A non group theoretic proof of completeness of arbitrary coherent states $D(\alpha) \mid f >$

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Abstract

A new proof for the completeness of the coherent states $D(\alpha) \mid f >$ for the Heisenberg Weyl group and the groups $SU(2)$ and $SU(1,1)$ is presented. Generalizations of these results and their consequences are disussed.

Introduction

Resolution of the identity operator in terms of the eigenstates of suitable operators proves to be an important calculational tool in quantum mechanics. One comes across numerous instances where quantum mechanical calculations are greatly simplified by a judicious use of the resolution of the identity in terms of the eigenstates of appropriate operators. Among the various resolutions of the identity, the one which has played a key role in quantum optics is that in terms of the coherent states $\mid \alpha >$ [1-3], the eigenstates of the annihilation operator

$$\frac{1}{\pi}\int d^2\alpha \mid \alpha >< \alpha \mid = \mathbf{I} \quad , \tag{1}$$

where

$$\mid \alpha >= D(\alpha) \mid 0 > \quad ; \quad D(\alpha) = \exp(\alpha a^\dagger - \alpha^* a) \quad ; \quad [a, a^\dagger] = \mathbf{I} \ . \tag{2}$$

The coherent states $\mid \alpha >$ together with (1) have not only led to new calculational techniques but also led to new conceptual developments such as the notion of quasi probability distributions.

The proof of (1) found in most text books on quantum optics and quantum mechanics proceeds by expanding $\mid \alpha >$ in terms of Fock states and carrying out the α-integration and by using the completeness of Fock states. In recent times states like $D(\alpha) \mid n >$, the displaced number states [4-6], have been used in quantum optics and it is known that these also form a complete set for each n [5]. In fact, from a group theoretic point of view [7,8] one has a more general result

$$\frac{1}{\pi}\int d^2\alpha D(\alpha) \mid f >< f \mid D^\dagger(\alpha) = \mathbf{I} \quad , \tag{3}$$

where $\mid f >$, referred to as the fiducial state, is any fixed normalizable state. (In (3) it has been assumed that $\mid f >$ is normalized to unity.) The states

$$\mid \alpha; f >= D(\alpha) \mid f > \quad , \tag{4}$$

are referred to as generalized coherent states. (To avoid confusion with other notions of generlized coherent states, we would, hereafter, refer to them as f-coherent states.) The choice $\mid f >=\mid n >$ in (3), for instance, leads to the resolution of the identity in terms of the displaced number states. The group theoretical proof of (3), using Schur's Lemma, is based on the following observations

(a) $D(\beta)$ provide an irreducible representation (upto a phase) of the Heisenberg Weyl group.

(b) the operator

$$X_1(f) \equiv \frac{1}{\pi} \int d^2\alpha D(\alpha) \mid f >< f \mid D^\dagger(\alpha) \quad , \tag{5}$$

commutes with the $D(\beta)$'s and hence, by Schur's Lemma, is proportional to the identity operator

$$X_1(f) = c(f)\mathbf{I} \quad , \tag{6}$$

(c) the constant $c(f)$ can be calculated by taking the matrix element of $X_1(f)$ between any normalizable state. (For consistency, $c(f)$ should be $< \infty$ which, for coherent states for certain groups leads to restrictions on the fiducial states.) For the Heisenberg-Weyl group, it is easy to show that for any fiducial state $\mid f >; < f \mid f >= 1$, $c(f) = 1$ and hence one has (3). By expanding $\mid f >$ in terms of Fock states (3) may equivalently be written as

$$\frac{1}{\pi} \int d^2\alpha \mid \alpha; n >< \alpha; m \mid = \mathbf{I}\delta_{nm} \quad ; \quad \mid \alpha; n >\equiv D(\alpha) \mid n > \quad . \tag{7}$$

The considerations given above apply to other groups like $SU(2)$ and $SU(1,1)$ as well [7,8]. For the case of $SU(2)$

$$[S_+, S_-] = 2S_z \quad ; \quad [S_z, S_\pm] = \pm S_\pm \quad , \tag{8}$$

one has

$$X_2(m) \equiv \frac{2S+1}{4\pi} \int \frac{d^2\zeta}{(1+ \mid \zeta \mid^2)^2} \quad \mid \zeta; m >< \zeta; m \mid = \mathbf{I} \quad , \tag{9}$$

where

$$\mid \zeta; m > \equiv D(\xi) \mid S, m > \quad ; \quad D(\xi) = \exp(\xi S_+ - \xi^* S_-) \quad , \tag{10}$$

and $\mid S, m >$ are eigenstates of S^2 and S_z. The variables ζ and ξ are related to each other as follows

$$\xi = \frac{\theta}{2} e^{-i\phi} \quad ; \quad \zeta = \tan \frac{\theta}{2} e^{-i\phi} \quad , \tag{11}$$

and the integration in (9) is over the entire ζ-plane.

Similarly, for $SU(1,1)$

$$[K_-, K_+] = 2K_z \quad ; \quad [K_z, K_\pm] = \pm K_\pm \quad , \tag{12}$$

realized via

$$K_+ = \frac{1}{2} a^{\dagger 2} \quad ; \quad K_- = \frac{1}{2} a^2 \quad ; \quad K_z = \frac{1}{2}(a^\dagger a + \frac{1}{2}) \quad , \tag{13}$$

one has

$$X_3(n) \equiv \frac{1}{2\pi} \int \frac{d^2\zeta}{(1- \mid \zeta \mid^2)^2} \mid \zeta; 2n+1 >< \zeta; 2n+1 \mid = \mathbf{I}_{odd} \quad , \tag{14}$$

where

$$\mid \zeta; 2n+1 > \equiv D(\xi) \mid 2n+1 > \; ; \; D(\xi) = \exp(\xi K_+ - \xi^* K_-) \; ; \; K_z \mid 2n+1 >= (n + \frac{3}{4}) \mid 2n+1 > \quad , \tag{15}$$

and ζ and ξ are related to each other as follows

$$\xi = \mid \xi \mid e^{-i\phi} \quad ; \quad \zeta = \tanh \mid \xi \mid e^{-i\phi} \quad . \tag{16}$$

The operator $\mathbf{I}_{odd}$ in (14) denotes the unit operator in the odd sector of the Fock space.

$$\mathbf{I}_{odd} \equiv \sum_{k=0}^{\infty} \mid 2k+1 >< 2k+1 \mid \quad , \tag{17}$$

and the integration in (14) is over the unit disc centered at the origin in the complex ζ-plane.

New proof of completeness of f-coherent states

We first consider (3). To prove (3) in a rather elegant way we make use of the following results:

(i) resolution of the identity (1) in terms of coherent states.

(ii) the fact that an operator is uniquely determined by its diagonal elements [9].

$$<\beta \mid G \mid \beta>= 1 \text{ for all } \beta \quad \text{if and only if} \quad G = \mathbf{I} \quad . \tag{18}$$

Now consider the operator $X_1(f)$

$$X_1(f) \equiv \frac{1}{\pi} \int d^2\alpha D(\alpha) \mid f >< f \mid D^\dagger(\alpha) \quad . \tag{19}$$

Consider the diagonal elements of $X_1(f)$

$$\begin{aligned} <\beta \mid X_1(f) \mid \beta> &= \frac{1}{\pi} \int d^2\alpha <\beta \mid D(\alpha) \mid f >< f \mid D^\dagger(\alpha) \mid \beta> \quad , \\ &= \frac{1}{\pi} \int d^2\alpha < 0 \mid D^\dagger(\beta) D(\alpha) \mid f >< f \mid D^\dagger(\alpha) D(\beta) \mid 0 > \quad , \end{aligned} \tag{20}$$

which on using the algebraic property of the displacement operator $D(\alpha)$

$$D^\dagger(\beta) D(\alpha) = D(\alpha - \beta) \exp[(\beta^* \alpha - \beta \alpha^*)/2] \quad , \tag{21}$$

reduces to

$$<\beta \mid X_1(f) \mid \beta>= \frac{1}{\pi} \int d^2\alpha \, |< 0 \mid D^\dagger(\beta - \alpha) \mid f >|^2 \quad . \tag{22}$$

On rewriting the integrand (22) in terms of coherent states and changing the variable of integration (22) becomes

$$\begin{aligned} <\beta \mid X_1(f) \mid \beta> &= \frac{1}{\pi} \int d^2\alpha \, |<\beta - \alpha \mid f >|^2 \\ &= \frac{1}{\pi} \int d^2\alpha < f \mid \alpha >< \alpha \mid f > \\ &= < f \mid \frac{1}{\pi} \int d^2\alpha \mid \alpha >< \alpha \mid f >= 1 \quad . \end{aligned} \tag{23}$$

Thus the diagonal coherent elements of $X_1(f)$ for all values of β are equal to unity and therefore using the property (18) we conclude that

$$X_1(f) = \mathbf{I} \quad . \tag{24}$$

This constitutes a direct proof of the completeness of the f-coherent states of the Heisenberg-Weyl group.

Next we consider the $SU(2)$ case. In this the analogues of (i) and (ii) above are

(i) completeness of the atomic coherent states $\mid \zeta; -S >$ [10]

$$\frac{2S+1}{4\pi} \int \frac{d^2\zeta}{(1+\mid \zeta \mid^2)^2} \ \mid \zeta; -S >< \zeta; -S \mid = \mathbf{I} \quad , \tag{25}$$

(ii) $$< \zeta; -S \mid G \mid \zeta; -S > = 1 \text{ for all } \zeta \text{ if an only if } G = \mathbf{I}. \tag{26}$$

We consider the diagonal matrix elements of $X_2(m)$ defined in (9) between the atomic coherent states $\mid \zeta'; -S >$. We follow the same procedure as above and use the following algebraic properties.

$$D(\xi_1)D(\xi_2) = D(\xi_3) \exp[i\Phi(\xi_1, \xi_2)S_z] \quad , \tag{27}$$

where

$$\Phi(\xi_1, \xi_2) = \frac{1}{i} \ln \left[\frac{1 - \zeta_1\zeta_2^*}{1 - \zeta_1^*\zeta_2} \right] \quad , \tag{28}$$

and

$$\zeta_3 = \frac{\zeta_1 + \zeta_2}{1 - \zeta_1^*\zeta_2} \quad . \tag{29}$$

Further, under the change of variables from ζ_2 to ζ_3 the measure of integration in (9) is invariant

$$\frac{d^2\zeta_2}{(1+\mid \zeta_2 \mid^2)^2} = \frac{d^2\zeta_3}{(1+\mid \zeta_3 \mid^2)^2} \quad . \tag{30}$$

Using these relations we obtain

$$< \zeta'; -S \mid X_2 \mid \zeta'; -S > = \frac{2S+1}{4\pi} \int \frac{d^2\zeta''}{(1+\mid \zeta'' \mid^2)^2} < S, N \mid \zeta''; -S >< \zeta''; -S \mid S, N > \quad , \tag{31}$$

which, on using the completeness of the atomic coherent states yields

$$< \zeta'; -S \mid X_2(m) \mid \zeta'; -S >= 1 \quad \text{for all} \quad \zeta' \quad , \tag{32}$$

and hence $X_2(m) = \mathbf{I}$. It is important to note that the fiducial state in this case must be an eigenstate of S_z otherwise the phase factor which arises from the use of (27) will not cancel.

Similarly, in the $SU(1,1)$ case, we use the following algebraic properties.

$$D(\xi_1)D(\xi_2) = D(\xi_3)\exp[i\Phi(\xi_1,\xi_2)K_z] \quad , \tag{33}$$

where

$$\Phi(\xi_1,\xi_2) = \frac{1}{i}\ln\left[\frac{1+\zeta_1\zeta_2}{1+\zeta_1^*\zeta_2}\right] \quad , \tag{34}$$

$$\zeta_3 = \frac{\zeta_1+\zeta_2}{1+\zeta_1^*\zeta_2} \quad . \tag{35}$$

The measure of integration is invariant under the change of variables from ζ_2 to ζ_3

$$\frac{d^2\zeta_2}{(1-\mid\zeta_2\mid^2)^2} = \frac{d^2\zeta_3}{(1-\mid\zeta_3\mid^2)^2} \quad . \tag{36}$$

On using the completeness of $\mid \zeta; 1 >$, one can show that

$$< \zeta; 1 \mid X_3(n) \mid \zeta; 1 >= 1 \quad \text{for all} \quad \zeta \quad , \tag{37}$$

and hence $X_3(n) = \mathbf{I}$.

Outlook:

We have thus shown that

$$\int d\mu(\zeta)D(\zeta) \mid f >< f \mid D^\dagger(\zeta) = \mathbf{I} \quad , \tag{38}$$

for the f-coherent states for the three groups considered above. The relation (38) is amenable to further generalisations. In the case of Heisenberg- Weyl group, by expanding the state $\mid f >$ in (38) in terms of the number states $\mid n >$ one obtains

$$\int d\mu(\zeta)D(\zeta) \mid m >< n \mid D^\dagger(\zeta) = \mathbf{I}\delta_{mn} \quad , \tag{39}$$

and hence

$$\int d\mu(\zeta) D(\zeta) \mid f_1 >< f_2 \mid D^\dagger(\zeta) = \mathbf{I} < f_1 \mid f_2 > \quad . \tag{40}$$

In view of (39), one has

$$\int d\mu(\zeta) D(\zeta) \rho_o D^\dagger(\zeta) = \mathbf{I} \quad , \tag{41}$$

where ρ_o is an arbitrary density matrix. For $SU(2)$ and $SU(1,1)$, (38) implies (41) with ρ_o subject to the conditions

$$[\rho_o, S_z] = 0 \ \text{ and } \ [\rho_o, K_z] = 0 \quad , \tag{42}$$

respectively. It may be noted that, in the context of Heisenberg-Weyl group, resolutions of the identity of the type (41) have been derived by Vourdas and Bishop [11] for two specific choices of ρ_o. The fact that, for the Heisenberg-Weyl group (41) is valid for an arbitrary ρ_o does not seem to be generally appreciated.

The results given above enable us to derive interesting identities involving orthogonal polynomials. For example the following integral[1] involving the Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$ [12]

$$\frac{1}{2}\left[\frac{\Gamma(n+1)\ \Gamma(p+3/2)}{\Gamma(p+1)\ \Gamma(n+3/2)}\right] \int_o^1 \frac{dx}{(1-x)^{1/2}}\ x^{p-n}\left[P_n^{p-n,1/2)}(1-2x)\right]^2 = 1 \quad , \tag{43}$$

can be derived from (38) by applying it to the $SU(1,1)$ case and using the relations[2]

$$< 2m+1 \mid D(\xi) \mid 2n+1 > \ = e^{-i(m-n)\phi}\left[\frac{\Gamma(n+1)\Gamma(m+3/2)}{\Gamma(m+1)\ \Gamma(n+3/2)}\right]^{1/2} (\mid \zeta \mid)^{m-n}(1- \mid \zeta \mid^2)^{3/4}$$

[1]A direct proof of (43) appears to be difficult. We have succeeded in proving it using Racah identities [13].

[2]Expressions for these matrix elements in terms of associated Legendre functions may be found in [7].

$$P_n^{(m-n,1/2)}(1-2\mid\zeta\mid^2) \quad \text{for} \quad m \geq n \quad , \tag{44}$$

$$= e^{-i(n-m)\phi}\left[\frac{\Gamma(m+1)\ \Gamma(n+3/2)}{\Gamma(n+1)\ \Gamma(m+3/2)}\right]^{1/2}(-\mid\zeta\mid)^{n-m}(1-\mid\zeta\mid^2)^{3/4}$$

$$P_m^{(n-m,1/2)}(1-2\mid\zeta\mid^2) \quad \text{for} \quad m \leq n \quad , \tag{45}$$

In conclusion, we also note the possibility of using relations like (1) to construct new classes of quasi-probability distributions. Thus, for instance, for any density operator ρ, one can define a generalised Q-function as follows

$$Q(\zeta) = Tr[\rho D(\zeta)\rho_o D^\dagger(\zeta)] \tag{46}$$

We hope to discuss this in detail elsewhere.

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