \lambda_{\rm d})(\lambda_{\rm b} - \lambda_{\rm d})(\lambda_{\rm b} - \lambda_{\rm s}). \tag{7}$$

The bases when $M_{\rm u}$ or $M_{\rm d}$ is diagonal are of special interest for the mass matrices considered in Secs. IV and V. For the case, $M_{\rm u}$ diagonal and $M_{\rm d} = M$, Eq. (5) reduces to [10]

$$J(V) = \frac{\mathrm{Im}(M_{12}\, M_{23}\, M_{13}^*)}{D(\lambda_j)}. \tag{8}$$

There is a similar formula for the case $M_{\rm d}$ diagonal. This result shows that to obtain CP-violation, the mass matrix for up-quark (down-quark) must have $\mathrm{Im}(M_{12}\, M_{23}\, M_{13}^*)$ non-zero in a basis in which the down-quark (up-quark) mass matrix is diagonal. Consequently, the Fritzsch type of mass matrices can only be used in the physical basis.

In general, our procedure in each case is to first determine the elements of the quark mass matrices in terms of the eigenvalues and then to form a χ^2-function which contains eleven summands. The first five compare the theoretical expressions as functions of the elements of the quark mass matrices of the four best measured moduli, namely, $|V_{\rm ud}|$, $|V_{\rm us}|$, $|V_{\rm cd}|$, and $|V_{\rm cs}|$, and of the Jarlskog invariant $J(V)$, with their experimental values [5]. The last six summands constrain the quark mass matrices eigenvalues to the experimentally deduced quark masses values at the M_Z energy scale [4].

III. FRITZSCH TYPE MASS MATRICES

We consider first the well-known case of the Fritzsch mass matrices [6, 7, 8], given by the hermitian matrices

$$M_{\rm u} = \begin{pmatrix} 0 & A & 0 \\ A^* & 0 & B \\ 0 & B^* & C \end{pmatrix}, \qquad M_{\rm d} = \begin{pmatrix} 0 & A' & 0 \\ A'^* & 0 & B' \\ 0 & B'^* & C' \end{pmatrix}. \tag{9}$$

Without lack of generality we may take C and C' to be positive. Since we may rotate M_{u} and M_{d} with the same unitary matrix X without changing the physics [13, 14], we can make M_{u} real with positive elements by choosing

$$X = \begin{pmatrix} e^{-i\phi_A} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{i\phi_B} \end{pmatrix}, \tag{10}$$

with ϕ_A and ϕ_B the phases of A and B, respectively. Then, M_{u} and M_{d} have eight parameters, A, B, C, $|A'|$, $|B'|$, C', and the phases $\phi_{A'}$ and $\phi_{B'}$.

From the characteristic equation of M_{u}, we have

$$\begin{gathered} C = \lambda_{\mathrm{u}} + \lambda_{\mathrm{c}} + \lambda_{\mathrm{t}}, \\ -A^2 - B^2 = \lambda_{\mathrm{u}}\lambda_{\mathrm{c}} + \lambda_{\mathrm{u}}\lambda_{\mathrm{t}} + \lambda_{\mathrm{c}}\lambda_{\mathrm{t}}, \\ -A^2 C = \lambda_{\mathrm{u}}\lambda_{\mathrm{c}}\lambda_{\mathrm{t}}. \end{gathered} \tag{11}$$

Solving for A and B yields

$$\begin{gathered} A = \left[-\frac{\lambda_{\mathrm{u}}\lambda_{\mathrm{c}}\lambda_{\mathrm{t}}}{\lambda_{\mathrm{u}} + \lambda_{\mathrm{c}} + \lambda_{\mathrm{t}}} \right]^{1/2}, \\ B = \left[-\frac{(\lambda_{\mathrm{t}} + \lambda_{\mathrm{c}})(\lambda_{\mathrm{t}} + \lambda_{\mathrm{u}})(\lambda_{\mathrm{c}} + \lambda_{\mathrm{u}})}{\lambda_{\mathrm{u}} + \lambda_{\mathrm{c}} + \lambda_{\mathrm{t}}} \right]^{1/2}. \end{gathered} \tag{12}$$

Similarly for M_{d}, the parameters $|A'|$, $|B'|$, and C' are obtained from (11) and (12) by replacing (A,B,C) by ($|A'|$,$|B'|$,C') and (λ_{u},λ_{c},λ_{t}) by (λ_{d},λ_{s},λ_{b}).

According to (2), the magnitudes of the unitary quark mixing matrix elements are given in the Fritzsch case by

$$\begin{aligned} |V_{\alpha j}|^2 \;=\; & \left[(\lambda_\alpha - \lambda_\beta)(\lambda_\alpha - \lambda_\gamma)(\lambda_j - \lambda_k)(\lambda_j - \lambda_l)\right]^{-1} \times \\ & \Big\{ \left(\lambda_\beta\lambda_\gamma + A^2 + B^2\right)\left(\lambda_k\lambda_l + |A'|^2 + |B'|^2\right) + \left(\lambda_\beta\lambda_\gamma + A^2\right)\left(\lambda_k\lambda_l + |A'|^2\right) \\ & + \left[(\lambda_\alpha + \lambda_\beta)(\lambda_\alpha + \lambda_\gamma) + B^2 \right] \left[(\lambda_j + \lambda_k)(\lambda_j + \lambda_l) + |B'|^2 \right] \\ & + 2\,(\lambda_\beta + \lambda_\gamma)(\lambda_k + \lambda_l)\, A|A'|\cos(\phi_{A'}) \\ & + 2\lambda_\alpha\lambda_j\, B|B'|\cos(\phi_{B'}) + 2B\, A|B'||A'|\cos(\phi_{A'} + \phi_{B'}) \Big\}, \end{aligned} \tag{13}$$

where (α,β,γ) is any permutation of (u,c,t) and (j,k,l) any permutation of (d,s,b). By unitarity only four of the nine $|V_{\alpha j}|^2$ are independent. As mentioned in Sec. I, we shall use the four best experimentally measured magnitudes $|V_{\mathrm{ud}}|$, $|V_{\mathrm{us}}|$, $|V_{\mathrm{cd}}|$, and $|V_{\mathrm{cs}}|$.

Finally, the Jarslkog invariant $J(V)$ given by Eq. (5) translates for the Fritzsch case into,

$$
\begin{aligned}
J(V) \;=\; & -\big[(\lambda_{\rm t}-\lambda_{\rm c})(\lambda_{\rm t}-\lambda_{\rm u})(\lambda_{\rm c}-\lambda_{\rm u})(\lambda_{\rm b}-\lambda_{\rm s})(\lambda_{\rm b}-\lambda_{\rm d})(\lambda_{\rm s}-\lambda_{\rm d})\big]^{-1} \times \\
& \bigg\{ \Big[B\,|B'|\sin(\phi_{B'}) - A\,|A'|\sin(\phi_{A'}) \Big] \\
& \quad \times \Big[A^2|B'|^2 + B^2|A'|^2 - 2\,A\,B\,|A'|\,|B'|\cos(\phi_{A'}+\phi_{B'}) \Big] \\
& \quad + A\,|A'|\sin(\phi_{A'}) \Big[C^2|B'|^2 + B^2C'^2 - 2\,C\,B\,C'\,|B'|\cos(\phi_{B'}) \Big] \bigg\} \,. \qquad (14)
\end{aligned}
$$

The mass matrices $M_{\rm u}$ and $M_{\rm d}$ in Eq. (9) do not have positive definite eigenvalues. However, a viable mass matrix does not need to have positive definite eigenvalues [7]. These eigenvalues are real but not necessarily positive. Thus, $\lambda_{\rm u}^2 = m_{\rm u}^2$, $\lambda_{\rm d}^2 = m_{\rm d}^2$, etc., where $m_{\rm u}$ is the (positive) mass of the up quark, etc.

In this case it is possible to fix the relative phases between the eigenvalues and the quark masses. For the up quark sector (and analogously for the down one) we need a solution with the mass hierarchy $|\lambda_{\rm u}| << |\lambda_{\rm c}| << |\lambda_{\rm t}|$. From the first relation (11) we see that $\lambda_{\rm t} = m_{\rm t}$, C being positive. This coupled with the second relation in (11) require $\lambda_{\rm u} = m_{\rm u}$ and $\lambda_{\rm c} = -m_{\rm c} < 0$. Then, for the Fritzsch case the relative phases between λ's and m's are

$$
(\lambda_{\rm u}\,,\lambda_{\rm c}\,,\lambda_{\rm t}) = (m_{\rm u}\,,-m_{\rm c}\,,m_{\rm t}) \quad \text{and} \quad (\lambda_{\rm d}\,,\lambda_{\rm s}\,,\lambda_{\rm b}) = (m_{\rm d}\,,-m_{\rm s}\,,m_{\rm b})\,. \qquad (15)
$$

From the above formulation, we are now in position to apply the procedure described at the end of Sec. I to determine the parameters of the quark mass matrices in the Fritzsch case. The parameters to be estimated are the six eigenvalues ($\lambda_{\rm u}$,$\lambda_{\rm c}$,$\lambda_{\rm t}$) and ($\lambda_{\rm d}$,$\lambda_{\rm s}$,$\lambda_{\rm b}$), and the two phases $\phi_{A'}$ and $\phi_{B'}$.

Using Eqs. (12) for the up and down quark sectors the $|V_{\alpha\,j}|$ and $J(V)$ can be expressed as functions of the six eigenvalues and the phases $\phi_{A'}$ and $\phi_{B'}$. We now fit these theoretical expressions to the experimental values of the four moduli $|V_{\rm ud}|$, $|V_{\rm us}|$, $|V_{\rm cd}|$, and $|V_{\rm cs}|$, and $J(V)$ [5]. In doing so we constrain the eigenvalues to the experimentally determined values of the quark masses at the M_Z energy scale [4] displayed in Column 2 of Table I, with the relative phases as given by (15). The fitted values obtained for the eigenvalues are given in Column 3 of Table I. Column 3 also gives the values predicted for $\phi_{A'}$, $\phi_{B'}$, the four moduli and $J(V)$. The corresponding $\Delta\chi^2$ for the eigenvalues, $|V_{\alpha\,j}|$, and $J(V)$ are given in the last column. Note that $\phi_{A'}$ and $\phi_{B'}$ are unknown to begin with. The total $\chi^2/(\text{dof}) = 4.23/3 = 1.41$.

Also, from the relations in Eq. (12) for the up and down quark sectors and the entries of Table I we can determine for the derived parameters A, B, C, $|A'|$, $|B'|$, and C', their "experimental" values (using the experimental constraints on the quark masses) and their predicted values (using the fitted values of the eigenvalues), along with their corresponding $\Delta\chi^2$ contributions. These numbers are shown in Table II.

We observe from Table I that the fitted values obtained for the phases $\phi_{A'}$ and $\phi_{B'}$ are compatible with $-\pi/2$ and 0, respectively. In Tables III and IV we display the corresponding results obtained for this particular choice. In this six parameters fit (the six eigenvalues) the total $\chi^2/(\text{dof}) = 4.84/5 = 0.97$.

IV. STECH TYPE MASS MATRIX

The second case we consider is the model of Stech [9],

$$M_{\rm u}=\begin{pmatrix}\lambda_{\rm u} & 0 & 0\\ 0 & \lambda_{\rm c} & 0\\ 0 & 0 & \lambda_{\rm t}\end{pmatrix}, \qquad M_{\rm d}=p\,M_{\rm u}+i\begin{pmatrix}0 & a & d\\ -a & 0 & b\\ -d & -b & 0\end{pmatrix}. \tag{16}$$

The mass matrices are hermitian, the λ's are the eigenvalues of $M_{\rm u}$, and p is constant. a, b, and d are real and a and b can be made positive like A and B in Eq. (9). The Stech model has seven parameters.

Using the characteristic equation of $M_{\rm d}$, we may express p, a and b in terms of the eigenvalues of $M_{\rm u}$ and $M_{\rm d}$ and the parameter d,

$$p=\frac{\lambda_{\rm d}+\lambda_{\rm s}+\lambda_{\rm b}}{\lambda_{\rm u}+\lambda_{\rm c}+\lambda_{\rm t}}, \tag{17}$$

$$a^2=\frac{-\lambda_{\rm u}\,E_1+E_2-(\lambda_{\rm c}-\lambda_{\rm u})\,d^2}{\lambda_{\rm t}-\lambda_{\rm u}}, \tag{18}$$

$$b^2=\frac{\lambda_{\rm t}\,E_1-E_2-(\lambda_{\rm t}-\lambda_{\rm c})\,d^2}{\lambda_{\rm t}-\lambda_{\rm u}}, \tag{19}$$

where,

$$E_1=\frac{1}{2}[\lambda_{\rm d}^2+\lambda_{\rm s}^2+\lambda_{\rm b}^2-p^2(\lambda_{\rm u}^2+\lambda_{\rm c}^2+\lambda_{\rm t}^2)], \quad E_2=\frac{1}{p}(p^3\lambda_{\rm u}\,\lambda_{\rm c}\,\lambda_{\rm t}-\lambda_{\rm d}\,\lambda_{\rm s}\,\lambda_{\rm b}). \tag{20}$$

From (2), the moduli of the unitary quark mixing matrix elements in this case are,

$$\begin{aligned}|V_{\alpha\,j}|^2 \;=\; & \big[(\lambda_k-\lambda_j)(\lambda_l-\lambda_j)\big]^{-1}\times\\ & \big[(\lambda_k-p\,\lambda_\alpha)(\lambda_l-p\,\lambda_\alpha)+(a^2+d^2)\,\delta_{\alpha,{\rm u}}+(a^2+b^2)\,\delta_{\alpha,{\rm c}}+(b^2+d^2)\,\delta_{\alpha,{\rm t}}\big]\,,\end{aligned} \tag{21}$$

again (j,k,l) is any permutation of (d,s,b) and we shall use only the four independent expressions corresponding to the best experimentally measured magnitudes $|V_{\rm ud}|$, $|V_{\rm us}|$, $|V_{\rm cd}|$, and $|V_{\rm cs}|$.

For the Stech case the Jarslkog invariant $J(V)$ given by (5) is simply [15]

$$J(V)=\frac{a\,b\,d}{(\lambda_{\rm b}-\lambda_{\rm s})(\lambda_{\rm b}-\lambda_{\rm d})(\lambda_{\rm s}-\lambda_{\rm d})}. \tag{22}$$

In order to determine the parameters of the mass matrices in the Stech case, notice first that using Eqs. (17) - (20) the original set of seven parameters ($\lambda_{\rm u}$, $\lambda_{\rm c}$, $\lambda_{\rm t}$, p, a, b, d) can be replaced by the set ($\lambda_{\rm u}$, $\lambda_{\rm c}$, $\lambda_{\rm t}$, $\lambda_{\rm d}$, $\lambda_{\rm s}$, $\lambda_{\rm b}$, d).

As mentioned before, $\lambda_{\rm u}^2=m_{\rm u}^2$, $\lambda_{\rm d}^2=m_{\rm d}^2$, etc., and we need a solution with the mass hierarchies $|\lambda_{\rm u}| << |\lambda_{\rm c}| << |\lambda_{\rm t}|$ for the up quark sector and $|\lambda_{\rm d}| << |\lambda_{\rm s}| << |\lambda_{\rm b}|$ for the down one. In this case the relative phases between the λ's and the quark masses can not be fixed *a priori* and all the two possible signs in front of each of the quark masses must be explored.

As in Sec. III, in the χ^2 function we shall compare the theoretical expressions of $|V_{\rm ud}|$, $|V_{\rm us}|$, $|V_{\rm cd}|$, $|V_{\rm cs}|$ (from (21)), and $J(V)$ (from (22)) as functions of the six eigenvalues and

d (using (17) - (20)) with their experimental counterparts [5]. The $|\lambda|$'s are constrained to the experimentally determined quark masses [4].

Our best fit for the Stech case is obtained for the identification (15) of the relative phases between λ's and m's, that is,

$$(\lambda_{\rm u}\,, \lambda_{\rm c}\,, \lambda_{\rm t}) = (m_{\rm u}\,, -m_{\rm c}\,, m_{\rm t})\,, \quad (\lambda_{\rm d}\,, \lambda_{\rm s}\,, \lambda_{\rm b}) = (m_{\rm d}\,, -m_{\rm s}\,, m_{\rm b})\,. \tag{23}$$

The corresponding numerical results are displayed in Table V. The total $\chi^2/(\text{dof}) = 9.10/4 = 2.28$. Using the experimental constraints and the values of the fitted parameters of Table V, in Table VI we show the experimental, predicted, and $\Delta\chi^2$ values for the derived parameter p of Eq. (17), and the predicted values for a and b of Eqs. (18) and (19), respectively.

V. NEW TYPE OS MASS MATRICES

Recently [10] mass matrices which are a simple generalization of the Fritzsch mass matrix with one extra parameter in $M_{\rm u}$ and $M_{\rm d}$ were considered. For want of a short name we call it the CGS type mass matrix. The extra parameter comes from choosing the 13 (and 31) matrix element to be non-zero and complex. This choice gives CP-violation (see Eq. (8)), unlike the Fritzsch case, even if one works in either up-quark or down-quark diagonal basis. In subsections V A and V B we consider this type of matrix in the down-quark and up-quark diagonal basis, respectively. In Sec. VI we do present fits with the CGS-type matrices in the physical basis.

We now consider a basis in which the up-quark (down-quark) mass matrix $M_{\rm u}$ ($M_{\rm d}$) is diagonal and the remaining mass matrix $M_{\rm d}$ ($M_{\rm u}$) is hermitian of the CGS-type, namely,

$$M = \begin{pmatrix} 0 & a & d \\ a^* & 0 & b \\ d^* & b^* & c \end{pmatrix}. \tag{24}$$

For $d = 0$ this reduces to the Fritzsch-type mass matrix and will give $J(U) = 0$, where U is the corresponding diagonalizing unitary matrix.

Let $\lambda_{1,2,3}$ the eigenvalues of M, from the characteristic equation we have

$$\begin{aligned} c &= \lambda_1 + \lambda_2 + \lambda_3\,, \\ -\left(|a|^2 + |b|^2 + |d|^2\right) &= \lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3\,, \\ -c|a|^2 + 2\text{Re}(abd^*) &= \lambda_1\lambda_2\lambda_3\,. \end{aligned} \tag{25}$$

As before, for the quark sector we need the mass hierarchy $|\lambda_1| << |\lambda_2| << |\lambda_3|$ and from Eqs. (25) it is required that $\lambda_1, \lambda_3 > 0$ and $\lambda_2 < 0$, assuming $c > 0$, for both up and down quarks. For simplicity we take a and b to be real and positive and d as pure imaginary. Solving for a, b, and c yields

$$a = \left[-\frac{\lambda_1\lambda_2\lambda_3}{\lambda_1 + \lambda_2 + \lambda_3}\right]^{1/2}\,, \qquad c = \lambda_1 + \lambda_2 + \lambda_3\,, \tag{26}$$

$$b = \left[-\frac{(\lambda_3 + \lambda_2)(\lambda_3 + \lambda_1)(\lambda_2 + \lambda_1)}{\lambda_1 + \lambda_2 + \lambda_3} - |d|^2 \right]^{1/2} . \tag{27}$$

The CGS-type mass matrix in the up or down quark diagonal basis has in principle, only 4 parameters, the eigenvalues λ_1, λ_2, and λ_3 of M, and the magnitude of d. Nevertheless, the total number of parameters is 7 because this number comes from the non-zero elements of the two mass matrices. These are 3 parameters from the diagonal mass matrix plus the 4 in the other non-diagonal mass matrix. The number of constraints is 11 as before. We now investigate the viability of M in both the up-quark and down-quark diagonal basis.

A. CGS-type mass matrix in down quark diagonal basis

In this case $M = M_{\rm u}$ is the up-quark mass matrix which is diagonalized by $V_{\rm u}$. So the CKM-matrix $V = V_{\rm u}$ since $V_{\rm d} = I$. According to (2), the magnitudes of the unitary quark mixing matrix elements are given by

$$\begin{aligned} |V_{\alpha j}|^2 \;=\; & \left[(\lambda_\beta - \lambda_\alpha)(\lambda_\gamma - \lambda_\alpha)\right]^{-1} \times \\ & \left\{(a^2 + |d|^2 + \lambda_\beta\lambda_\gamma)\,\delta_{j,{\rm d}} + (a^2 + b^2 + \lambda_\beta\lambda_\gamma)\,\delta_{j,{\rm s}} \right. \\ & \left. + \left[b^2 + |d|^2 + (\lambda_\beta + \lambda_\alpha)(\lambda_\gamma + \lambda_\alpha)\right]\delta_{j,{\rm b}}\right\} , \end{aligned} \tag{28}$$

where (α,β,γ) is any permutation of (u,c,t). As mentioned in Sec. I, we shall use the four best experimentally measured magnitudes $|V_{\rm ud}|$, $|V_{\rm us}|$, $|V_{\rm cd}|$, and $|V_{\rm cs}|$.

Note that $\lambda_{\rm c} < 0$ would imply $J(V) = J(V_{\rm u}) < 0$, so we choose $d = -i|d|$ in this case. The Jarslkog invariant $J(V)$ given by (5), is then [15]

$$J(V) = -\frac{a\,b\,|d|}{(\lambda_{\rm t} - \lambda_{\rm c})(\lambda_{\rm t} - \lambda_{\rm u})(\lambda_{\rm c} - \lambda_{\rm u})} . \tag{29}$$

The relative phases between the eigenvalues and the up quark masses are

$$(\lambda_{\rm u}\,, \lambda_{\rm c}\,, \lambda_{\rm t}) = (m_{\rm u}\,, -m_{\rm c}\,, m_{\rm t}) . \tag{30}$$

The parameters to be estimated in this case are the three eigenvalues ($\lambda_{\rm u}$,$\lambda_{\rm c}$,$\lambda_{\rm t}$) and $|d|$.

In the χ^2 function we shall compare the theoretical expressions of $|V_{\rm ud}|$, $|V_{\rm us}|$, $|V_{\rm cd}|$, $|V_{\rm cs}|$ (from (28)), and $J(V)$ (from (29)) as functions of the above parameters (using relations (26) and (27)), with their experimental counterparts [5]. We shall also constrain the λ's to the experimentally determined quark masses [4] with the relative phases as given by (30).

The results of our best fit with the four parameters and the eight constraints used are displayed in Table VII. The total $\chi^2/(\text{dof}) = 5.92/4 = 1.48$. Using the experimental constraints and the values of the fitted parameters of Table VII, in Table VIII we show the experimental, predicted, and $\Delta\chi^2$ values for the derived parameters a and c of Eqs. (26) and the predicted value for b from Eq. (27).

B. CGS-type mass matrix in up quark diagonal basis

In this case $M = M_\mathrm{d}$ is the down-quark mass matrix which is diagonalized by V_d. So the CKM-matrix $V = V_\mathrm{d}^\dagger$ since $V_\mathrm{u} = I$. According to (2), the magnitudes of the unitary quark mixing matrix elements are given by

$$\begin{aligned} |V_{\alpha j}|^2 \;=\; & \big[(\lambda_k - \lambda_j)(\lambda_l - \lambda_j)\big]^{-1} \times \\ & \big\{(a^2 + |d|^2 + \lambda_k\lambda_l)\,\delta_{\alpha,\mathrm{u}} + (a^2 + b^2 + \lambda_k\lambda_l)\,\delta_{\alpha,\mathrm{c}} \\ & + \big[b^2 + |d|^2 + (\lambda_k + \lambda_j)(\lambda_l + \lambda_j)\big]\,\delta_{\alpha,\mathrm{t}}\big\}\,, \end{aligned} \tag{31}$$

where (j,k,l) is any permutation of (d,s,b). As mentioned in Sec. I, we shall use the four best experimentally measured magnitudes $|V_\mathrm{ud}|$, $|V_\mathrm{us}|$, $|V_\mathrm{cd}|$, and $|V_\mathrm{cs}|$.

Note that $\lambda_\mathrm{s} < 0$ imply $J(V) = J(V_\mathrm{d}^\dagger) > 0$, so we choose $d = i|d|$ in this case. The Jarslkog invariant $J(V)$ given by (5), is then [15]

$$J(V) = -\frac{a\,b\,|d|}{(\lambda_\mathrm{b} - \lambda_\mathrm{s})(\lambda_\mathrm{b} - \lambda_\mathrm{d})(\lambda_\mathrm{s} - \lambda_\mathrm{d})}. \tag{32}$$

The relative phases between the eigenvalues and the down quark masses are

$$(\lambda_\mathrm{d}\,, \lambda_\mathrm{s}\,, \lambda_\mathrm{b}) = (m_\mathrm{d}\,, -m_\mathrm{s}\,, m_\mathrm{b})\,. \tag{33}$$

The parameters to be estimated in this case are the three eigenvalues (λ_d,λ_s,λ_b) and $|d|$.

In the χ^2 function we shall compare the theoretical expressions of $|V_\mathrm{ud}|$, $|V_\mathrm{us}|$, $|V_\mathrm{cd}|$, $|V_\mathrm{cs}|$ (from (31)), and $J(V)$ (from (32)) as functions of the above parameters (using (26) and (27)), with their experimental counterparts [5]. We shall also constrain the λ's to the experimentally determined quark masses [4] with the relative phases as given by (33).

The results of our best fit with the four parameters and the eight constraints used are displayed in Table IX. The total $\chi^2/(\mathrm{dof}) = 15.50/4 = 3.88$. Using the experimental constraints and the values of the fitted parameters of Table IX, in Table X we show the experimental, predicted, and $\Delta\chi^2$ values for the derived parameters a and c of Eqs. (26) and the predicted value for b from Eq. (27).

VI. CGS-TYPE MASS MATRIX IN PHYSICAL BASIS

We conclude by considering a small variation of the Fritzsch case, given by the hermitian matrices

$$M_\mathrm{u} = \begin{pmatrix} 0 & a & 0 \\ a & 0 & b \\ 0 & b & c \end{pmatrix}, \qquad M_\mathrm{d} = \begin{pmatrix} 0 & a' & i|d'| \\ a' & 0 & b' \\ -i|d'| & b' & c' \end{pmatrix}. \tag{34}$$

The parameters a, b, c, a', b', and c' are considered real and positive. Here M_d is CGS-type and M_u is Fritzsch-type so that we have 7 real parameters.

From the characteristic equations of M_u and M_d we can solve for the matrix elements in terms of the corresponding eigenvalues and $|d'|$,

$$a = \left[-\frac{\lambda_\mathrm{u}\lambda_\mathrm{c}\lambda_\mathrm{t}}{\lambda_\mathrm{u}+\lambda_\mathrm{c}+\lambda_\mathrm{t}} \right]^{1/2} , \qquad c = \lambda_\mathrm{u}+\lambda_\mathrm{c}+\lambda_\mathrm{t} ,$$

$$b = \left[-\frac{(\lambda_\mathrm{t}+\lambda_\mathrm{c})(\lambda_\mathrm{t}+\lambda_\mathrm{u})(\lambda_\mathrm{c}+\lambda_\mathrm{u})}{\lambda_\mathrm{u}+\lambda_\mathrm{c}+\lambda_\mathrm{t}} \right]^{1/2} , \tag{35}$$

and

$$a' = \left[-\frac{\lambda_\mathrm{d}\lambda_\mathrm{s}\lambda_\mathrm{b}}{\lambda_\mathrm{d}+\lambda_\mathrm{s}+\lambda_\mathrm{b}} \right]^{1/2} , \qquad c' = \lambda_\mathrm{d}+\lambda_\mathrm{s}+\lambda_\mathrm{b} ,$$

$$b' = \left[-\frac{(\lambda_\mathrm{b}+\lambda_\mathrm{s})(\lambda_\mathrm{b}+\lambda_\mathrm{d})(\lambda_\mathrm{s}+\lambda_\mathrm{d})}{\lambda_\mathrm{d}+\lambda_\mathrm{s}+\lambda_\mathrm{b}} - |d'|^2 \right]^{1/2} . \tag{36}$$

From (2), the magnitudes of the unitary quark mixing matrix elements in this case are

$$\begin{aligned} |V_{\alpha j}|^2 \;=\; & \big[(\lambda_\beta-\lambda_\alpha)(\lambda_\gamma-\lambda_\alpha)(\lambda_k-\lambda_j)(\lambda_l-\lambda_j)\big]^{-1} \times \\ & \big[(a\,b'+b\,a')^2 + 2(a^2a'^2+b^2b'^2) + (a^2+b^2)|d'|^2 \\ & + 2\,a\,a'(\lambda_\beta+\lambda_\gamma)(\lambda_k+\lambda_l) + \lambda_\beta\lambda_\gamma\lambda_k\lambda_l \\ & + \lambda_k\lambda_l(a^2+b^2+\lambda_\alpha^2) + \lambda_\beta\lambda_\gamma(a'^2+b'^2+|d'|^2+\lambda_j^2) \\ & + c\,\lambda_\alpha(b'^2+|d'|^2) + c'\lambda_j\,b^2 + \lambda_\alpha\lambda_j(c\,c'+2b\,b')\big] , \end{aligned} \tag{37}$$

where (α,β,γ) is any permutation of (u,c,t) and (j,k,l) any permutation of (d,s,b). We shall use the four best experimentally measured magnitudes $|V_\mathrm{ud}|$, $|V_\mathrm{us}|$, $|V_\mathrm{cd}|$, and $|V_\mathrm{cs}|$.

The Jarslkog invariant $J(V)$ given by Eq. (5) translates into,

$$J(V) = -\frac{b\,|d'|\,\big[(a'b-a\,b')\,(b\,c'-b'c)+a\,c\,|d'|^2\big]}{(\lambda_\mathrm{b}-\lambda_\mathrm{s})(\lambda_\mathrm{b}-\lambda_\mathrm{d})(\lambda_\mathrm{s}-\lambda_\mathrm{d})(\lambda_\mathrm{t}-\lambda_\mathrm{c})(\lambda_\mathrm{t}-\lambda_\mathrm{u})(\lambda_\mathrm{c}-\lambda_\mathrm{u})} . \tag{38}$$

As in the Fritzsch case it is possible to fix the relative phases between the eigenvalues and the quark masses using Eqs. (35) and (36),

$$(\lambda_\mathrm{u}\,,\lambda_\mathrm{c}\,,\lambda_\mathrm{t}) = (m_\mathrm{u}\,,-m_\mathrm{c}\,,m_\mathrm{t}) \quad \text{and} \quad (\lambda_\mathrm{d}\,,\lambda_\mathrm{s}\,,\lambda_\mathrm{b}) = (m_\mathrm{d}\,,-m_\mathrm{s}\,,m_\mathrm{b}) . \tag{39}$$

The parameters to be estimated are the six eigenvalues (λ_u,λ_c,λ_t) and (λ_d,λ_s,λ_b), and $|d'|$.

In the χ^2 function we shall compare the theoretical expressions of $|V_\mathrm{ud}|$, $|V_\mathrm{us}|$, $|V_\mathrm{cd}|$, $|V_\mathrm{cs}|$ (from (37)), and $J(V)$ (from (38)) as functions of the above parameters (using relations (35) and (36)), with their experimental counterparts [5]. We shall also constrain the λ's to the experimentally determined quark masses [4] with the relative phases as given by (39).

The results of our best fit are displayed in Table XI. Using the experimental constraints and the values of the fitted parameters of Table XI, in Table XII we show the experimental, predicted, and $\Delta\chi^2$ values for the derived parameters a, b, c, a', and c' of Eqs. (35) and (36), and the predicted value for b' of Eq. (36).

For this case, the total $\chi^2/(\mathrm{dof}) = 1.89/4 = 0.47$ which is much better than the Fritzsch case or the Stech case. The fit for the mixed case, namely M_u CGS-type and M_d Fritzsch-type (with 7 real parameters) gives the same $\chi^2/(\mathrm{dof})$ as the fit given above. The reason is that there exists an unitary matrix Y such that it can rotate the mass matrices in Eq. (34)

to mass matrices $N_{\rm u} = Y^\dagger M_{\rm u} Y$ and $N_{\rm d} = Y^\dagger M_{\rm d} Y$ which are CGS-type and Fritzsch-type, respectively. Similarly, we can rotate $M_{\rm u}$ and $M_{\rm d}$ in Eq. (34), using an unitary matrix so that both the up and down quark mass matrices are of the CGS-type. Consequently, the case when both mass matrices are of the CGS-type with 8 parameters does not give any improvement in the quality of the fit.

VII. CONCLUDING REMARKS AND DISCUSSION OF RESULTS

Working with different specific textures of quark mass matrices, in this work we have determined bounds on the mass matrices elements for each case using as experimental restrictions the values of the four independent best measured moduli of the quark mixing matrix elements, the Jarlskog invariant $J(V)$, and the quark masses at the M_Z energy scale.

The results of the fits are presented in Tables I-XII. The $\chi^2/(\text{dof})$ for the various cases are tabulated in Table XIII for easy comparison. As can be seen, the fits for the Fritzsch and Stech type are comparable even though the former has 8 parameters compared to the 7 of the latter. The CGS-type matrix in the $M_{\rm u}$ diagonal basis is poorer despite having 7 parameters, but in the $M_{\rm d}$ diagonal basis it gives a fit comparable to the Fritzsch and Stech type matrices. However, the choice of different type of mass matrices in Sec. VI (last row of Table XIII) for up and down quarks gives a very good fit despite having only 7 parameters. This is very encouraging phenomenologically.

An important question is whether the bounds obtained on the structure of the quark mass matrices are affected by the evolution of the quark masses. To check the stability of our type of analysis we repeated it using as restrictions the values of the quark masses at the 2 GeV energy scale [4]. The results for $\chi^2/(\text{dof})$ for the various cases, at 2 GeV scale, are summarized in Table XIV. Results in Table XIV are very similar to the results in Table XIII implying that our results are not affected by the evolution of the quark masses from 2 to 91 GeV [16].

A simply way of understanding this behaviour is to note that if all quark masses are scaled by a common factor, then the algebraic expressions for the dimensionless numbers $J(V)$ and the four moduli $|V_{\alpha j}|$ ($\alpha = {\rm u, c}$; $j = {\rm d, s}$) will be unaffected. Actually, the ratio of the quark masses at 2 GeV and M_Z energy scales are $m_q(2\text{ GeV})/m_q(M_Z) = 1.71 - 1.74$ ($q = {\rm u, d, s, c, b}$) and 1.85 for $q = {\rm t}$ [4].

Phenomenologically, our results are encouraging. However, the basic problem remains, namely to find a fundamental theoretical mechanism to generate quark mass matrices.

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TABLE I: Fritzsch-type mass matrices in physical basis. In the first part of the table we show the experimental constraints imposed (where applicable) and the values obtained for the fitted parameters, along with their $\Delta\chi^2$ contribution. The eigenvalues λ are in MeV. The experimentally observed, predicted, and $\Delta\chi^2$ values of the four best measured magnitudes of the quark mixing matrix elements and the Jarlskog invariant are displayed in the second part of the table. The total $\chi^2/(\mathrm{dof}) = 4.23/3 = 1.41$.

Fitted parameters	Experimental constraint (15),[4]	Value obtained from fit	$\Delta\chi^2$
λ_{u}	$1.28^{+0.50}_{-0.43}$	$1.66^{+0.40}_{-0.36}$	0.58
λ_{c}	$-(0.624 \pm 0.083) \times 10^3$	$-(0.621 \pm 0.080) \times 10^3$	0.0013
λ_{t}	$(172.5 \pm 3.0) \times 10^3$	$(172.4 \pm 3.0) \times 10^3$	0.0011
λ_{d}	$2.91^{+1.24}_{-1.20}$	$2.29^{+0.50}_{-0.49}$	0.27
λ_{s}	$-(55^{+16}_{-15})$	$-(38.0^{+4.3}_{-5.8})$	1.28
λ_{b}	$(2.89 \pm 0.09) \times 10^3$	$(2.901 \pm 0.090) \times 10^3$	0.015
$\phi_{A'}$	—	$(-71^{+19}_{-23})^\circ$	—
$\phi_{B'}$	—	$(-4^{+18}_{-16})^\circ$	—
Observable	Experiment [5]	Prediction (13),(14)	$\Delta\chi^2$
$\lvert V_{\mathrm{ud}}\rvert$	0.97383 ± 0.00024	0.97391	0.11
$\lvert V_{\mathrm{us}}\rvert$	0.2272 ± 0.0010	0.2269	0.090
$\lvert V_{\mathrm{cd}}\rvert$	0.2271 ± 0.0010	0.2266	0.25
$\lvert V_{\mathrm{cs}}\rvert$	0.97296 ± 0.00024	0.97266	1.56
$J(V)$	$(3.08 \pm 0.18) \times 10^{-5}$	3.03×10^{-5}	0.077

TABLE II: Fritzsch-type mass matrices in physical basis. Experimentally observed, predicted, and $\Delta\chi^2$ values of the derived parameters A, B, C, and $|A'|$, $|B'|$, C', of Eqs. (12) in the up and down quark sectors, respectively. Experimental and predicted values were determined by using the entries of Table I for the experimental quark masses constraints and the fitted values of the eigenvalues, respectively.

Derived parameters	Experiment (MeV)	Prediction (MeV)	$\Delta\chi^2$
A	$28.3^{+5.8}_{-5.1}$	32.1	0.43
B	$(10.36 \pm 0.70) \times 10^3$	10.33×10^3	0.0018
C	$(171.9 \pm 3.0) \times 10^3$	171.8×10^3	0.0011
$\lvert A'\rvert$	$12.8^{+3.3}_{-3.2}$	9.4	1.13
$\lvert B'\rvert$	388^{+60}_{-56}	322	1.39
C'	$(2.838 \pm 0.091) \times 10^3$	2.865×10^3	0.088

TABLE III: Fritzsch-type mass matrices in physical basis with the phases fixed at $\phi_{A'} = -\pi/2$ and $\phi_{B'} = 0$. In the first part of the table we show the experimental constraints imposed (where applicable) and the values obtained for the fitted parameters, along with their $\Delta\chi^2$ contribution. The eigenvalues λ are in MeV. The experimentally observed, predicted, and $\Delta\chi^2$ values of the four best measured magnitudes of the quark mixing matrix elements and the Jarlskog invariant are displayed in the second part of the table. The total $\chi^2/(\text{dof}) = 4.84/5 = 0.97$.

Fitted parameters	Experimental constraint (15),[4]	Value obtained from fit	$\Delta\chi^2$
λ_u	$1.28^{+0.50}_{-0.43}$	$1.66^{+0.40}_{-0.36}$	0.58
λ_c	$-(0.624 \pm 0.083) \times 10^3$	$-(0.622 \pm 0.079) \times 10^3$	0.00058
λ_t	$(172.5 \pm 3.0) \times 10^3$	$(172.4 \pm 3.0) \times 10^3$	0.0011
λ_d	$2.91^{+1.24}_{-1.20}$	1.98 ± 0.22	0.60
λ_s	$-(55^{+16}_{-15})$	$-(38.3^{+4.0}_{-3.9})$	1.24
λ_b	$(2.89 \pm 0.09) \times 10^3$	$(2.902 \pm 0.090) \times 10^3$	0.018
Observable	Experiment [5]	Prediction (13),(14)	$\Delta\chi^2$
$\lvert V_{ud}\rvert$	0.97383 ± 0.00024	0.97393	0.17
$\lvert V_{us}\rvert$	0.2272 ± 0.0010	0.2269	0.090
$\lvert V_{cd}\rvert$	0.2271 ± 0.0010	0.2266	0.25
$\lvert V_{cs}\rvert$	0.97296 ± 0.00024	0.97264	1.78
$J(V)$	$(3.08 \pm 0.18) \times 10^{-5}$	3.02×10^{-5}	0.11

TABLE IV: Fritzsch-type mass matrices in physical basis with the phases fixed at $\phi_{A'} = -\pi/2$ and $\phi_{B'} = 0$. Experimentally observed, predicted, and $\Delta\chi^2$ values of the derived parameters A, B, C, and $|A'|$, $|B'|$, C', of Eqs. (12) in the up and down quark sectors, respectively. Experimental and predicted values were determined by using the entries of Table I for the experimental quark masses constraints and the fitted values of the eigenvalues, respectively.

Derived parameters	Experiment (MeV)	Prediction (MeV)	$\Delta\chi^2$
A	$28.3^{+5.8}_{-5.1}$	32.2	0.45
B	$(10.36 \pm 0.70) \times 10^3$	10.34×10^3	0.00082
C	$(171.9 \pm 3.0) \times 10^3$	171.8×10^3	0.0011
$\lvert A'\rvert$	$12.8^{+3.3}_{-3.2}$	8.8	1.56
$\lvert B'\rvert$	388^{+60}_{-56}	325	1.27
C'	$(2.838 \pm 0.091) \times 10^3$	2.865×10^3	0.088

TABLE V: Stech-type mass matrix in up quark diagonal basis. In the first part of the table we show the experimental constraints imposed (where applicable) and the values obtained for the fitted parameters, along with their $\Delta\chi^2$ contribution. The eigenvalues λ and d are in MeV. The experimentally observed, predicted, and $\Delta\chi^2$ values of the four best measured magnitudes of the quark mixing matrix elements and the Jarlskog invariant are displayed in the second part of the table. The total $\chi^2/(\mathrm{dof}) = 9.10/4 = 2.28$.

Fitted parameters	Experimental constraint (15),[4]	Value obtained from fit	$\Delta\chi^2$
λ_u	$1.28^{+0.50}_{-0.43}$	$1.29^{+0.23}_{-0.22}$	0.00040
λ_c	$-(0.624 \pm 0.083) \times 10^3$	$-(0.650^{+0.035}_{-0.034}) \times 10^3$	0.10
λ_t	$(172.5 \pm 3.0) \times 10^3$	$(172.4 \pm 2.8) \times 10^3$	0.0011
λ_d	$2.91^{+1.24}_{-1.20}$	0.9570 ± 0.0045	2.65
λ_s	$-(55^{+16}_{-15})$	$-(17.720^{+0.085}_{-0.086})$	6.18
λ_b	$(2.89 \pm 0.09) \times 10^3$	$(2.901^{+0.054}_{-0.052}) \times 10^3$	0.015
d	—	-9.07 ± 0.43	—
Observable	Experiment [5]	Prediction (21),(22)	$\Delta\chi^2$
$\lvert V_\mathrm{ud}\rvert$	0.97383 ± 0.00024	0.97385	0.0069
$\lvert V_\mathrm{us}\rvert$	0.2272 ± 0.0010	0.2272	0.00
$\lvert V_\mathrm{cd}\rvert$	0.2271 ± 0.0010	0.2269	0.040
$\lvert V_\mathrm{cs}\rvert$	0.97296 ± 0.00024	0.97288	0.11
$J(V)$	$(3.08 \pm 0.18) \times 10^{-5}$	3.08×10^{-5}	0.00

TABLE VI: Stech-type mass matrix in up quark diagonal basis. Experimental, predicted, and $\Delta\chi^2$ values for the derived parameter p of Eq. (17). The predicted values for a and b of Eqs. (18) and (19), respectively, are also displayed. Experimental and predicted values were determined by using the entries of Table V for the experimental quark masses constraints and the fitted values of the eigenvalues, respectively.

Derived parameters	Experiment	Prediction	$(\Delta\chi^2)$
p	$(16.51 \pm 0.60) \times 10^{-3}$	16.79×10^{-3}	0.22
a	—	4.12 MeV	—
b	—	130 MeV	—

TABLE VII: CGS-type mass matrix in down quark diagonal basis. In the first part of the table we show the experimental constraints imposed (where applicable) and the values obtained for the fitted parameters, along with their $\Delta\chi^2$ contribution. The eigenvalues λ and $|d|$ are in MeV. The experimentally observed, predicted, and $\Delta\chi^2$ values of the four best measured magnitudes of the quark mixing matrix elements and the Jarlskog invariant are displayed in the second part of the table. The total $\chi^2/(\text{dof}) = 5.92/4 = 1.48$.

Fitted parameters	Experimental constraint (15),[4]	Value obtained from fit	$\Delta\chi^2$
λ_u	$1.28^{+0.50}_{-0.43}$	$1.26^{+0.24}_{-0.19}$	0.0022
λ_c	$-(0.624 \pm 0.083) \times 10^3$	$-(0.490^{+0.060}_{-0.059}) \times 10^3$	2.61
λ_t	$(172.5 \pm 3.0) \times 10^3$	$(173.0 \pm 3.0) \times 10^3$	0.028
$\lvert d\rvert$	—	$(2.04^{+0.13}_{-0.14}) \times 10^3$	—
Observable	Experiment [5]	Prediction (28),(29)	$\Delta\chi^2$
$\lvert V_\text{ud}\rvert$	0.97383 ± 0.00024	0.97395	0.25
$\lvert V_\text{us}\rvert$	0.2272 ± 0.0010	0.2268	0.16
$\lvert V_\text{cd}\rvert$	0.2271 ± 0.0010	0.2265	0.36
$\lvert V_\text{cs}\rvert$	0.97296 ± 0.00024	0.97258	2.51
$J(V)$	$(3.08 \pm 0.18) \times 10^{-5}$	3.08×10^{-5}	0.00

TABLE VIII: CGS-type mass matrix in down quark diagonal basis. Experimental, predicted, and $\Delta\chi^2$ values for the derived parameters a and c, of Eqs. (26). The predicted value for b of Eq. (27) is also displayed. Experimental and predicted values were determined by using the entries of Table VII for the experimental quark masses constraints and the fitted values of the eigenvalues, respectively.

Derived parameters	Experiment (MeV)	Prediction (MeV)	$(\Delta\chi^2)$
a	$28.3^{+5.8}_{-5.1}$	24.9	0.44
c	$(171.9 \pm 3.0) \times 10^3$	172.5×10^3	0.040
b	—	8.97×10^3	—

TABLE IX: CGS-type mass matrix in up quark diagonal basis. In the first part of the table we show the experimental constraints imposed (where applicable) and the values obtained for the fitted parameters, along with their $\Delta\chi^2$ contribution. The eigenvalues λ and $|d|$ are in MeV. The experimentally observed, predicted, and $\Delta\chi^2$ values of the four best measured magnitudes of the quark mixing matrix elements and the Jarlskog invariant are displayed in the second part of the table. The total $\chi^2/(\mathrm{dof}) = 15.50/4 = 3.88$.

Fitted parameters	Experimental constraint (15),[4]	Value obtained from fit	$\Delta\chi^2$
λ_d	$2.91^{+1.24}_{-1.20}$	$0.318^{+0.099}_{-0.106}$	4.67
λ_s	$-(55^{+16}_{-15})$	$-(6.4 \pm 1.7)$	10.50
λ_b	$(2.89 \pm 0.09) \times 10^3$	$(2.895 \pm 0.090) \times 10^3$	0.0031
$\|d\|$	—	$9.2^{+2.1}_{-1.3}$	—
Observable	Experiment [5]	Prediction (31),(32)	$\Delta\chi^2$
$\|V_\mathrm{ud}\|$	0.97383 ± 0.00024	0.97387	0.028
$\|V_\mathrm{us}\|$	0.2272 ± 0.0010	0.2271	0.010
$\|V_\mathrm{cd}\|$	0.2271 ± 0.0010	0.2269	0.040
$\|V_\mathrm{cs}\|$	0.97296 ± 0.00024	0.97284	0.25
$J(V)$	$(3.08 \pm 0.18) \times 10^{-5}$	3.08×10^{-5}	0.00

TABLE X: CGS-type mass matrix in up quark diagonal basis. Experimental, predicted, and $\Delta\chi^2$ values for the derived parameters a and c, of Eqs. (26). The predicted value for b of Eq. (27) is also displayed. Experimental and predicted values were determined by using the entries of Table IX for the experimental quark masses constraints and the fitted values of the eigenvalues, respectively.

Derived parameters	Experiment (MeV)	Prediction (MeV)	$(\Delta\chi^2)$
a	$12.8^{+3.3}_{-3.2}$	1.4	12.69
c	$(2.838 \pm 0.091) \times 10^3$	2.889×10^3	0.31
b	—	133	—

TABLE XI: Fritzsch- and CGS-type mass matrices in physical basis. In the first part of the table we show the experimental constraints imposed (where applicable) and the values obtained for the fitted parameters, along with their $\Delta\chi^2$ contribution. The eigenvalues λ and $|d'|$ are in MeV. The experimentally observed, predicted, and $\Delta\chi^2$ values of the four best measured magnitudes of the quark mixing matrix elements and the Jarlskog invariant are displayed in the second part of the table. The total $\chi^2/(\mathrm{dof}) = 1.89/4 = 0.47$.

Fitted parameters	Experimental constraint (15),[4]	Value obtained from fit	$\Delta\chi^2$
λ_u	$1.28^{+0.50}_{-0.43}$	$1.30^{+0.49}_{-0.43}$	0.0016
λ_c	$-(0.624 \pm 0.083) \times 10^3$	$-(0.643 \pm 0.081) \times 10^3$	0.052
λ_t	$(172.5 \pm 3.0) \times 10^3$	$(172.4 \pm 3.0) \times 10^3$	0.0011
λ_d	$2.91^{+1.24}_{-1.20}$	$2.79^{+0.39}_{-0.38}$	0.010
λ_s	$-(55^{+16}_{-15})$	$-(35.7 \pm 4.6)$	1.66
λ_b	$(2.89 \pm 0.09) \times 10^3$	$(2.899 \pm 0.090) \times 10^3$	0.010
$\lvert d'\rvert$	—	$8.3^{+1.5}_{-1.1}$	—
Observable	Experiment [5]	Prediction (37),(38)	$\Delta\chi^2$
$\lvert V_\mathrm{ud}\rvert$	0.97383 ± 0.00024	0.97385	0.0069
$\lvert V_\mathrm{us}\rvert$	0.2272 ± 0.0010	0.2272	0.00
$\lvert V_\mathrm{cd}\rvert$	0.2271 ± 0.0010	0.2269	0.040
$\lvert V_\mathrm{cs}\rvert$	0.97296 ± 0.00024	0.97288	0.11
$J(V)$	$(3.08 \pm 0.18) \times 10^{-5}$	3.08×10^{-5}	0.00

TABLE XII: Fritzsch- and CGS-type mass matrices in physical basis. Experimentally observed, predicted, and $\Delta\chi^2$ values of the derived parameters a, b, c, and a' and c', of Eqs. (35) and (36) in the up and down quark sectors, respectively. The predicted value for b' of Eqs. (36) is also displayed. Experimental and predicted values were determined by using the entries of Table I for the experimental quark masses constraints and the fitted values of the eigenvalues, respectively.

Derived parameters	Experiment (MeV)	Prediction (MeV)	$\Delta\chi^2$
a	$28.3^{+5.8}_{-5.1}$	28.9	0.011
b	$(10.36 \pm 0.70) \times 10^3$	10.52×10^3	0.052
c	$(171.9 \pm 3.0) \times 10^3$	171.8×10^3	0.0011
a'	$12.8^{+3.3}_{-3.2}$	10.0	0.76
c'	$(2.838 \pm 0.091) \times 10^3$	2.866×10^3	0.095
b'	—	309	—

TABLE XIII: Comparison of the $\chi^2/(\text{dof})$ for the various types of mass matrices and quark bases considered using as experimental constraints the determined values of the quark masses at the M_Z energy scale, the magnitudes of the quark mixing matrix elements V_{ud}, V_{us}, V_{cd}, and V_{cs}, and the Jarlskog invariant $J(V)$. Note that the number of parameters comes from the non-zero elements of the two mass matrices. These are 3 parameters from a diagonal mass matrix plus those in the other non-diagonal mass matrix. There are 11 summands in the χ^2 function as explained at the end of Sec. II.

Table	Type of mass matrix	Basis	Number of parameters	$\chi^2/(\text{dof})$
I, II	Fritzsch	Physical ($\phi_{A'}$ and $\phi_{B'}$ free)	8	$4.23/3 = 1.41$
III, IV		Physical ($\phi_{A'} = -\pi/2$, $\phi_{B'} = 0$)	6	$4.84/5 = 0.97$
V, VI	Stech	M_{u} diagonal	7	$9.10/4 = 2.28$
VII, VIII	CGS	M_{d} diagonal	7	$5.92/4 = 1.48$
IX, X		M_{u} diagonal	7	$15.50/4 = 3.88$
XI, XII	M_{u} Fritzsch-type and M_{d} CGS-type	Physical	7	$1.89/4 = 0.47$

TABLE XIV: Comparison of the $\chi^2/(\text{dof})$ for the various types of mass matrices and quark bases considered using as experimental constraints the determined values of the quark masses at the 2 GeV energy scale ($m_{\text{u}} = 2.2^{+0.8}_{-0.7}$ MeV, $m_{\text{d}} = 5.0 \pm 2.0$ MeV, $m_{\text{s}} = 95 \pm 25$ MeV, $m_{\text{c}} = 1.07 \pm 0.12$ GeV, $m_{\text{b}} = 5.04^{+0.16}_{-0.15}$ GeV, $m_{\text{t}} = 318.9^{+13.1}_{-12.3}$ GeV) [4], the magnitudes of the quark mixing matrix elements V_{ud}, V_{us}, V_{cd}, and V_{cs}, and the Jarlskog invariant $J(V)$. Note that the number of parameters comes from the non-zero elements of the two mass matrices. These are 3 parameters from a diagonal mass matrix plus those in the other non-diagonal mass matrix. There are 11 summands in the χ^2 function as explained at the end of Sec. II.

Table	Type of mass matrix	Basis	Number of parameters	$\chi^2/(\text{dof})$
I, II	Fritzsch	Physical ($\phi_{A'}$ and $\phi_{B'}$ free)	8	$4.80/3 = 1.60$
III, IV		Physical ($\phi_{A'} = -\pi/2$, $\phi_{B'} = 0$)	6	$5.49/5 = 1.10$
V, VI	Stech	M_{u} diagonal	7	$11.00/4 = 2.75$
VII, VIII	CGS	M_{d} diagonal	7	$5.39/4 = 1.35$
IX, X		M_{u} diagonal	7	$17.99/4 = 4.50$
XI, XII	M_{u} Fritzsch-type and M_{d} CGS-type	Physical	7	$2.47/4 = 0.62$