Grand canonical partition functions for multi level para Fermi systems of any order

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Abstract

A general formula for the grand canonical partition function for a para Fermi system of any order and of any number of levels is derived.

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Parastatistics[1,3] was introduced by Green[1] long ago as a generalisation of Bose and Fermi statistics . Subequently it has found many interesting applications to the paraquark model[4] and to parastring models[5].Green's generalisation, carried out at the level of the algebra of creation and annihilation operators, involves introducing trilinear relations in place of the bilinear relations which characterize Bose and Fermi systems. The Fock space of an M-level para Bose system of order p, where p is any positive integer, is characterized by the trilinear relations

$$[a_k, \{a_l, a_m\}] = 0 \ ; \ [a_k, \{a_l^\dagger, a_m^\dagger\}] = 2\delta_{kl}a_m^\dagger + 2\delta_{km}a_l^\dagger \ ; \ [a_k, \{a_l^\dagger, a_m\}] = 2\delta_{kl}a_m \quad , \tag{1}$$

and the supplementary conditions

$$a_k a_l^\dagger |0>= p\delta_{kl}|0> \quad . \tag{2}$$

Here the subscripts k, l, m take on values $1 \cdots M$. Similarly, the trilinear relations

$$[a_k, [a_l, a_m]] = 0 \ ; \ [a_k, [a_l^\dagger, a_m^\dagger]] = 2\delta_{kl}a_m^\dagger - 2\delta_{km}a_l^\dagger \ ; \ [a_k, [a_l^\dagger, a_m]] = 2\delta_{kl}a_m \quad , \tag{3}$$

together with the supplementary conditions (2) define a para Fermi systems of order p. Bose and Fermi statistics arise from these as a special case correspnding to $p = 1$. A convenient and a physically appealing representation of para systems is provided by the Green decomposition. Here the annihilation (creation) operators a_i ($a_i^\dagger$) for a para system of order p are expressed as sums of annihilation (creation) operators operators $a_{i\alpha}$ ($a_{i\alpha}^\dagger$) which carry an extra label α taking values $1, \cdots, p$.

$$a_i = \sum_{\alpha=1}^{p} a_{i\alpha} \quad , \quad a_i^\dagger = \sum_{\alpha=1}^{p} a_{i\alpha}^\dagger \quad ; \quad a_{i\alpha}|0>= 0 \quad . \tag{4}$$

The operators $a_{i\alpha}$ and ${a^\dagger}_{i\alpha}$ obey commutation relations which are partly bosonic and partly fermionic. For a para Bose system of order p these anomalous commutation relations are

$$\begin{gathered}[a_{i\alpha}, a_{j\alpha}] = 0 \quad ; \quad [a_{i\alpha}, a_{j\alpha}^\dagger] = \delta_{ij} \\ \{a_{i\alpha}, a_{j\beta}\} = \{a_{i\alpha}, a_{j\beta}^\dagger\} = 0 \ \ if \ \alpha \neq \beta \quad .\end{gathered} \tag{5}$$

For a para Fermi system of order p, the corresponding relations are

$$\{a_{i\alpha}, a_{j\alpha}\} = 0 \quad ; \quad \{a_{i\alpha}, a^\dagger_{j\alpha}\} = \delta_{ij}$$
$$[a_{i\alpha}, a_{j\beta}] \;\; ; \;\; [a_{i\alpha}, a^\dagger_{j\beta}] = 0 \;\; if \;\; \alpha \neq \beta \quad . \tag{6}$$

Given the relations (1) and (3) and the supplementary condition (2) one can build a Fock space for parbose and para Fermi systems by repeated applications of the creation operators on the vacuum state. For the Fock space thus obtained, two natural question arise.

[1] What is the dimensionality of the N-particle subspace of the Fock space thus obtained?

and, at a finer level,

[2] How many independent states are there corresponding to a given set of occupation $n_1 \cdots n_M$.

One needs answers to these questions,in particular, that of the latter, in order to be able to construct the canonical and grand canonical partition functions for parasystems. Though parastatistics was introduced nearly forty years ago, it is seems surprising that, until recently, the only results that were available for a non trivial parasystem pertained to the following special cases:

(a) Single level parasystems of arbitrary order[6,7]

(b) Two level parasystems of arbitrary order[8,9]

(c) Parasystems of order two consisting of arbitrary number of levels[10].

The task for the general case i.e. for the case of a parasytem of arbitarary number of levels and of arbitarary order was completed recently by one of us[11]. This work encompasses not only parastatistics of any order but also all statistics that can be defined on the basis of the permutation group including those for which no simple definition in terms of the algebra of creation and annihilation operators is possible. This was achieved by following the approach to parastatistics pioneered by Messiah and Greenberg[12] and further developed by Hartle Stolt and Taylor[13]. In this approach parastatistics arises in the quantum mechanical description of an assembly of N-identical particles with the permutation group S_N playing a central role in defining various kinds of statistics including the parastatistics of Green. In this work, it was shown that the the general structure of the canonical partition function for an ideal system corresponding to any quantum statistics based on the permutation group is as follows

$$Z_N(x_1, \cdots, x_M) = \sum_\lambda s_\lambda(x_1, \cdots, x_M) \quad , \tag{7}$$

where $x_i \equiv \exp(-\beta\epsilon_i)$; $i = 1, \cdots, M$, ϵ_i denote the energies corresponding to the states $i = 1, \cdots, M$ and $\lambda \equiv (\lambda_1, \cdots, \lambda_M); \lambda_1 \geq \lambda_\geq \cdots \lambda_M$ denotes a partition of N. The functions $s_\lambda(x_1, \cdots, x_M)$ denote the Schur functions[14,15] which are homogeneous symmetric polynomials of degree N in the variables $x_i, \cdots, x_M$. Explicitly the Schur functions are given by

$$s_\lambda(x_1, \cdots, x_M) = \frac{\det(x_i^{\lambda_j + M - j})}{\det(x_i^{M-j})} \quad ; \quad 1 \leq i, j \leq M \quad . \tag{8}$$

An alternative definition of Schur functions in terms of the monomial symmetric functions[14,15] is as follows

$$s_\chi(x_1, \cdots, x_M) = \sum_\lambda K_{\chi\lambda} m_\lambda(x_1, \cdots, x_M) \quad , \tag{9}$$

where $K_{\chi\lambda}$ denote the Kostka numbers[14,15].

The canonical partition functions corresponding to various statistics based on the permutation group are obtained by putting appropriate restrictions on the partitions λ that occur on the R.H.S. of (7). Parabose case of order p arises when we retrict the sum in (7) to only those partitions of N whose length $l(\lambda)$ (the number of the non-zero λ_i's) is less than equal to p. Similarly, para Fermi case of order p arises when we restrict λ in (7) to those partitions for which $\lambda_1 \leq p$.

In this letter we shall confine ourselves to para Fermi systems of order p. The canonical partition function for such a system is given by

$$Z_N^{PF}(x_1, \cdots, x_M; p) = \sum_{\substack{\lambda \\ \lambda_1 \leq p}} s_\lambda(x_1, \cdots, x_M) \quad , \tag{10}$$

and hence the grand canonical partition function is given by

$$\mathcal{Z}^{PF}(x_1,\cdots,x_M,\mu;p)=\sum_N exp(\mu\beta N)Z_N^{PF}(x_1,\cdots,x_M;p) \quad . \tag{11}$$

Using (10) and the fact that the Schur functions are homogeneous polynomials of degree N, (11) can be written as

$$\mathcal{Z}^{PF}(x_1,\cdots,x_M,\mu;p)=\sum_N \sum_{\substack{\lambda \\ \lambda_1\le p}} s_\lambda(X_1,\cdots,X_M) \quad , \tag{12}$$

where $X_i \equiv \exp(-\beta(\epsilon_i-\mu))$. The sum on the R.H.S of (12) is explicitly known[14] and is given by

$$\mathcal{Z}^{PF}(X_1,\cdots,X_M;p)=\frac{\det(X_j^{2M+p+1-i}-X_j^i)}{\det(X_j^{2M+1-i}-X_j^i)} \;\; ; \;\; 1\le i,j\le M \quad . \tag{13}$$

We now consider some special cases

Case I: p=1, M arbitrary

This case corresponds to the Fermi statistics. In this case (13) becomes

$$\mathcal{Z}^{F}(X_1,\cdots,X_M)=\frac{\det(X_j^{2M+2-i}-X_j^i)}{\det(X_j^{2M+1-i}-X_j^i)} \;\; ; \;\; 1\le i,j\le M \quad . \tag{14}$$

In $\det(X_j^{2M+2-i}-X_j^i)$, adding to each row the one that succeeds it and using

$$(X_j^{2M+2-i}-X_j^i)+(X_j^{2M+1-i}-X_j^{i+1})=(1+X_j)(X_j^{2M+1-i}-X_j^i) \quad , \tag{15}$$

$$(X_j^{M+2}-X_j^M)=(1+X_j)(X_j^{M+1}-X_j^M) \quad , \tag{16}$$

one finds that

$$\det(X_j^{2M+2-i}-X_j^i)=\prod_j(1+X_j) \;\; \det(X_j^{2M+1-i}-X_j^i) \quad , \tag{17}$$

and hence

$$\mathcal{Z}^{F}(X_1,\cdots,X_M)=\prod_j(1+X_j) \quad . \tag{18}$$

Case II: $p\rightarrow\infty$, M arbitrary

In this limiting case, christened HST statistics in ref 11, (13) becomes

$$\mathcal{Z}^{HST}(X_1,\cdots,X_M) = (-1)^M \frac{\det(X_j^i)}{\det(X_j^{2M+1-i} - X_j^i)} \quad ; \quad 1 \le i,j \le M \quad , \tag{19}$$

which on using

$$\det(X_j^i) = \prod_j {X_j}^j \prod_{i<j} (1 - X_i/X_j) \quad , \tag{20}$$

$$\det(X_j^{2M+1-i} - X_j^i) = \prod_j {X_j}^j \prod_j (X_j - 1) \prod_{i<j} (1 - X_i/X_j)(1 - X_iX_j) \quad , \tag{21}$$

yields

$$\mathcal{Z}^{HST}(X_1,\cdots,X_M) = \prod_i \frac{1}{(1-X_i)} \prod_{i<j} \frac{1}{(1-X_iX_j)} \quad . \tag{22}$$

Case III: p arbitrary, M =1

For $M = 1$, with p arbitrary, (13) gives

$$\mathcal{Z}^{PF}(X;p) = \frac{(1-X^{p+1})}{(1-X)} \quad . \tag{23}$$

Case IV: p arbitrary, M =2

In this case, (13) gives the following expression for the grand canonical partition function

$$\mathcal{Z}^{PF}(X_1,X_2;p) = \frac{1}{(1-X_1)} \frac{1}{(1-X_2)} \left[\frac{(1-(X_1X_2)^{p+2})}{(1-X_1X_2)} - \frac{({X_1}^{p+2} - {X_2}^{p+2})}{(X_1 - X_2)} \right] \quad . \tag{24}$$

This result is the same as that given in ref 9 but differs from that in ref 8. The mistake in ref 8 appears to be that the results for para Fermi systems were obtained from those for para Bose systems by simply putting occupancy restrictions without taking into account the fact that the number of states corresponding to a given set of occupation numbers in the two cases is also different.

The R.H.S. of (24) may be simplified to yield

$$\begin{aligned}\mathcal{Z}^{PF}(X_1,X_2;p) = & \left(\sum_{l=0}^{p} {X_1}^l\right) \left(\sum_{l=0}^{p} {X_2}^l\right) + X_1X_2 \left(\sum_{l=0}^{p-2} {X_1}^l\right) \left(\sum_{l=0}^{p-2} {X_2}^l\right) \\ & + (X_1X_2)^2 \left(\sum_{l=0}^{p-4} {X_1}^l\right) \left(\sum_{l=0}^{p-4} {X_2}^l\right) + \cdots \quad .\end{aligned} \tag{25}$$

Case IV: p arbitrary , M =3

For $M = 3$, (13) gives

$$\mathcal{Z}^{PF}(X_1, X_2, X_3; p) = \frac{\det(X_j^{p+7-i} - X_j^i)}{\det(X_j^{2M+1-i} - X_j^i)} \quad ; \quad 1 \leq i, j \leq 3 \quad . \tag{26}$$

Setting $p = 2$ and simplifying the R.H.S. one obtains

$$\begin{aligned} \mathcal{Z}^{PF}(X_1, X_2, X_3; 2) = \ & 1 + (X_1 + X_2 + X_3) + (X_1 + X_2 + X_3)^2 \\ & + (X_1X_2 + X_2X_3 + X_3X_1)[(1 + X_1)(1 + X_2)(1 + X_3) - 1] \\ & + (X_1X_2X_3)^2 \quad . \end{aligned} \tag{27}$$

Similarly for $p = 3$, one obtains

$$\begin{aligned} \mathcal{Z}^{PF}(X_1, X_2, X_3; 3) = \ & (1 + X_1)(1 + X_2)(1 + X_3)[1 + (X_1 + X_2 + X_3) \\ & + ({X_1}^2 + {X_2}^2 + {X_3}^2 + X_1X_2 + X_2X_3 + X_3X_1) + X_1X_2X_3 \\ & + ({X_1}^2{X_2}^2 + {X_2}^2{X_3}^2 + {X_3}^2{X_1}^2 \\ & + X_1X_2X_3(X_1 + X_2 + X_3)) + (X_1X_2X_3)^2] \quad . \end{aligned} \tag{28}$$

To conclude, we have given in (13) the general formula for the grand canonical partition function for a multi level para Fermi system of arbitrary order p and have shown how all hitherto known results for para Fermi systems corresponding to specific values of p and M arise from it as special cases.

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