Eigenstates of linear combinations of two boson creation and annihilation operators : An algebraic approach

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Abstract

Eigenstates of the linear combinations $a^2 + \beta a^{\dagger 2}$ and $ab + \beta a^\dagger b^\dagger$ of two boson creation and annihilation operators are presented. The algebraic procedure given here is based on the work of Shanta et al. [Phys. Rev. Lett. **72**, 1447, 1994] for constructing eigenstates of generalized annihilation operators. Expressions for the overlaps of these states with the number states, the coherent states and the squeezed states are given in a closed form.

PACS No. : 42.50 Dv 03.65 Fd

1. Introduction

In a recent work[1] it was shown that for an operator F made up of powers of a single bosonic annihilation operator or of products of commuting bosonic annihilation operators one can construct a canonical conjugate $G_i^\dagger$ such that $[F, G_i^\dagger] = 1$ on all states in a specific sector S_i of the Fock space. The eigenstates of the generalized annihilation operators F and G_i were constructed and it was shown that these coherent states come in pairs and are in some sense duals of each other. Thus, for instance, it was shown that the cat states[2] and the Yuen states[3] form a dual pair and so do the pair coherent states[4–6] and the Caves-Shumaker states.[7] This construction has also been recently extended to deformed algebras.[8]

In this work we show that a two fold application of the construction developed in ref. 1 can also be put to use for a purely algebraic construction of the eigenstates of operators of the type $F + \beta F^\dagger$. In particular, we consider two operators of this type.

(i) $\mathcal{F}_1 = (a^2 + \beta a^{\dagger 2})$; $[a, a^\dagger] = 1$

(ii) $\mathcal{F}_2 = (ab + \beta a^\dagger b^\dagger)$; $[a, a^\dagger] = 1$, $[b, b^\dagger] = 1$, $[a, b] = 0$.

It is well known that the operators $\frac{1}{2}a^2, \frac{1}{2}a^{\dagger 2}$ and $\frac{1}{2}(a^\dagger a + \frac{1}{2})$ when identified respectively with K_-, K_+ and K_z furnish a realization of the $su(1,1)$ algebra $[K_z, K_\pm] = \pm K_\pm$, $[K_-, K_+] = 2K_z$. Similarly the operators $K_- = ab$, $K_+ = a^\dagger b^\dagger$ and $K_z = \frac{1}{2}(aa^\dagger + bb^\dagger)$ provide a two mode realization of the same algebra. In view of this, one is considering here the eigenvalue problem for the operator $K_- + \beta K_+$. The eigenfunctions of the operator $\mathcal{F}_1$ have recently been constructed by Nieto and Truax[9] and by Satya Prakash and Agarwal[10] by solving the appropriate differential euqation. The latter authors have also investigated the non-classical aspects of these states. A similar analysis has been carried out for the eigenfunctions of $\mathcal{F}_2$ corresponding to the zero eigenvalue of $aa^\dagger - bb^\dagger$.[11] Here we present a complete solution to the problem using an entirely algebraic approach.

2. Eigenstates of The Operator $\mathcal{F}_1$

The solution of the eigenvalue problem

$$\mathcal{F}_1 \mid \psi >= \lambda \mid \psi > \qquad \mathcal{F}_1 = (a^2 + \beta a^{\dagger 2}) \quad , \tag{1}$$

involves two steps

1. Construction of the state $\mid \psi >_o$ annihilated by $\mathcal{F}_1$

$$\mathcal{F}_1 \mid \psi >_o = 0 \quad . \tag{2}$$

2. Construction of the canonical conjugate $\mathcal{G}$ to $\mathcal{F}_1$ satisfying $[\mathcal{F}_1, \mathcal{G}^\dagger] = 1$.

Construction of the state $\mid \psi >_o$ annihilated by $\mathcal{F}_1$:

We rewrite (2) as

$$a^2 \mid \psi >_o = -\beta a^{\dagger 2} \mid \psi >_o \quad , \tag{3}$$

and apply a^2 on both sides. The resulting equation can be written as

$$F \mid \psi >_o = -\beta \mid \psi >_o \quad , \quad F = \frac{1}{(n_1+1)(n_a+2)} a^4 \quad , \quad n_a \equiv a^\dagger a \quad . \tag{4}$$

Thus the task of solving (3) reduces to constructing the solutions of the eigenvalue problem (4). It must be borne in mind that the solutions of (3) satisfy (4) but not vice versa. After constructing the solutions of (4) one has to discard those which are not solutions of (3).

To find the eigenstates of F in (4) we follow the procedure of ref. 1. The states annihilated by F are $\mid 0 >$, $\mid 1 >$, $\mid 2 >$ and $\mid 3 >$. Successive applications of $F^\dagger$ on these four generate the four sectors

$$S_0 = \{\mid 4n >\}, \ S_1 = \{\mid 4n+1 >\}, \ S_2 = \{\mid 4n+2 >\}, \ S_3 = \{\mid 4n+3 >\} \quad , \tag{5}$$

where $n = 0, 1, \ldots\ldots.$ As shown in Appendix A, the canonical conjugates $G_i^\dagger$ of F, satisfying $[F, G_i^\dagger] = 1$ in the sector S_i are

$$G_o^\dagger = \frac{a^{\dagger 4}}{4} \frac{1}{(n_a + 3)} \quad , \tag{6}$$

$$G_1^\dagger = \frac{a^{\dagger 4}}{4}\frac{1}{(n_a+4)} \quad , \tag{7}$$
$$G_2^\dagger = \frac{a^{\dagger 4}}{4}\frac{(n_a+2)}{(n_a+4)(n_a+3)} \quad , \tag{8}$$
$$G_3^\dagger = \frac{a^{\dagger 4}}{4}\frac{(n_a+1)}{(n_a+4)(n_a+3)} \quad . \tag{9}$$

The general solution of (4) may thus be written as

$$\begin{aligned} \mid \psi >_o \; &= C_o \exp(-\beta G\dagger_o) \mid 0 > + C_1 \exp(-\beta G_1^\dagger) \mid 1 > + C_2 \exp(-\beta G_2^\dagger) \mid 2 > \\ &+ C_3 \exp(-\beta G_3^\dagger) \mid 3 > \quad . \end{aligned} \tag{10}$$

Of these four independent solutions of (4) the latter two, which are specific linear combinations of the states in the sectors S_2 and S_3 respectively, are not solutions of (3) as can readily be verified. The general solution of (2) is thus

$$\mid \psi >_o = C_o \exp(-\beta G_o^\dagger) \mid 0 > + C_1 \exp(-\beta G_1^\dagger) \mid 1 > \quad . \tag{11}$$

This completes the first step towards the solution of the eigenvalue problem (1).

Construction of the canonical conjugate $\mathcal{G}^\dagger$ of $\mathcal{F}_1$

We begin by noticing that the canonical conjugates $g_i^\dagger$, $i = 0, 1$ of a^2

$$g_o^\dagger = \frac{a^{\dagger 2}}{2}\frac{1}{(n_a+1)} \quad , \tag{12}$$
$$g_1^\dagger = \frac{a^{\dagger 2}}{2}\frac{1}{(n_a+2)} \quad , \tag{13}$$

satifying

$$[a^2, g_i^\dagger] = 1 \quad , \tag{14}$$

in the sectors

$$S_o = \{\mid 2m >\} \quad , \quad S_1 = \{\mid 2m+1 >\} \quad , \quad m = 0, 1 \ldots\ldots \quad , \tag{15}$$

respectively also satisfy

$$[a^{\dagger 2}, g_i^\dagger] = 4 g_i^{\dagger 2} \quad . \tag{16}$$

This suggests the following form for $\mathcal{G}_i^\dagger$, the canonical conjugate of $\mathcal{F}_1 \equiv a^2 + \beta a^{\dagger 2}$, we are looking for

$$\mathcal{G}_i^\dagger = \sum_{n=1}^{\infty} b_n (g_i^\dagger)^n \quad . \tag{17}$$

On requiring that

$$[a^2 + \beta a^{\dagger 2}, \mathcal{G}_i^\dagger] = 1 \quad , \tag{18}$$

we find that

$$b_{2m} = 0 \quad , \quad b_{2m+1} = \frac{(-4\beta)^m}{(2m+1)} \quad , \tag{19}$$

so that

$$\mathcal{G}_i^\dagger = \frac{1}{\sqrt{4\beta}} \tan^{-1}(\sqrt{4\beta} g_i^\dagger) \quad . \tag{20}$$

The completes the construction of $\mathcal{G}_i^\dagger$. The general solution of the eigenvalue problem (1) is given by

$$\mid \psi >= C_o \mid \psi, e> + C_1 \mid \psi, 0> \quad , \tag{21}$$

where

$$\mid \psi, e> \ = \exp(\lambda \mathcal{G}_o^\dagger) \exp(-\beta G_o^\dagger) \mid 0> \quad , \tag{22}$$

$$\mid \psi, o> \ = \exp(\lambda \mathcal{G}_1^\dagger) \exp(-\beta G_1^\dagger) \mid 1> \quad . \tag{23}$$

3. Overlap between the eigenstates of $\mathcal{F}_1$ and the squeezed vacuum $\mid \mu >$:

As noted in [1] the squeezed vacuua

$$\mid \mu, e >= \exp \mu a^{\dagger 2} \mid 0> \quad ; \quad \mid \mu, o >= \exp \mu a^{\dagger 2} \mid 1> \quad , \tag{24}$$

in the even and odd sectors are respectively eigenstates of g_o and g_1.

$$g_o \mid \mu, e >= \mu \mid \mu, e> \quad ; \quad g_1 \mid \mu, o >= \mu \mid \mu, o> \quad , \tag{25}$$

Calculation of the overlap between the state $\mid \psi >$ and the squeezed vacuua would become rather easy if one could express the operators on the RHS of (22) and (23) entirely in terms of $g_i^\dagger$. The operators $\mathcal{G}_i^\dagger$ are related to $g_i^\dagger$ through (20). Further, using the fact that

$$G_o^\dagger = g_o^{\dagger 2}(n_a+1) \qquad ; \qquad G_1^\dagger = g_1^{\dagger 2}(n_a+2) \quad , \tag{26}$$

it can be shown that

$$\exp(-\beta G_o^\dagger) \mid 0 > \; = (1+4\beta g_o^{\dagger 2})^{-1/4} \mid 0 > \quad , \tag{27}$$

$$\exp(-\beta G_1^\dagger) \mid 1 > \; = (1+4\beta g_1^{\dagger 2})^{-3/4} \mid 1 > \quad . \tag{28}$$

The derivation of (27) entails expanding the exponential and

(a) using $[g_o^{\dagger 2}(n_a+1)]^q = (g_o^{\dagger 2})^q (n_a+1)(n_a+4)\cdots(n_a+4(q-3))$

(b) applying $(n_a+1)(n_a+4)\cdots(n_a+4(q-3))$ on $|0>$ and

(c) and summing up the resulting series using

$$(1+x)^{-\alpha} = \sum_{n=0}^{\infty} (-1)^n \frac{\Gamma(\alpha+n)}{\Gamma(\alpha)} \frac{x^n}{n!} \quad .$$

The derivation of (28) involves exactly the same steps.The states $\mid \psi, e >$ and $\mid \psi, o >$ can thus be written as

$$\mid \psi, e > \; = \exp\left(\frac{\lambda}{\sqrt{4\beta}} \tan^{-1}(\sqrt{4\beta} g_o^\dagger)\right)(1+4\beta g_o^{\dagger 2})^{-1/4} \mid 0 > \quad , \tag{29}$$

$$\mid \psi, o > \; = \exp\left(\frac{\lambda}{\sqrt{4\beta}} \tan^{-1}(\sqrt{4\beta} g_1^\dagger)\right)(1+4\beta g_1^{\dagger 2})^{-3/4} \mid 1 > \quad , \tag{30}$$

from which it follows that

$$< \mu, e \mid \psi, e > \; = \exp\left(\frac{\lambda}{\sqrt{4\beta}} \tan^{-1}(\sqrt{4\beta} \mu^*)\right)(1+4\beta \mu^{*2})^{-1/4} \quad , \tag{31}$$

$$< \mu, o \mid \psi, o > \; = \exp\left(\frac{\lambda}{\sqrt{4\beta}} \tan^{-1}(\sqrt{4\beta} \mu^*)\right)(1+4\beta \mu^{*2})^{-3/4} \quad . \tag{32}$$

These finite expressions for the ovelaps are valid for $\mid \mu\sqrt{\beta} \mid < 1/2$.

4. Overlap between the eigenstates of $\mathcal{F}_1$ and the coherent states $\mid \alpha >$:

The calculate the overlap between $\mid \psi >$ and the coherent states $\mid \alpha >$, it proves convenient to express the operators on the RHS of (29) and (30) in terms $a^\dagger$. Simple algebraic manipulations outlined in Appendix B, enable us to rewrite (29) and (30) as follows

$$| \psi, e> \ = \exp\left(-\frac{i}{2}\sqrt{\beta}a^{\dagger 2}\right) M\left(\frac{1}{4} - \frac{i\lambda}{4\sqrt{\beta}}, \frac{1}{2}, i\sqrt{\beta}a^{\dagger 2}\right) | 0> \quad , \tag{33}$$

$$| \psi, o> \ = \exp\left(-\frac{i}{2}\sqrt{\beta}a^{\dagger 2}\right) M\left(\frac{3}{4} - \frac{i\lambda}{4\sqrt{\beta}}, \frac{3}{2}, i\sqrt{\beta}a^{\dagger 2}\right) | 1> \quad . \tag{34}$$

With $| \psi, e>$ and $| \psi, o>$ written in this way, one immediately obtains

$$< \alpha | \psi, e> \ = \exp\left(-\frac{i}{2}\sqrt{\beta}\alpha^{*2}\right) M\left(\frac{1}{4} - \frac{i\lambda}{4\sqrt{\beta}}, \frac{1}{2}, i\sqrt{\beta}\alpha^{*2}\right) \exp(- | \alpha |^2 /2) \quad , \tag{35}$$

$$< \alpha | \psi, o> \ = \alpha^* \exp\left(-\frac{i}{2}\sqrt{\beta}\alpha^{*2}\right) M\left(\frac{3}{4} - \frac{i\lambda}{4\sqrt{\beta}}, \frac{3}{2}, i\sqrt{\beta}\alpha^{*2}\right) \exp(- | \alpha |^2 /2) \quad , \tag{36}$$

where $M(a, b, z)$ denote the confluent hypergeometric functions. From (35) and (36) one can readily calculate the Q-function for these states.

5. Overlap with number states:

We first consider $| \psi, e>$. Expanding the RHS of (33) in powers of $a^\dagger$ we obtain

$$| \psi, e>= \frac{\Gamma(\frac{1}{2})}{\Gamma(\frac{1}{4} - \frac{i\lambda}{4\sqrt{\beta}})} \sum_{k=0}^{\infty}\sum_{l=0}^{\infty} \frac{1}{k!}\left(-\frac{1}{2}\right)^k \frac{\Gamma(\frac{1}{4} - \frac{i\lambda}{4\sqrt{\beta}} + l)}{\Gamma(l + \frac{1}{2})l!} (i\sqrt{\beta})^{k+l}(a^\dagger)^{2(k+l)} | 0> \ , \tag{37}$$

which, in turn, yields

$$< 2n | \psi, e >= \frac{\Gamma(\frac{1}{2})}{\Gamma(\frac{1}{4} - \frac{i\lambda}{4\sqrt{\beta}})} \left(\frac{-i\sqrt{\beta}}{2}\right)^n \sqrt{2n!} \sum_{l=0}^{n} \frac{\Gamma(\frac{1}{4} - \frac{i\lambda}{4\sqrt{\beta}} + l)}{\Gamma(l + \frac{1}{2})l!(n - l)!} (-2)^l \ . \tag{38}$$

The expression on the RHS may be expressed in terms of the hypergeometric functions as follows

$$< 2n | \psi, e >= \left(\frac{-i\sqrt{\beta}}{2}\right)^n \frac{\sqrt{(2n)!}}{n!} F\left(-n, \frac{1}{4} - \frac{i\lambda}{4\sqrt{\beta}}; \frac{1}{2}; 2\right) \quad . \tag{39}$$

Similarly

$$< 2n + 1 | \psi, o >= \left(\frac{-i\sqrt{\beta}}{2}\right)^n \frac{\sqrt{(2n + 1)!}}{n!} F\left(-n, \frac{3}{4} - \frac{i\lambda}{4\sqrt{\beta}}; \frac{3}{2}; 2\right) \quad . \tag{40}$$

6. Coordinate space wave function:

Using (33) and (34) we can easily derived the expressions for the coordinate space wave functions for the states $| \psi, e>$ and $| \psi, o>$. Details are given in Appendix C. The (un normalized) wave functions turn out to be

$$< x \mid \psi, e > \; = \exp\left[-\frac{1}{2}\left(\frac{1+i\sqrt{\beta}}{1-i\sqrt{\beta}}\right)x^2\right] M\left(\frac{1}{4}-\frac{i\lambda}{4\sqrt{\beta}}, \frac{1}{2}, \frac{2i\sqrt{\beta}}{1+\beta}x^2\right) \quad , \tag{41}$$

$$< x \mid \psi, o > \; = x\exp\left[-\frac{1}{2}\left(\frac{1+i\sqrt{\beta}}{1-i\sqrt{\beta}}\right)x^2\right] M\left(\frac{3}{4}-\frac{i\lambda}{4\sqrt{\beta}}, \frac{3}{2}, \frac{2i\sqrt{\beta}}{1+\beta}x^2\right) \quad . \tag{42}$$

7. Eigenstates of the operator $\mathcal{F}_2$:

We next consider the eigenvalue problem for the operator $\mathcal{F}_2$.

$$\mathcal{F}_2 \mid \phi >= \lambda \mid \phi > \qquad ; \qquad \mathcal{F}_2 = (ab + \beta a^\dagger b^\dagger) \quad . \tag{43}$$

As before this task can be broken up into two steps - construction of the state $\mid \phi >_o$ annihilated by $\mathcal{F}_2$

$$\mathcal{F}_2 \mid \phi >_o = 0 \quad , \tag{44}$$

and the construction of the canonical conjugate $\mathcal{G}^\dagger$ of $\mathcal{F}_2$.

We rewrite (43) as

$$ab \mid \phi >_o = -\beta a^\dagger b^\dagger \mid \phi >_o \quad , \tag{45}$$

and apply ab on both sides. The resulting equation can be written as

$$F \mid \phi >_o = -\beta \mid \phi >_o \qquad , \qquad F = \frac{1}{(n_a+1)(n_b+1)}a^2 b^2 \quad . \tag{46}$$

Now the states annihilated by F are $\mid 0, p >$, $\mid q, 0 >$, $p \geq 0$, $q > 0$, and $\mid 1, q >$ and $\mid q, 1 >; p \geq 1, q > 1$. Successive applications of $F^\dagger$ on these generate the following sectors

$$S_{o,p} = \{\mid 2n, 2n+p >\}, p \geq 0 \qquad ; \qquad S_{q,o} = \{\mid 2n+q, 2n >\}, q > 0 \;;$$

$$S_{1,p} = \{\mid 2n, +1, 2n+p >\}, p \geq 1 \qquad ; \qquad S_{q,1} = \{\mid 2n+q, 2n+1 >\}, q > 1 \;,$$

where $n = 0, 1, \cdots$. As shown in Appendix A, the canonical conjugates $G_i^\dagger$ of F, satisfying

$$[F, G_i^\dagger] = 1 \quad , \tag{47}$$

in these sectors are found to be

$$G_o^\dagger = \frac{1}{2} a^{\dagger 2} b^{\dagger 2} \frac{(n_a+2)}{(n_a+2)(n_b+2)} \quad , \tag{48}$$

$$G_1^\dagger = \frac{1}{2} a^{\dagger 2} b^{\dagger 2} \frac{(n_b+2)}{(n_a+2)(n_b+2)} \quad , \tag{49}$$

$$G_2^\dagger = \frac{1}{2} a^{\dagger 2} b^{\dagger 2} \frac{(n_a+1)}{(n_a+2)(n_b+2)} \quad , \tag{50}$$

$$G_3^\dagger = \frac{1}{2} a^{\dagger 2} b^{\dagger 2} \frac{(n_b+1)}{(n_a+2)(n_b+2)} \quad . \tag{51}$$

The general solution of (46) may then be written as

$$\begin{aligned} \mid \phi >_o \; &= \sum_{p=o}^{\infty} C_{o,p} \exp(-\beta G_o^\dagger) \mid 0, p > + \sum_{q=1}^{\infty} C_{1,q} \exp(-\beta G_1^\dagger) \mid q, 0 > \\ &= \sum_{p=1}^{\infty} C_{2,p} \exp(-\beta G_2^\dagger) \mid 1, p > + \sum_{q=2}^{\infty} C_{3,q} \exp(-\beta G_3^\dagger) \mid q, 1 > \quad . \end{aligned} \tag{52}$$

The last two terms in (52) satisfy (46) but not (45) and should therefore be discarded. The general solution of (45) is thus

$$\mid \phi >_o = \sum_{p=o}^{\infty} C_{o,p} \exp(-\beta G_o^\dagger) \mid 0, p > + \sum_{q=1}^{\infty} C_{1,q} \exp(-\beta G_1^\dagger) \mid q, 0 > \quad . \tag{53}$$

The completes the first step towards the solution of the eigenvalue problem (53). Notice that $\mid \phi >_o$ is a linear combination of states in the sectors $S_{o,p}$ and $S_{q,o}$. To construct $\mathcal{G}^\dagger$, we notice that the canonical conjugates of ab in the sectors $S_{o,p}$ and $S_{q,o}$ are respectively given by

$$\mathrm{g}_o^\dagger = a^\dagger b^\dagger \frac{1}{(n_b+1)} \quad , \tag{54}$$

$$\mathrm{g}_1^\dagger = a^\dagger b^\dagger \frac{1}{(n_a+1)} \quad , \tag{55}$$

satisfying

$$[ab, \mathrm{g}_i^\dagger] = 1 \quad . \tag{56}$$

These canonical conjugates also satisfy

$$[a^\dagger b^\dagger, \mathrm{g}_i^\dagger] = \mathrm{g}_i^{\dagger 2} \quad . \tag{57}$$

As before, taking $\mathcal{G}_i^\dagger$ to be of the form

$$\mathcal{G}_i^\dagger = \sum_{n=1}^{\infty} b_n (g_i^\dagger)^n \quad , \tag{58}$$

and requiring that

$$[ab + \beta a^\dagger b^\dagger, \mathcal{G}_i^\dagger] = 1 \quad . \tag{59}$$

we find that

$$b_{2m} = 0 \qquad , \qquad b_{2m+1} = \frac{(-\beta)^m}{(2m+1)} \quad , \tag{60}$$

so that

$$\mathcal{G}_i^\dagger = \frac{1}{\sqrt{\beta}} \tan^{-1}(\sqrt{\beta} g_i^\dagger) \quad . \tag{61}$$

The general solution of the eigenvalue problem (43) is then given by

$$| \phi > = \sum_{p=o}^{\infty} C_{o,p} \mid \phi; 0, p > + \sum_{q=1}^{\infty} C_{1,q} \mid \phi; q, 0 > \quad , \tag{62}$$

where

$$| \phi; 0, p > \ \equiv \exp(\lambda \mathcal{G}_o^\dagger) \exp(-\beta G_o^\dagger) \mid 0, p > \quad , \tag{63}$$

$$| \phi; q, 0 > \ \equiv \exp(\lambda \mathcal{G}_1^\dagger) \exp(-\beta G_1^\dagger) \mid q, 0 > \quad . \tag{64}$$

8. Overlap between the eigenstates of $\mathcal{F}_2$ and the generalised Caves Schumaker States:

It was noted in ref 1. that the (un normalized) states

$$| \mu; 0, p > \ = \exp \mu a^\dagger b^\dagger \mid 0, p > \ ; \quad p = 0, 1, \cdots, \tag{65}$$

$$| \mu; q, 0 > \ = \exp \mu a^\dagger b^\dagger \mid q, 0 > \ ; \quad q = 1, 2 \cdots. \tag{66}$$

(hereafter referred to as the generalised Caves-Schumaker states) are eigenstates of $g_o^\dagger$ and $g_1^\dagger$ respectively. These states, which contain the Caves-Schumaker state $\exp \mu a^\dagger b^\dagger \mid 0, 0 >$, are the counterparts of the states $| \mu, e >$ and $| \mu; o >$ of section 3 and have been studied in detail in ref 12. To express the overlap of these with the states $| \phi; 0, p >$ and $| \phi; q, 0 >$, we expresses the latter in terms $g_o^\dagger$ and $g_1^\dagger$ as follows

$$| \phi;0,p> \ = \exp\left(\frac{\lambda}{\sqrt{\beta}}\tan^{-1}(\sqrt{\beta}g_o^\dagger)\right)(1+\beta g_o^{\dagger 2})^{-(p+1)/2} \ | \ 0,p> \ , \tag{67}$$

$$| \phi;q,0> \ = \exp\left(\frac{\lambda}{\sqrt{\beta}}\tan^{-1}(\sqrt{\beta}g_1^\dagger)\right)(1+\beta g_1^{\dagger 2})^{-(q+1)/2} \ | \ q,0> \ . \tag{68}$$

The steps involved in deriving (67) and (68) are exactly the same as those used in obtaining (27) and (28). From the expressions one readily obtains

$$< \mu;0,p|\phi;0,p> \ = \exp\left(\frac{\lambda}{\sqrt{\beta}}\tan^{-1}(\sqrt{\beta}\mu^*)\right)(1+\beta\mu^{*2})^{-(p+1)/2} \ , \tag{69}$$

$$< \mu;q,0|\phi;q,0> \ = \exp\left(\frac{\lambda}{\sqrt{\beta}}\tan^{-1}(\sqrt{\beta}\mu^*)\right)(1+\beta\mu^{*2})^{-(q+1)/2} \ . \tag{70}$$

These finite expressions for the ovelaps are valid for $|\mu\sqrt{\beta}| < 1$

9. Overlap between the eigenstates of $\mathcal{F}_2$ and the coherent states $|\gamma,\delta>$: the Q functions

Using the results given in Appendix B, one finds that $|\phi,0,p>$ and $|\phi;q,0>$ can be expressed in terms of the operators $a^\dagger$ and $b^\dagger$ as follows

$$|\phi;0,p> \ = \exp(-i\sqrt{\beta}a^\dagger b^\dagger)M\left(\frac{p+1}{2}-\frac{i\lambda}{2\sqrt{\beta}}\,,\,p+1\,,\,2i\sqrt{\beta}a^\dagger b^\dagger\right)|0,p> \ , \tag{71}$$

$$|\phi;q,0> \ = \exp(-i\sqrt{\beta}a^\dagger b^\dagger)M\left(\frac{q+1}{2}-\frac{i\lambda}{2\sqrt{\beta}}\,,\,q+1\,,\,2i\sqrt{\beta}a^\dagger b^\dagger\right)|q,0> \ . \tag{72}$$

The overlaps of these states with the two mode coherent states $|\gamma,\delta>$ are therefore given by

$$\begin{aligned} < \gamma,\delta|\phi;0,p> \ &= \exp(-i\sqrt{\beta}\gamma^*\delta^*)\exp(-(|\gamma|^2+|\delta|^2)/2) \\ &\times M\left(\frac{p+1}{2}-\frac{i\lambda}{2\sqrt{\beta}}\ ,\ p+1\ ,\ 2i\sqrt{\beta}\gamma^*\delta^*\right) \ , \end{aligned} \tag{73}$$

$$\begin{aligned} < \gamma,\delta|\phi;q,0> \ &= \exp(-i\sqrt{\beta}\gamma^*\delta^*)\exp(-(|\gamma|^2+|\delta|^2)/2) \\ &\times M\left(\frac{q+1}{2}-\frac{i\lambda}{2\sqrt{\beta}}\ ,\ q+1\ ,\ 2i\sqrt{\beta}\gamma^*\delta^*\right) \ . \end{aligned} \tag{74}$$

From these expressions one can readily calculate the corresponding Q-functions.

10. Overlap with number states:

We first consider $| \ \phi;0,p>$. Expanding the RHS of (71) in powers of $a^\dagger b^\dagger$ we obtain

$$| \phi; 0, p >= \frac{\Gamma(p+1)}{\Gamma(\frac{p+1}{2} - \frac{i\lambda}{2\sqrt{\beta}})} \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \frac{1}{k!} (-1)^k (2)^l \frac{\Gamma(\frac{p+1}{2} - \frac{i\lambda}{2\sqrt{\beta}} + l)}{\Gamma(l+p+1) l!} (i\sqrt{\beta})^{k+l} (a^\dagger b^\dagger)^{(k+l)} \mid 0, p > \ , \tag{75}$$

which, in turn, yields

$$< n, n+p \mid \phi; 0, p >= \frac{\Gamma(p+1)}{\Gamma(\frac{p+1}{2} - \frac{i\lambda}{2\sqrt{\beta}})} (-i\sqrt{\beta})^n \sqrt{n!(n+p)!} \sum_{l=0}^{n} \frac{\Gamma(\frac{p+1}{2} - \frac{i\lambda}{2\sqrt{\beta}} + l)}{\Gamma(l+p+1) l! (n-l)!} (-2)^l \ . \tag{76}$$

The expression on the RHS may be expressed in terms of the hypergeometric functions as follows

$$< n, n+p \mid \phi; 0, p >= (-i\sqrt{\beta})^n \frac{\sqrt{n!(n+p)!}}{n!} F\left(-n, \frac{p+1}{2} - \frac{i\lambda}{2\sqrt{\beta}}; p+1; 2\right) \ . \tag{77}$$

Similarly

$$< n+q, n \mid \phi; q, 0 >= (-i\sqrt{\beta})^n \frac{\sqrt{n!(n+q)!}}{n!} F\left(-n, \frac{q+1}{2} - \frac{i\lambda}{2\sqrt{\beta}}; q+1; 2\right) \ . \tag{78}$$

11. Concluding Remarks

To conclude, we have given a purely algebraic method for constructing the eigenstates of the operators $\mathcal{F}_1 = (a^2 + \beta a^{\dagger 2})$ and $\mathcal{F}_2 = (ab + \beta a^\dagger b^\dagger)$. The operator $\mathcal{F}_2$ on making the canonical transformation

$$a = (c + id)/\sqrt{2} \ , \ b = (c - id)/\sqrt{2} \ ; \ a^\dagger = (c^\dagger + id^\dagger)/\sqrt{2} \ , \ b^\dagger = (c^\dagger - id^\dagger)/\sqrt{2} \ , \tag{79}$$

can be written as

$$\mathcal{F}_2 = \frac{1}{2}(c^2 + d^2) + \beta \frac{1}{2}(c^{\dagger 2} + d^{\dagger 2}) \ , \tag{80}$$

and is therefore the sum of two operators of $\mathcal{F}_1$ type. Thus in constructing the eigenstates of $\mathcal{F}_2$ given by (43) one has also constructed the eigenstates of the operator $\mathcal{F}_2$ given by (80). The construction presented here directly expresses the eigenstates of these operators in the exponential form i.e., as states obtained by applying exponentials of certain operators on the appropriate “vacuua”. As is well known, the Yuen states and the Caves-Schumaker states are eigenstates of linear combinations of creation and annihilation operators. The states constructed here may be considered as natural generalisations of these in the sense that they are eigenstates of operators which involve linear combinations of squares or products creation and annihilation operators and, like the Yuen and the Caves-Shumaker states, may find useful applications in quantum optics.

Appendix - A

The canonical conjugates $G_i^\dagger$ of any single mode annihilation operator of the form $f(n_a)a^p$ where $f(x)$ has no zeros at integer values of x (including zero) are given by the formula[1]

$$G_i^\dagger = \frac{1}{p} F^\dagger \frac{1}{FF^\dagger}(n_a + p - i) \ ; \ i = 0, \cdots, p-1 \ . \tag{A.1}$$

Thus,for instance, for the operator F in (4) given by

$$F = \frac{1}{(n_a+1)(n_a+2)} a^4 \ , \tag{A.2}$$

one has

$$FF^\dagger = \frac{(n_a+4)(n_a+3)}{(n_a+1)(n_a+2)} \ , \tag{A.3}$$

and hence

$$G_i^\dagger = \frac{1}{4} a^{\dagger 4} \frac{(n_a + 4 - i)}{(n_a+4)(n_a+3)} \ . \tag{A.4}$$

Setting $i = 0, 1, 2, 3$ one obtains the expressions in (6)-(9). Similarly, for $F = a^2$, one obtains (12) and (13).

Consider now a two mode annihilation operator consisting of products of single mode annihilation operators of the above type.

$$F = F_1(a) F_2(b) \ , \tag{A.5}$$

where

$$F_1(a) = f_1(n_a) a^k \ ; \ F_2(b) = f_2(n_b) b^l \ . \tag{A.6}$$

The vacuua of F are $|i, p>, \ ; i = 0, \cdots, k-1$ and $|q, j>, \ ; j = 0, \cdots, l-1$. The canonical conjugates of F in the sectors built on $|i, p>$ are given by

$$G_i^\dagger = \left[\frac{1}{k} F_1^\dagger \frac{1}{F_1 F_1^\dagger}(n_a + k - i)\right] \left[F_2^\dagger \frac{1}{F_2 F_2^\dagger}\right] \ . \tag{A.7}$$

Similarly, the canonical conjugates of F in the sectors built on $|q, j> , j = 0, \cdots, l-1$ are given by

$$G_j^\dagger = \left[\frac{1}{l} F_2^\dagger \frac{1}{F_2 F_2^\dagger}(n_b + l - j)\right] \left[F_1^\dagger \frac{1}{F_1 F_1^\dagger}\right] \ . \tag{A.8}$$

Thus, for instance, for the operator F in (46) one has

$$F_1 = \frac{1}{n_a + 1} a^2 \;\; ; \;\; F_2 = \frac{1}{n_b + 1} b^2 \;\; , \qquad (A.9)$$

for which

$$F_1 F_1^\dagger = \frac{(n_a + 2)}{(n_a + 1)} \;\; ; \;\; F_2 F_2^\dagger = \frac{(n_b + 2)}{(n_b + 1)} \;\; , \qquad (A.10)$$

and hence, for the sectors built on $|i, p>$

$$G_i^\dagger = \frac{1}{2} a^{\dagger 2} \frac{1}{(n_a + 2)} (n_a + 2 - i) b^{\dagger 2} \frac{1}{(n_b + 2)} \;\; . \qquad (A.11)$$

Setting $i = 0, 1$, one obtains (48) and (49). Similarly (A.8) yields (50) and (51). The same considerations as above, applied to the operator $F = ab$, yield (50) and (55).

Appendix - B

In this Appendix we show that the expressions for states $|\psi; e>$ and $|\psi, o>$ the given by (29) and (30) which involve the exponential of an inverse tan function respectively can be rewritten as in (33) and (34). Similarly, we show that the states $|\phi; 0, p>$ and $|\phi; q, 0>$ in (67) and (68) can be rewritten as in (71) and (72).

Consider, for instance, $|\phi; 0, p>$

$$|\phi; 0, p >= \exp\left(\frac{\lambda}{\sqrt{\beta}} \tan^{-1}(\sqrt{\beta} g_o^\dagger)\right)(1 + \beta g_o^{\dagger 2})^{-(p+1)/2} |0, p> \quad . \qquad (B.1)$$

Defining

$$z \equiv \lambda / \sqrt{\beta} \qquad \text{and} \qquad x \equiv \sqrt{\beta} g_0^\dagger \;\; , \qquad (B.2)$$

(B.1) can be written as

$$|\phi; 0, p >= \exp(z \tan^{-1} x)(1 + x^2)^{-(p+1)/2} |0, p> \quad . \qquad (B.3)$$

Using the identity

$$\tan^{-1} x = \frac{1}{2i} \ln\left(\frac{1 + ix}{1 - ix}\right) \;\; ,$$

one finds that

$$\exp(z \tan^{-1} x)(1 + x^2)^{-(p+1)/2} = (1 + ix)^{-(p+1)} \left(1 - \frac{2ix}{1 + ix}\right)^{(iz - (p+1))/2} \quad . \qquad (B.4)$$

Using

$$(1+x)^{-\alpha} = \sum_{n=0}^{\infty} \frac{\Gamma(\alpha+n)}{\Gamma(\alpha)} \frac{(-x)^n}{n!} \quad , \tag{B.5}$$

the RHS of (B.4) can be expanded in powers of x as

$$\begin{aligned} &\exp(z \tan^{-1} x)(1+x^2)^{-(p+1)/2} = \\ &\sum_{q=0}^{\infty}\sum_{k=0}^{\infty} \frac{(2i)^q(-i)^k \Gamma(q+(p+1-iz)/2)\Gamma(p+q+k+1)}{\Gamma((p+1-iz)/2)\Gamma(p+q+1)q!k!} x^{k+q} \quad . \end{aligned} \tag{B.6}$$

The action of x^{k+q} on $|0,p>$ yields

$$\begin{aligned} x^{k+q}|0,q> &= (\sqrt{\beta}a^\dagger)^{k+q}\left(b^\dagger \frac{1}{n_b+1}\right)^{k+q}|0,p> \\ &= \frac{\Gamma(p+1)}{\Gamma(k+q+p+1)}(\sqrt{\beta}a^\dagger b^\dagger)^{k+q}|0,p> \quad . \end{aligned} \tag{B.7}$$

Using (B.6) and (B.7) in (B.1) one obtains

$$\begin{aligned} |\phi;0,p> &= \left[\sum_{k=0}^{\infty} \frac{(-i\sqrt{\beta}a^\dagger b^\dagger)^k}{k!}\right] \\ &\times \left[\frac{\Gamma(p+1)}{\Gamma((p+1-iz)/2)} \sum_{q=0}^{\infty} \frac{\Gamma(q+(p+1-iz)/2)}{\Gamma(p+q+1)q!}(2i\sqrt{\beta}a^\dagger b^\dagger)^q\right]|0,q> , \qquad (B.8) \\ &= \exp(-i\sqrt{\beta}a^\dagger b^\dagger) M\left(\frac{p+1}{2} - \frac{i\lambda}{2\sqrt{\beta}} \ , \ p+1 \ , \ 2i\sqrt{\beta}a^\dagger b^\dagger\right)|0,p> , \qquad (B.9) \end{aligned}$$

which is the same as (67).

Following the same procedure one can derive (33), (34) and (72) from (29), (30) and (68).

Appendix - C

In this Appendix we derive the expressions in (41) and (42) for the coordinate space wave functions for $|\psi,e>$ and $|\psi,o>$. We first consider $|\psi;e>$. From (33) we have

$$\begin{aligned} &< x|\psi,e> \\ &= < x|\exp\left(-\frac{i}{2}\sqrt{\beta}a^{\dagger 2}\right) M\left(\frac{1}{4} - \frac{i\lambda}{4\sqrt{\beta}} \ , \ \frac{1}{2} \ , \ i\sqrt{\beta}a^{\dagger 2}\right)|0> \ , \\ &= \frac{\Gamma(\frac{1}{2})}{\Gamma(\frac{1}{4}-\frac{i\lambda}{4\sqrt{\beta}})} \sum_{l=0}^{\infty} \frac{\Gamma(\frac{1}{4}-\frac{i\lambda}{4\sqrt{\beta}}+l)}{\Gamma(l+\frac{1}{2})l!}(-1)^l < x|(-i\sqrt{\beta}a^{\dagger 2})^l \exp\left(-\frac{1}{2}\sqrt{\beta}a^{\dagger 2}\right)|0> \ , \end{aligned}$$

$$= \frac{\Gamma(\frac{1}{2})}{\Gamma(\frac{1}{4} - \frac{i\lambda}{4\sqrt{\beta}})} \sum_{l=0}^{\infty} \frac{\Gamma(\frac{1}{4} - \frac{i\lambda}{4\sqrt{\beta}} + l)}{\Gamma(l + \frac{1}{2}) l!} (-1)^l \left[\frac{\partial^l}{\partial \mu^l} < x | \exp(-i\sqrt{\beta}\mu a^{\dagger 2}) | 0 >< \right]_{\mu=1/2} . \qquad (C.1)$$

To proceed further, we need to know $< x|\exp(-i\sqrt{\beta}\mu a^{\dagger 2})|0 >$. This can be done by

(a) expanding the exponential

(b) using the fact that

$$< x | a^{\dagger 2m} | 0 >= \frac{1}{[\Gamma(\frac{1}{2})]^{\frac{1}{2}}} \exp\left(-\frac{x^2}{2}\right) (-2)^m m! L_m^{-\frac{1}{2}}(x^2) \ , \qquad (C.2)$$

(c) and recognizing the series thus obtained as the generating function of the associated Laguerre functions. This leads to

$$< x | \exp(-i\sqrt{\beta}\mu a^{\dagger 2}) | 0 >= \frac{1}{\sqrt{\Gamma(\frac{1}{2})(1 - 2i\mu\sqrt{\beta})}} \exp\left(-\frac{x^2}{2}\right) \exp\left(\frac{2i\mu\sqrt{\beta}x^2}{2i\mu\sqrt{\beta} - 1}\right) . \qquad (C.3)$$

We rewrite this expression as

$$\begin{aligned} < x | \exp(-i\sqrt{\beta}\mu a^{\dagger 2}) | 0 > &= \frac{1}{\sqrt{\Gamma(\frac{1}{2})(1 - 2i\mu\sqrt{\beta})}} \exp\left(-\frac{1}{2}\frac{(1 + i\sqrt{\beta})}{(1 - i\sqrt{\beta})} x^2\right) \\ &\times \exp\left(\frac{i\sqrt{\beta}(1 - 2\mu)x^2}{(i\sqrt{\beta} - 1)(2i\mu\sqrt{\beta} - 1)}\right) , \end{aligned} \qquad (C.4)$$

and define

$$z = \left(\frac{2i\sqrt{\beta}}{1 - i\sqrt{\beta}}\right)\left(\mu - \frac{1}{2}\right) \ , \qquad (C.5)$$

to obtain

$$\begin{aligned} &\left[\frac{\partial^l}{\partial \mu^l} < x | \exp(-i\sqrt{\beta}\mu a^{\dagger 2}) | 0 >\right]_{\mu=1/2} \\ &= \frac{1}{\sqrt{\Gamma(\frac{1}{2})(1 - i\sqrt{\beta})}} \exp\left(-\frac{1}{2}\frac{(1 + i\sqrt{\beta})}{(1 - i\sqrt{\beta})} x^2\right) \left(\frac{2i\sqrt{\beta}}{1 - i\sqrt{\beta}}\right)^l \left[\frac{\partial^l}{\partial z^l} \frac{1}{\sqrt{(1 - z)}} \right. \\ &\times \left. \exp\left(\frac{z}{z - 1}\left(\frac{x^2}{1 - i\sqrt{\beta}}\right)\right)\right]_{z=0} , \\ &= \frac{1}{\sqrt{\Gamma(\frac{1}{2})(1 - i\sqrt{\beta})}} \exp\left(-\frac{1}{2}\frac{(1 + i\sqrt{\beta})}{(1 - i\sqrt{\beta})} x^2\right) \left(\frac{2i\sqrt{\beta}}{1 - i\sqrt{\beta}}\right)^l l! L_l^{-\frac{1}{2}}\left(\frac{x^2}{1 - i\sqrt{\beta}}\right) . \end{aligned} \qquad (C.6)$$

Substituting this in (C.1) we get

$$
\begin{aligned}
< x|\psi, e > \; &= \frac{\sqrt{\Gamma(\frac{1}{2})}}{\Gamma(\frac{1}{4} - \frac{i\lambda}{4\sqrt{\beta}})} \frac{1}{\sqrt{(1-i\sqrt{\beta})}} \exp\left(-\frac{1}{2}\frac{(1+i\sqrt{\beta})}{(1-i\sqrt{\beta})}x^2\right) \\
&\times \sum_{l=0}^{\infty} \frac{\Gamma(\frac{1}{4} - \frac{i\lambda}{4\sqrt{\beta}} + l)}{\Gamma(l+\frac{1}{2})} \left(\frac{-2i\sqrt{\beta}}{1-i\sqrt{\beta}}\right)^l L_l{}^{-\frac{1}{2}} \left(\frac{x^2}{1-i\sqrt{\beta}}\right) \quad . \qquad (C.7)
\end{aligned}
$$

Putting the expansion

$$
L_l{}^{-\frac{1}{2}} \left(\frac{x^2}{1-i\sqrt{\beta}}\right) = \sum_{m=0}^{l} \frac{\Gamma(l+\frac{1}{2})}{\Gamma(m+\frac{1}{2})\Gamma(l-m+1)m!} \left(\frac{-x^2}{1-i\sqrt{\beta}}\right)^m \quad , \qquad (C.8)
$$

for the Laguerre polynomials in (C.8) and rearranging the double series we get

$$
\begin{aligned}
< x|\psi, e > \; &= \frac{\sqrt{\Gamma(\frac{1}{2})}}{\Gamma(\frac{1}{4} - \frac{i\lambda}{4\sqrt{\beta}})} \frac{1}{\sqrt{(1-i\sqrt{\beta})}} \exp\left(-\frac{1}{2}\frac{(1+i\sqrt{\beta})}{(1-i\sqrt{\beta})}x^2\right) \\
&\times \sum_{m=0}^{\infty} \frac{1}{\Gamma((m+\frac{1}{2})m!} \left(\frac{2i\sqrt{\beta}x^2}{(1-i\sqrt{\beta})^2}\right)^m \\
&\times \sum_{k=0}^{\infty} \frac{\Gamma(\frac{1}{4} - \frac{i\lambda}{4\sqrt{\beta}} + m + k)}{k!} \left(\frac{-2i\sqrt{\beta}}{1-i\sqrt{\beta}}\right)^k \quad . \qquad (C.9)
\end{aligned}
$$

The sum over k is easily carried out using (B.5) and gives

$$
\begin{aligned}
&\sum_{k=0}^{\infty} \frac{\Gamma(\frac{1}{4} - \frac{i\lambda}{4\sqrt{\beta}} + m + k)}{k!} \left(\frac{-2i\sqrt{\beta}}{1-i\sqrt{\beta}}\right)^k \\
&\qquad = \Gamma(\frac{1}{4} - \frac{i\lambda}{4\sqrt{\beta}} + m) \left(\frac{1-i\sqrt{\beta}}{1+i\sqrt{\beta}}\right)^{(\frac{1}{4} - \frac{i\lambda}{4\sqrt{\beta}} + m)} \quad . \qquad (C.10)
\end{aligned}
$$

Substituting this in (C.9) and recognizing the infinite series as that for a confluent hypergeometric function, we finally obtain

$$
< x|\psi, e >= \exp\left(-\frac{1}{2}\frac{(1+i\sqrt{\beta})}{(1-i\sqrt{\beta})}x^2\right) M\left(\frac{1}{4} - \frac{i\lambda}{4\sqrt{\beta}}, \frac{1}{2}, \frac{2i\sqrt{\beta}x^2}{1+\beta}\right) \quad , \qquad (C.11)
$$

where we have omitted factors independent of x. Proceedings in exactly the same way one can derive the expression (42) for $< x \mid \psi; o >$.

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