Constraints on mass matrices due to measured property of the mixing matrix

S. Chaturvedi *
School of Physics
University of Hyderabad, Hyderabad 500 046, India
Virendra Gupta †
Departamento de Fisíca Aplicada, CINVESTAV-Unidad Mérida
A.P. 73 Cordemex 97310 Mérida, Yucatan, Mexico

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Abstract

It is shown that two specific properties of the unitary matrix V can be expressed directly in terms of the matrix elements and eigenvalues of the hermitian matrix M which is diagonalized by V. These are the asymmetry $\Delta(V) = |V_{12}|^2 - |V_{21}|^2$, of V with respect to the main diagonal and the Jarlskog invariant $J(V) = \mathrm{Im}(V_{11}V_{12}^*V_{21}^*V_{22})$. These expressions for $\Delta(V)$ and $J(V)$ provide constraints on possible mass matrices from the available data on V.

*scsp@uohyd.ernet.in
†virendra@aruna.mda.cinvestav.mx

1 Introduction

Flavor mixing in both the quarks and lepton sector has been firmly established experimentally for a long time, However, there is still no deep understanding of the observed mixings.

In the Standard Model, the unitary mixing matrices arise from the diagonalization of the corresponding hermitian mass matrices. In the lepton sector one usually works in the basis in which the charged lepton mass matrix is diagonal, so that the neutrino flavor mixing is described by a single unitary matrix [1] which diagonalizes the neutrino mass matrix. In the quark sector [2], in the physical basis, the CKM-mixing matrix $V = U^{\dagger}U'$, where the unitary matrices U and U' diagonalize the up-quark and down-quark mass matrices respectively. One can also work in a basis in which the up-quark (down-quark) mass matrix M (M') is diagonal. In these bases, the mixing matrix in the quark sector (like the neutrino sector) will come from a single mass matrix. Clearly, if we knew the mass matrices fully then the corresponding mixing matrices are completely determined. In practice, the mass matrices are guessed at, while experiment can only determine the numerical values of the matrix elements of the mixing matrix. Our objective is to learn something about the structure of the underlying mass matrix M from the knowledge of the mixing matrix V which diagonalizes it. Can a general property of V imply a constraint on M ? In particular we show that the asymmetry $\Delta(V)$ w.r.t. the main diagonal and, $J(V)$, the Jarlskog invariant [3] which is a measure of CP-violation can be directly expressed in terms of the eigenvalues and matrix elements of M! These can provide simple criterion for selecting suitable mass matrices.

2 Derivations of the formulas for $\Delta(V)$ and $J(V)$

Consider a 3×3 mass matrix M which is diagonalized by the unitary matrix V, so that

$$M = V \ \widehat{M} \ V^{\dagger}, \tag{1}$$

where $\widehat{M} = \mathrm{diag}(m_1, m_2, m_3)$. We can also write

$$M = m_1 \ N_1 + m_2 \ N_2 + m_3 \ N_3, \tag{2}$$

where N_α are the projectors of M. They satisfy,

$$N_\alpha \ N_\beta = N_\alpha \ \delta_{\alpha\beta} \quad \text{and} \qquad (N_\alpha)_{k\ell} = V_{k\alpha} \ V^{*}_{\ell\alpha}. \tag{3}$$

Furthermore, in terms of M and its eigenvalues,

$$N_\alpha = \frac{(m_\beta - M)(m_\gamma - M)}{(m_\beta - m_\alpha)(m_\gamma - m_\alpha)}, \quad \alpha \neq \beta \neq \gamma, \tag{4}$$

with $\alpha,\ \beta,\ \gamma$ taking values from 1 to 3. It is clear from Eq.(3) that

$$|V_{k\alpha}|^2 = (N_\alpha)_{kk}. \tag{5}$$

Through this equation each $|V_{k\alpha}|$ can be calculated in terms of the eigenvalues[1] and matrix elements of M.

(a) Formula for $\Delta(V)$

The asymmetry with respect to the main diagonal of V is given by

$$\Delta(V) \equiv |V_{12}|^2 - |V_{21}|^2 = |V_{23}|^2 - |V_{32}|^2 = |V_{31}|^2 - |V_{13}|^2. \tag{6}$$

The last two equations follow from the unitarity of V, namely, $VV^\dagger = V^\dagger V = I$. Using Eqs.(4, 5), simple algebra gives,

$$\Delta(V) = \frac{1}{D(m)} \{ \sum_k \Big(m_k\ (M^2)_{kk} - m_k^2\ M_{kk} \Big) \}, \tag{7}$$

where

$$\begin{aligned} D(m) &\equiv \begin{vmatrix} 1 & 1 & 1 \\ m_1 & m_2 & m_3 \\ m_1^2 & m_2^2 & m_3^2 \end{vmatrix} \\ &= (m_2 - m_1)(m_3 - m_1)(m_3 - m_2). \end{aligned} \tag{8}$$

Our result in Eq.(7) tells us, given m_i and M, whether V will be symmetric ($\Delta(V) = 0$) or not. We note that the asymmetry for the CKM-matrix, $\Delta(V)$ is intriguingly small!

(b) Formula for $J(V)$

We use the definition

$$J(V) = \mathrm{Im}(V_{11} V_{12}^* V_{21}^* V_{22}). \tag{9}$$

The imaginary parts of eight other plaquettes are just $\pm J(V)$ because V is unitary [3]. To derive our result we note that

$$M_{12} M_{23} M_{13}^* = \sum_{k,\ell,n} m_k m_\ell m_n V_{1k} V_{2k}^* V_{2\ell} V_{3\ell}^* V_{1n}^* V_{3n}, \tag{10}$$

[1] Our results require non-degenerate eigenvalues. This is true for the quarks.

since from Eq.(1), $M_{ij} = \sum_k m_k V_{ik} V^*_{jk}$. Now use the unitarity relation $V^*_{1\ell} V_{1n} + V^*_{2\ell} V_{2n} + V^*_{3\ell} V_{3n} = \delta_{\ell n}$ and take imaginary parts to obtain

$$\begin{aligned} \mathrm{Im}(M_{12} M_{23} M^*_{13}) &= \sum_{k,\ell} m_k m_\ell^2 \ \mathrm{Im}(V_{1k} V^*_{2k} V_{2\ell} V^*_{1\ell}) \\ &- \left[\sum_n m_n(|V_{1n}|^2 + |V_{2n}|^2)\right] \cdot \sum_{k,\ell} m_k m_\ell \ \mathrm{Im}(V_{1k} V^*_{2k} V_{2\ell} V^*_{1\ell}) \end{aligned} \tag{11}$$

The imaginary parts on the RHS, for various plaquettes of V, yield $\pm J(V)$ for various values of k and ℓ. As a result the second term sums up to zero. One thus obtains

$$J(V) = \frac{\mathrm{Im}(M_{12} M_{23} M^*_{13})}{D(m)}. \tag{12}$$

This remarkable result shows that if $M_{12} M_{23} M^*_{13}$ is real for a given M, then the Jarlskog invariant for the matrix V which diagonalizes it vanishes. Thus to obtain CP-violation, the mass matrix for up-quark (down- quark) must have $\mathrm{Im}(M_{12} M_{23} M^*_{13})$ non-zero in a basis in which down-quark (up-quark) mass matrix is diagonal. Equivalently, $\Theta \equiv \theta_{12} + \theta_{23} - \theta_{13} \neq n\pi$ ($n = 0, 1, 2, \cdots$), here θ_{ij} is the phase of M_{ij}. This is reminiscent of the fact [4] that physically the relevant phase for CP-violation in CKM-matrix is $\Phi \equiv \phi_{12} + \phi_{23} - \phi_{13}$, where ϕ_{ij} is the phase of V_{ij}. The reason is that Φ is invariant under re-phasing transformations. Clearly, under re-phasing transformations Θ is also invariant (See Eq.(11)). Furthermore it can be shown directly that if any one of the three off-diagonal elements of V is zero then $J(V)$ vanishes [5]. Thus the appearance of the numerator in Eq.(12) is understandable.

The use of Eqs.(7) and (12) for the quark sector will be considered elsewhere.

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References

[1] Z. Maki, M. Nakagawa and S. Sakata, Prog. Theor. Phys. **28**, 247 (1962); V. N. Gribov and B. M. Pontecorvo, Phys. Lett. B **28**, 493 (1969).

[2] M.Kobayashi and T.Maskawa,Prog.Theor.Phys. D **35**, 652 (1973); N. Cabibbo Phys. Rev. Lett. **10**, 531 (1953); L. L. Chau and W.-T. Keung, Phys. Rev. Lett. **53**, 1802 (1984).

[3] C. Jarlskog, Phys. Rev. Lett. **55**, 1039 (1985).

[4] M. Gronau and J. Schechter, Phys. Rev. Lett. **54**, 385 (1985); Phys. Rev. Lett. **54**, 2176 (1985). Other references can be found in these.

[5] Let $M_{ij} = m_1 V_{i1} V_{j1}^* + m_2 V_{i2} V_{j2}^* + m_3 V_{i3} V_{j3}^* = 0$, $i \neq j$. Multiply by $V_{i3}^* V_{j3}$ and take imaginary parts to obtain $(m_1 - m_2) J(V) = 0$. Similarly, multiplying by $V_{i2}^* V_{j2}$ and $V_{i1}^* V_{j1}$ and taking imaginary parts one obtains $(m_1 - m_3) J(V) = 0$ and $(m_2 - m_3) J(V) = 0$. Thus either $m_1 = m_2 = m_3$, that is, M is trivial or $J(V) = 0$.

[6] H. Fritzsch, Phys. Lett. B **70**, 436 (1977); Phys. Lett. B **166**, 423(1986).

[7] T. Kitazoe and K. Tanaka, Phys. Rev. **18**, 3476 (1978); H. Georgi and D. Nanapoulos, Phys. Lett. B **82**, 392 (1979); B. Stech, Phys. Lett. B **130**, 189 (1983).