Characterization of spectral triples: A combinatorial approach

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Abstract

We describe a general technique to study Dirac operators on noncommutative spaces under some additional assumptions. The main idea is to capture the compact resolvent condition in a combinatorial set up. Using this, we then prove that for a certain class of representations of the C^*-algebra $C(SU_q(\ell+1))$, any Dirac operator that diagonalises with respect to the natural basis of the underlying Hilbert space must have trivial sign.

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1 Introduction

A spectral triple is the starting point in noncommutative geometry (NCG) where a geometric space is described by a triple $(\mathcal{A},\mathcal{H},D)$, with $\mathcal{A}$ being an involutive algebra represented as bounded operators on a Hilbert space $\mathcal{H}$, and D being a selfadjoint operator with compact resolvent and having bounded commutators with the algebra elements. This D should be nontrivial in the sense that the associated Kasparov module should give a nontrivial element in K-homology. Observe that the self-adjoint operator D in a spectral triple comes with two very crucial restrictions on it, namely, it has to have compact resolvent, and must have bounded commutators with algebra elements. Various analytic consequences of the compact resolvent condition (growth properties of the commutators of the algebra elements with the sign of D) have been used in the past by various authors. Here we will take a new approach that will help us exploit it from a combinatorial point of view. The idea is very simple. Given a selfadjoint operator with compact resolvent, one can associate with it a certain graph in a natural way. This makes it possible to do a detailed combinatorial analysis of the growth restrictions (on the eigenvalues of D) that come from the boundedness of the commutators, and to characterize the sign of the operator D completely.

In the next section, we will outline the strategy. It should be noted that this technique has already been used implicitly in characterizing spectral triples for the quantum $SU(2)$ group by the authors in [1] and [2]. Here we will present the scheme in a more explicit way and use it in the remaining sections to study a more complicated and important case. The case that we treat is analogous to the one for $SU_q(2)$ treated in [2]. We will take a large class of representations of the C^*-algebra $C(SU_q(\ell+1))$, which includes the irreducibles in particular, and use the general scheme described in section 1 to prove that for a large majority of these representations, any Dirac operator that diagonalises nicely with respect to the canonical orthonormal basis must have trivial sign.

2 The combinatorial set up

Suppose $\mathcal{A}$ is a C^*-algebra represented on a Hilbert space, and suppose we want to have an idea about all operators D that will make $(\mathcal{A}, \mathcal{H}, D)$ into a spectral triple. Of course, in this generality, the problem would be intractable in most cases. We will impose some extra conditions on this D that will be natural from the context. This would give some information about the spectral resolution $\sum_{\gamma\in\Gamma} d_\gamma P_\gamma$, more specifically some idea about the set Γ and a diagonalising basis for D. Note that since D is known to be self-adjoint with discrete spectrum, there always exists such a basis. Next, let c be a positive real. Construct a graph $\mathcal{G}_c$ by taking the vertex set V to be Γ and by joining two points γ and γ' in $V = \Gamma$ by an edge if $|d_\gamma - d_{\gamma'}| < c$. Define $V^+ = \{\gamma \in V : d_\gamma > 0\}$ and $V^- = \{\gamma \in V : d_\gamma < 0\}$. One can assume without loss in generality that the null space of D is trivial, as this can be achieved just by a compact perturbation. Thus (V^+, V^-) gives us a partition of the vertex set $\Gamma = V$. Call two paths $(v_1, v_2, \ldots, v_m)$ and $(w_1, w_2, \ldots, w_n)$ in $\mathcal{G}_c$ (or more generally in any graph $\mathcal{G}$) **disjoint** if the sets $\{v_1, v_2, \ldots, v_m\}$ and $\{w_1, w_2, \ldots, w_n\}$ do not intersect. Now observe that there can not exist infinitely many disjoint paths from V^+ to V^-. This is because if $(v_1, v_2, \ldots, v_m)$ is a path from V^+ to V^-, then for some v_i, one must have $d_{v_i} \in [-c, c]$. Thus if there are infinitely many disjoint paths from V^+ to V^-, it would contradict the compact resolvent condition on D.

We say that a partition (V_1, V_2) of the vertex set in a graph admits an **infinte ladder** if there are infinitely many disjoint paths from V_1 to V_2. We call a partition (V_1, V_2) **sign-determining** if it does not admit an infinite ladder. Thus the partition (V^+, V^-) of the vertex set in the graph $\mathcal{G}_c$ is sign-determining.

Since we do not know the operator D apriori, we proceed from the other direction. Using the action of the algebra elements on the basis elements of $\mathcal{H}$ and using the boundedness of their commutators with D, we get certain growth restrictions on the d_γ's. These will give us some information about the edges in the graph. We exploit this knowledge to characterize those partitions (V_1, V_2) of the vertex set that do not admit any infinite ladder. This amounts to characterizing the sign of the operator D, in the sense that the sign of D must be of the form $\sum_{\gamma\in V_1} P_\gamma - \sum_{\gamma\in V_2} P_\gamma$.

Note here that whether or not a partition admits an infinite ladder will depend on the value of c. For a specific value of c, the graph $\mathcal{G}_c$ may have no edges, or too few edges (if the singular values of D happen to grow too fast). In such a case, there would exist too many partitions that are sign-determining, and as a result, will not be very useful. Therefore we will be interested only in those partitions that remain sign-determining for all sufficiently large c.

3 The group $SU_q(\ell+1)$

Let $\mathfrak{g}$ be a complex simple Lie algebra of rank ℓ. let $((a_{ij}))$ be the associated Cartan matrix, q be a real number lying in the interval $(0,1)$ and let $q_i = q^{(\alpha_i,\alpha_i)/2}$, where α_i's are the simple roots of $\mathfrak{g}$. Then the quantised universal envelopping algebra (QUEA) $U_q(\mathfrak{g})$ is the algebra generated by E_i, F_i, K_i and K_i^{-1}, $i = 1,\ldots,\ell$, satisfying the following relations

$$K_iK_j = K_jK_i, \quad K_iK_i^{-1} = K_i^{-1}K_i = 1,$$

$$K_iE_jK_i^{-1} = q_i^{\frac{1}{2}a_{ij}}E_j, \quad K_iF_jK_i^{-1} = q_i^{-\frac{1}{2}a_{ij}}F_j,$$

$$E_iF_j - F_jE_i = \delta_{ij}\frac{K_i^2 - K_i^{-2}}{q_i - q_i^{-1}},$$

$$\sum_{r=0}^{1-a_{ij}} (-1)^r \binom{1-a_{ij}}{r}_{q_i} E_i^{1-a_{ij}-r}E_jE_i^r = 0 \quad \forall\, i \neq j,$$

$$\sum_{r=0}^{1-a_{ij}} (-1)^r \binom{1-a_{ij}}{r}_{q_i} F_i^{1-a_{ij}-r}F_jF_i^r = 0 \quad \forall\, i \neq j,$$

where $\binom{n}{r}_q$ denote the q-binomial coefficients. Hopf *-structure comes from the following maps:

$$\Delta(K_i) = K_i \otimes K_i, \quad \Delta(K_i^{-1}) = K_i^{-1} \otimes K_i^{-1},$$

$$\Delta(E_i) = E_i \otimes K_i + K_i^{-1} \otimes E_i, \quad \Delta(F_i) = F_i \otimes K_i + K_i^{-1} \otimes F_i,$$

$$\epsilon(K_i) = 1, \quad \epsilon(E_i) = 0 = \epsilon(F_i),$$

$$S((K_i) = K_i^{-1}, \quad S(E_i) = -q_iE_i, \quad S(F_i) = -q_i^{-1}F_i,$$

$$K_i^* = K_i, \quad E_i^* = -q_i^{-1}F_i, \quad F_i^* = -q_iE_i.$$

In the type A case, the associated Cartan matrix is given by

$$a_{ij} = \begin{cases} 2 & \text{if } i = j, \\ -1 & \text{if } i = j \pm 1, \\ 0 & \text{otherwise,} \end{cases}$$

and $(\alpha_i,\alpha_i) = 2$ so that $q_i = q$ for all i. The QUEA in this case is denoted by $u_q(su(\ell+1))$.

Take the collection of matrix entries of all finite-dimensional unitarizable $u_q(su(\ell+1))$-modules. The algebra generated by these gets a natural Hopf*-structure as the dual of $u_q(su(\ell+1))$. One can also put a natural C^*-norm on this. Upon completion with respect to this norm, one gets a unital C^*-algebra that plays the role of the algebra of continuous functions on $SU_q(\ell+1)$. For a detailed account of this, refer to chapter 3, [10]. In [12], Woronowicz gave a different description of this C^*-algebra. which was later shown by Rosso ([11]) to be equivalent to the earlier one.

For remainder of this article, we will take G to be $SU_q(\ell+1)$ and $\mathcal{A}$ will be the C^*-algebra of continuous functions on G.

4 Irreducible representations

All irreducible representations of the C^*-algebra $\mathcal{A}$ are well-known ([10]). Let us briefly recall those here. The Weyl group for $SU_q(\ell+1)$ is isomorphic to the permutations group $\mathfrak{S}_{\ell+1}$ on $\ell+1$ symbols. Denote by s_i the transposition $(i, i+1)$. Then $\{s_1, \ldots, s_\ell\}$ form a set of generators for $\mathfrak{S}_{\ell+1}$. Any $\omega \in \mathfrak{S}_{\ell+1}$ can be written as a product

$$\omega = (s_{k_\ell} s_{k_\ell+1} \ldots s_\ell)(s_{k_{\ell-1}} s_{k_{\ell-1}+1} \ldots s_{\ell-1}) \ldots (s_{k_2} s_2)(s_{k_1}),$$

where k_i's are integers satisfying $0 \leq k_i \leq i$, with the understanding that $k_i = 0$ means that the string $(s_{k_i} s_{k_i+1} \ldots s_i)$ is missing. It follows from the strong deletion condition in the characterization of Coxeter system by Tits (see [8]) that the expression for ω given above is a reduced word in the generators s_i. We will denote the length of an element ω by $\ell(\omega)$.

Let S and N be the following operators on $L_2(\mathbb{Z})$:

$$Se_n = e_{n-1}, \quad Ne_n = ne_n.$$

We will denote by the same symbols their restrictions to $L_2(\mathbb{N})$ whenever there is no chance of ambiguity. Denote by ψ_{s_i} the following representation of $\mathcal{A}$ on $L_2(\mathbb{N})$:

$$\psi_{s_i}(u_{ab}) = \begin{cases} \sqrt{I - q^{2N+2}}S & \text{if } a = b = i, \\ S^*\sqrt{I - q^{2N+2}} & \text{if } a = b = i+1, \\ -q^{N+1} & \text{if } a = i,\ b = i+1, \\ q^N & \text{if } a = i+1,\ b = i, \\ \delta_{ab} I & \text{otherwise.} \end{cases}$$

Now suppose $\omega \in \mathfrak{S}_{\ell+1}$ is given by $s_{i_1} s_{i_2} \ldots s_{i_k}$. Define ψ_ω to be $\psi_{s_{i_1}} * \psi_{s_{i_2}} * \ldots * \psi_{s_{i_k}}$ (for two representations ϕ and ψ, $\phi * \psi$ denote the representation $(\phi \otimes \psi)\Delta$).

Next, let $\mathbf{z} = (z_1, \ldots, z_\ell) \in (S^1)^\ell$. Define

$$\chi_{\mathbf{z}}(u_{ab}) = \begin{cases} z_a \delta_{ab} & \text{if } a = 1, \\ \bar{z}_\ell \delta_{ab} & \text{if } a = \ell+1, \\ \bar{z}_{a-1} z_a \delta_{ab} & \text{otherwise.} \end{cases}$$

Define χ to be the integral $\int_{\mathbf{z} \in (S^1)^\ell} \chi_{\mathbf{z}} d\mathbf{z}$. Finally, define $\pi_{\omega,\mathbf{z}} = \psi_\omega * \chi_{\mathbf{z}}$ and $\pi_\omega = \psi_\omega * \chi$. It is known ([10]) that $\pi_{\omega,\mathbf{z}}$'s constitute all the ireducible representations of the C^*-algebra $\mathcal{A}$.

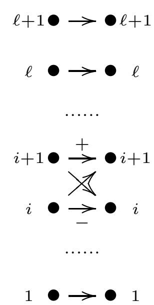
Let us introduce a few notations that will be handy later. For a subset $\Lambda = \{i_1, \ldots, i_k\} \subseteq \{1, 2, \ldots, \ell\}$, where $i_1 < i_2 < \ldots < i_k$, denote by s_Λ the element $s_{i_1} s_{i_2} \ldots s_{i_k}$ of $\mathfrak{S}_{\ell+1}$. Call a subset J of $\{1, 2, \ldots, \ell\}$ an **interval** if it is of the form $\{j, j+1, \ldots, j+s\}$. Then for any element ω of the Weyl group, there are intervals $\Lambda_1, \Lambda_2, \ldots, \Lambda_t$ with $\max \Lambda_r > \max \Lambda_s$ for $r > s$ such that

$$\omega = s_{\Lambda_t} s_{\Lambda_{t-1}} \ldots s_{\Lambda_1}. \tag{4.1}$$

Moreover, as long as we demand that Λ_j's are intervals and obey $\max \Lambda_r > \max \Lambda_s$ for $r > s$, an element ω determines the subsets $\Lambda_t, \ldots, \Lambda_1$ uniquely. Let Λ be the disjoint union of the Λ_j's, that is, $\Lambda = \cup_{j=1}^t \{(j, i) : i \in \Lambda_j\}$. Write $\Lambda_0 = \{(0, i) : i = 1, 2, \ldots, \ell\}$. Often we will identify Λ_0 with the set $\{1, 2, \ldots, \ell\}$. Let $\Gamma = \mathbb{N}^\Lambda \times \mathbb{Z}^{\Lambda_0} = \mathbb{N}^{\Lambda_t} \times \mathbb{N}^{\Lambda_{t-1}} \times \ldots \times \mathbb{N}^{\Lambda_1} \times \mathbb{Z}^{\Lambda_0}$. The Hilbert space on which π_ω acts is $L_2(\Gamma)$. We will denote by $\{e_\gamma : \gamma \in \Gamma\}$ the canonical orthonormal basis for this Hilbert space.

5 Diagram representation of π_ω

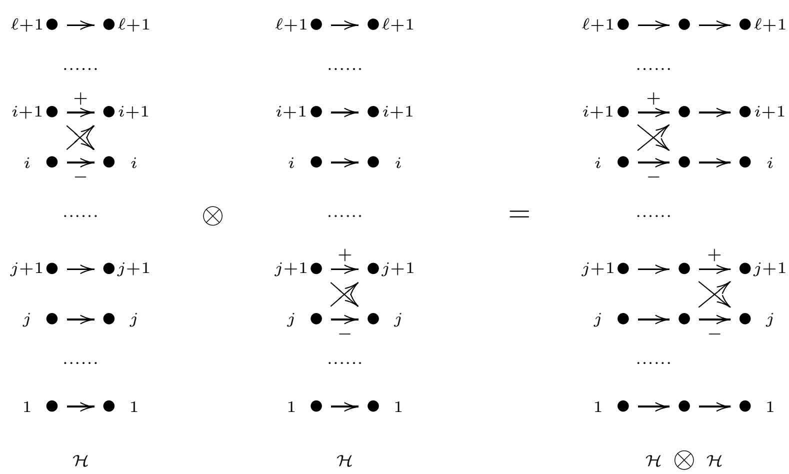
Let us describe how to use a diagram to represent the irreducible ψ_{s_i}.



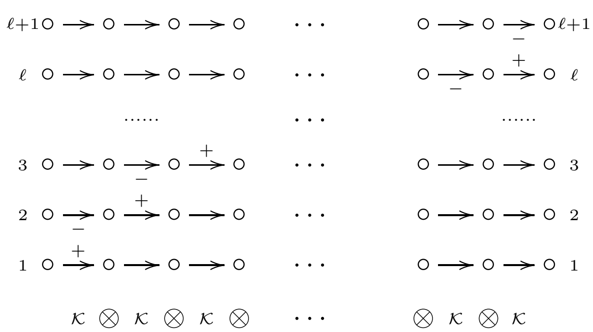
$\mathcal{H}$

In this diagram, each path from a node k on the left to a node l on the right stands for an operator on $\mathcal{H} = L_2(\mathbb{N})$. A horizontal unlabelled line stands for the identity operator, a horizontal line labelled with a $+$ sign stands for $S^* \sqrt{I - q^{2N+2}}$ and one labelled with a $-$ sign stands for $\sqrt{I - q^{2N+2}} S$. A diagonal line going upward represents $-q^{N+1}$ and a diagonal line going downward represents q^N. Now $\psi_{s_i}(u_{kl})$ is the operator represented by the path from k to l, and is zero if there is no such path. Thus, for example, $\psi_{s_i}(u_{11})$ is I, $\psi_{s_i}(u_{12})$ is zero, whereas $\psi_{s_i}(u_{ii+1}) = -q^{N+1}$, if $i > 1$.

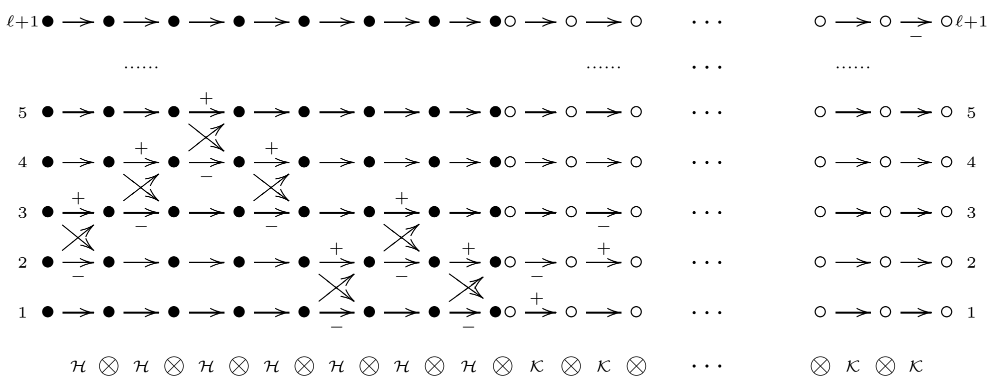
Next, let us explain how to represent $\psi_{s_i} * \psi_{s_j}$. Simply put the two diagrams representing ψ_{s_i} and ψ_{s_j} adjacent to each other, and identify, for each row, the node on the right side of the diagram for ψ_{s_i} with the node on the left in the diagram for ψ_{s_j}. Now, $\psi_{s_i} * \psi_{s_j}(u_{kl})$ would be an operator on $L_2(\mathbb{N}) \otimes L_2(\mathbb{N})$ determined by all the paths from the node k on the left to the node l on the right. It would be zero if there is no such path and if there are more than one paths, then it would be the sum of the operators given by each such path. Thus, we have the following operation on the elementary diagrams described above:



Next, we come to χ. The underlying Hilbert space now is $L_2(\mathbb{Z}^{\Lambda_0}) \cong L_2(\mathbb{Z})^{\otimes \ell}$ (to avoid any ambiguity, we have used hollow circles to denote the nodes as opposed to the bullets used in the earlier case); an unlabelled horizontal arrow stands for I in the corresponding component of $L_2(\mathbb{Z})^{\otimes \ell}$, an arrow labelled with a '+' above it indicates S^* and one labelled '$-$' below it stands for S. As earlier, $\chi(u_{kl})$ stands for the operator on $L_2(\mathbb{Z})^{\otimes \ell}$ represented by the path from k on the left to l on the right. In the diagram below, $\mathcal{K}$ will stand for $L_2(\mathbb{Z})$.



Finally, we come to the description of π_ω. As we have already remarked, reduced expression for ω is of the form $\omega = (s_{k_n} s_{k_n+1} \ldots s_n)(s_{k_{n-1}} \ldots s_{n-1}) \ldots (s_{k_2} s_2)(s_{k_1})$. To get the diagram for π_ω, we simply put the diagram for $\psi_{s_{k_n}} * \ldots * \psi_{s_{k_1}}$ and that for χ side by side and identify the nodes on the right of the first diagram with the corresponding ones on the left of the second diagram. Thus for example, if $\omega = (s_2 s_3 s_4)(s_3)(s_1 s_2)(s_1)$, then the following diagram represents π_ω:



The diagram for π_ω introduced above will play an important role in what follows.

6 Boundedness of commutators

Our goal is to study operators D on the space $\mathcal{H}_\omega = L_2(\Gamma)$ that diagonalize with respect to the natural canonical basis, and makes $(\pi_\omega(\mathcal{A}), \mathcal{H}_\omega, D)$ a spectral triple. Since D is a self-adjoint operator with discrete spectrum, it is of the form $\sum_{\gamma\in\Gamma} d(\gamma)e_\gamma$, where $d(\gamma) \geq 0$ for all γ.

Definition 6.1 A **move** will mean a path from a node on the left to a node on the right in the diagram representing π_ω. More formally, a *move* is a $(t+1)$-tuple of pairs $((i_t, j_t), \ldots, (i_0, j_0))$ such that

1. $j_k = i_{k-1}$ for $k \geq 1$, $i_0 = j_0$,
2. for $k \geq 1$, $j_k < i_k$ implies $j_k = i_k - 1$ and $j_k \in \Lambda_k$,
3. for $k \geq 1$, $j_k > i_k$ implies $i_k, i_k + 1, \ldots, j_k - 1 \in \Lambda_k$.

(the pair (i_k, j_k) will be referred as the k^{th} segment of the *move* lying in the k^{th} string from the right).
Observe that the above three conditions imply in particular that $1 \leq i_k, j_k \leq \ell + 1$ for all k. We will use the special notation H_r for the *move* for which each i_k and j_k equals r.

Given a *move* p, we will next define an element $m_p \in \mathbb{Z}^\Lambda \times \mathbb{Z}^{\Lambda_0} = \mathbb{Z}^{\Lambda\cup\Lambda_0}$ whose coordinates are all 0 or ± 1. Let $p = ((i_t, j_t), \ldots, (i_0, j_0))$. Define m_p by the following prescription:

$$m_p(r,s) = \begin{cases} -1 & \text{if } r = 0, s = i_0 - 1 \text{ or } r \geq 1, s = j_r \geq i_r, \\ +1 & \text{if } r = 0, s = i_0 \text{ or } r \geq 1, j_r \geq i_r = s + 1, \\ 0 & \text{otherwise.} \end{cases}$$

Thus for $r > 0$, $m_p(r, \cdot)$ will look like

$$\begin{cases} (0, 0, \ldots, 0) & \text{if } i_r < \min \Lambda_r \text{ or } i_r > \max \Lambda_r + 1 \text{ or } j_r = i_r - 1, \\ (-1, 0, \ldots, 0) & \text{if } i_r = j_r = \min \Lambda_r, \\ (0, 0, \ldots, 0, 1) & \text{if } i_r = j_r = \max \Lambda_r + 1, \\ (\underbrace{0, \ldots, 0}_{i_r - 2}, 1, \underbrace{0, \ldots, 0}_{j_r - i_r}, -1, 0, \ldots, 0) & \text{if } \min \Lambda_r < i_r \le j_r, \end{cases}$$

and $m_p(0, \cdot)$ will be of the form

$$\begin{cases} (1, 0, \ldots, 0) & \text{if } i_0 = j_0 = 1, \\ (0, 0, \ldots, 0, -1) & \text{if } i_0 = j_0 = \ell + 1, \\ (\underbrace{0, \ldots, 0}_{i_0 - 2}, -1, 1, 0, \ldots, 0) & \text{if } 1 < i_0 = j_0 < \ell + 1. \end{cases}$$

We will often refer to this associated element m_p when we talk about a *move* p.

Let us denote by P_{ij} the set of moves from node i on the left to node j on the right. For a move p, denote by T_p the corresponding operator on $\mathcal{H}^{\otimes \ell(\omega)} \otimes \mathcal{K}^{\otimes \ell}$. Then

$$\pi_\omega(u_{ij}) = \sum_{p \in P_{ij}} T_p. \tag{6.1}$$

Denote by W_p the operator obtained from T_p by replacing $\sqrt{I - q^{2N+2}}S$ by S and $S^* \sqrt{I - q^{2N+2}}$ by S^*. One can show easily that m_p is the unique element in $\mathbb{Z}^\Lambda \times \mathbb{Z}^{\Lambda_0}$ whose entries are all 0 or ± 1 such that $\langle W_p e_\gamma, e_{\gamma + m_p} \rangle \ne 0$ for some $\gamma \in \Gamma$.

Lemma 6.2 *Let $p, p' \in P_{ij}$. If p and p' are different, then for some (r, n), where $1 \le r \le t$ and $n \in \Lambda_r$, one has either $m_p(r, n) = 0, m_{p'}(r, n) = \pm 1$ or $m_p(r, n) = \pm 1, m_{p'}(r, n) = 0$.*

Proof: Since p and p' both belong to P_{ij} and are different, $m_p(r, n) \ne m_{p'}(r, n)$ for some pair (r, n). Now look at the coordinate where they are unequal for the first time (from the left), that is, let (r, n) be the pair such that

$$r = \max\{1 \le j \le t : m_p(j, i) \ne m_{p'}(j, i) \text{ for some } i\}, \quad n = \min\{i \in \Lambda_r : m_p(r, i) \ne m_{p'}(r, i)\}.$$

It is easy to see now that for this pair (r, n), the required conclusion holds. $\square$

Lemma 6.3 *Let F be a finite set of moves. For $p \in F$, let D_p be a (not necessarily bounded) number operator, i.e. an operator of the form $e_\gamma \mapsto t_\gamma e_\gamma$. If $\sum_{p \in F} D_p W_p$ is bounded, then $D_p W_p$ is bounded for each $p \in F$.*

Proof: Take $p' \in F$. Assume that $|F| > 1$. We will show that boundedness of $\sum_{p \in F} D_p W_p$ implies that of $\sum_{p \in F'} D_p W_p$ for some subset F' of F such that $p' \in F'$ and $|F'| < |F|$.

Let $p'' \in F$ be an element of F other than p'. By the previous lemma, there is a pair (r,n) such that either $m_{p'}(r,n) = 0$ and $m_{p''}(r,n) = \pm 1$ or $m_{p'}(r,n) = \pm 1$ and $m_{p''}(r,n) = 0$. For $z \in S^1$, let U_z be the unitary operator on $L_2(\Gamma)$ given by $U_z e_\gamma = z^{\gamma(r,n)} e_\gamma$. Now the proof will follow from the boundedness of the operator $\int_{z\in S^1} U_z(\sum_{p\in F} D_p W_p) U_z^* \, dz$. $\square$

Proposition 6.4 *$[D, \pi_\omega(u_{ij})]$ is bounded for all i and j if and only if $[D, W_p]$ is bounded for all moves p.*

Proof: It is enough to show that if $[D, \pi_\omega(u_{ij})]$ is bounded, and if $p \in P_{ij}$, then $[D, W_p]$ is bounded. Since $\pi_\omega(u_{ij}) = \sum_{p\in P_{ij}} T_p$ and each $[D, T_p]$ is of the form $D_p W_p$, it follows from the forgoing lemma that each $[D, T_p]$ is bounded. Since $\sqrt{1 - q^{2n+2}}$ is a bounded quantity whose inverse is also bounded, it follows that $[D, T_p]$ is bounded if and only if $[D, W_p]$ is bounded. $\square$

Thus there is a positive constant c such that D will have bounded commutators with all the $\pi_\omega(u_{ij})$'s if and only if $\|[D, W_p]\| \leq c$.

Let $p = ((i_t, j_t), \ldots, (i_0, j_0))$ be a move. A coordinate (r,s) is said to be a **diagonal component** of p if either $i_r < j_r$ and $s \in \{i_r, i_r + 1, \ldots, j_r - 1\}$, or $j_r = i_r - 1 = s$. One can check that this would correspond exactly to the diagonal parts of the move in the diagram representing ω. Denote by $c(\gamma, p)$ the quantity $\sum_{(j,i)} \gamma(j,i)$, the sum being taken over all diagonal components of p.

Lemma 6.5 *$[D, W_p]$ is bounded if and only if $|d(\gamma + m_p) - d(\gamma)| \leq c q^{-c(\gamma,p)}$.*

Proof: Follows easily once one writes down the expression of the commutator. $\square$

An immediate corollary is the following.

Corollary 6.6 *Let H_i be as in definition 6.1. Then $|d(\gamma + H_i) - d(\gamma)| \leq c$ for all $\gamma \in \Gamma$ and $1 \leq i \leq \ell + 1$.*

7 The growth graph and sign characterization

Let us now form the graph $\mathcal{G}_c$ by connecting two vertices γ and γ' if $|d(\gamma) - d(\gamma')| \leq c$. Characterization of $\operatorname{sign} D$ will then proceed as outlined in the beginning of section 2.

Definition 7.1 For $i \in \Lambda_0$, let J_i be the set $\{j \geq 1 : i \in \Lambda_j\}$. The set $\mathscr{F} = \{\gamma \in \mathbb{Z}^\Lambda \times \mathbb{Z}^{\Lambda_0} : -\gamma(0,i) = \gamma(0,i-1) = \gamma(j,i) \text{ for all } j \in J_i\}$ will be called the **free plane**. For a point $\gamma \in \Gamma$, we call the set $\mathscr{F}_\gamma = \{\gamma + \gamma' \in \Gamma : \gamma' \in \mathscr{F}\}$ the **free plane passing through** γ.

Note that for $\gamma \in \mathscr{F}$, the coordinates $\gamma(j,i)$ are all equal for $j \in J_i$.

For $1 \leq i \leq \ell$, define j_i to be 0 if J_i is empty, and to be that element of J_i for which $\gamma(j_i, i) = \min\{\gamma(j,i) : j \in J_i\}$.

Remark 7.2 1. If J_i is nonempty, j_i need not be unique.
2. If $\gamma' \in \mathscr{F}_\gamma$, then $\min_j \gamma(j,i)$ and $\min_j \gamma'(j,i)$ are attained for the same set of values of j.

Note that given a $\gamma \in \Gamma$, elements in $\mathscr{F}_\gamma$ are determined by the coordinates (j_i, i), $i = 1, \ldots, \ell$.

Lemma 7.3 *Let $\gamma, \gamma' \in \Gamma$. Then either $\mathscr{F}_\gamma = \mathscr{F}_{\gamma'}$ or $\mathscr{F}_\gamma$ and $\mathscr{F}_{\gamma'}$ are disjoint.*

Proof: Proof folows from the observation that $\mathscr{F}_\gamma = \gamma + \mathscr{F}$ and $\mathscr{F}$ is a subgroup in $\mathbb{Z}^\Lambda \times \mathbb{Z}^{\Lambda_0}$.
$\square$

Lemma 7.4 *Let $\gamma \in \Gamma$, and $\gamma' \in \mathscr{F}_\gamma$. Let γ'' be the element in $\mathscr{F}_\gamma$ for which*

$$\gamma''(j_\ell, \ell) = \gamma'(j_\ell, \ell), \quad \gamma''(j_i, i) = 0 \text{ for all } i < \ell.$$

Then there is a path in $\mathscr{F}_\gamma$ joining γ' to γ'' such that throughout this path, the (j_ℓ, ℓ)-coordinate remains constant.

Proof: Apply successively the moves

$$\gamma(j_{\ell-1}, \ell-1)H_{\ell-1}, \quad (\gamma(j_{\ell-2}, \ell-2) + \gamma(j_{\ell-1}, \ell-1))H_{\ell-2}, \quad \ldots, \quad \left(\sum_{i=1}^{\ell-1} \gamma(j_i, i)\right) H_1.$$

As none of these moves touch the (j_ℓ, ℓ)-coordinate, it remains constant throughout the path.
$\square$

Lemma 7.5 *Let $\gamma \in \Gamma$. Then either $\mathscr{F}_\gamma^+$ is finite or $\mathscr{F}_\gamma^-$ is finite.*

Proof: Write $C(\gamma) = \gamma(j_\ell, \ell)$. We will first show that $C(\mathscr{F}_\gamma^+)$ and $C(\mathscr{F}_\gamma^-)$ can not both be infinite. Suppose if possible both $C(\mathscr{F}_\gamma^+)$ and $C(\mathscr{F}_\gamma^-)$ are infinite. Then there exist two sequences of elements γ_n and δ_n with $\gamma_n \in \mathscr{F}_\gamma^+$ and $\delta_n \in \mathscr{F}_\gamma^-$ such that

$$C(\gamma_1) < C(\delta_1) < C(\gamma_2) < C(\delta_2) < \cdots.$$

Now start at γ_n and employ lemma 7.4 to reach a point $\gamma_n' \in \mathscr{F}_\gamma^+$ such that $C(\cdot)$ remains constant throughout the path and for which

$$\gamma_n'(j_i, i) = 0 \text{ for all } i < \ell.$$

Similarly start at δ_n and employ lemma 7.4 to use a path where $C(\cdot)$ remains constant to reach a point $\delta_n' \in \mathscr{F}_\gamma^-$ for which

$$\delta_n'(j_i, i) = 0 \text{ for all } i < \ell.$$

Now use the move H_ℓ to go from γ'_n to δ'_n. The paths thus constructed are all disjoint, because the $C(\cdot)$ coordinate lies between $C(\gamma_n)$ and $C(\delta_n)$ throughtout. But that means $(\mathscr{F}_\gamma^+, \mathscr{F}_\gamma^-)$ admits an infinite ladder.

Next, suppose $C(\mathscr{F}_\gamma^-) \subseteq [-K, K]$. If $\{\gamma'(j_i, i) : \gamma' \in \mathscr{F}_\gamma\}$ is not bounded for some i with $1 \le i \le \ell - 1$, get a sequence of points $\gamma_n \in \mathscr{F}_\gamma$ such that $\gamma_n(j_i, i) < \gamma_{n+1}(j_i, i)$ for all n. Starting at each γ_n, apply the move $H_{\ell+1}$ enough (e.g. $2K + 1$) times to produce an infinite ladder. $\square$

Let us next define a set that will play the role of a complementary axis. Let

$$\mathscr{C} = \{\gamma \in \Gamma : \prod_{j \in J_i} \gamma(j, i) = 0 \text{ for all } i\}.$$

It follows from the sweepout argument used in the proof of lemma 7.4 that for any $\gamma' \in \Gamma$, there is a $\gamma \in \mathscr{C}$ such that $\gamma' \in \mathscr{F}_\gamma$. But it is not necessary that for two distinct elements γ and γ' in $\mathscr{C}$, $\mathscr{F}_\gamma$ and $\mathscr{F}_{\gamma'}$ are disjoint. However, this will not be of serious concern to us.

Let

$$i_{min} = \min\{i \in \Lambda_0 : |J_i| > 1\}, \quad j_{min} = \min J_{i_{min}}, \quad j_{max} = \max J_{i_{min}}.$$

Thus i_{min} is the minimum i for which s_i appears more than once in ω, j_{min} and j_{max} are the first and the last string where it appears. Suppose now that we have removed the horizontal arrows labelled $+$ or $-$ corresponding to all the s_i's for which $|J_i| = 1$. Note that this would in particular remove all labelled horizontal lines corresponding to s_i's for $i < i_{min}$. Suppose the $j_{min}{}^{\text{th}}$ segment of a move is (i_{min}, i_{min}). This will uniquely specify the 0^{th} segment which will be of the form (i_0, i_0) for some $i_0 \le i_{min}$. Now define

$$C_0(\gamma) := \gamma(j_{min}, i_{min}) + \gamma(0, i_0), \qquad C_1(\gamma) = \gamma(j_{max}, i_{min}) \tag{7.1}$$

for $\gamma \in \Gamma$.

Lemma 7.6 *Let $\gamma \in \mathscr{C}$. Define an element $\gamma' \in \Gamma$ by the following prescription:*

$$\gamma'(j, i) = 0 \textit{ for all } j \ge 1, \quad \gamma'(0, i) = \begin{cases} 0 & \textit{if } i \ne i_0, \\ C_0(\gamma) & \textit{if } i = i_0. \end{cases}$$

Then there is a path connecting γ to γ' such that $C_0(\cdot)$ remains constant throughout this path.

Proof: We will describe a recursive algorithm to go from γ to γ'. Observe that since $\gamma \in \mathscr{C}$, we have $\gamma(t, \max \Lambda_t) = 0$. To begin with, remove all the horizontal arrows labelled $+$ or $-$ corresponding to the s_i's for which $|J_i| = 1$, and work with the resulting diagram.

Now suppose we are at $\delta \in \Gamma$ which satisfies

$$\delta(j, i) = 0 \text{ for all } j > r, \quad \delta(r, i) = 0 \text{ for all } i > n \in \Lambda_r.$$

Step I.

Case I. $r = j_{min}$ and $n = i_{min}$: then apply the move whose $j_{min}{}^{\text{th}}$ segment is (i_{min}, i_{min}). Apply this $\delta(j_{min}, i_{min})$ times. This will make the (j_{min}, i_{min})-coordinate zero and the $(0, i_0)$-coordinate $C_0(\gamma)$. Now proceed to step II.

Case II. $r \neq j_{min}$ or $n \neq i_{min}$: Proceed with the following algorithm.

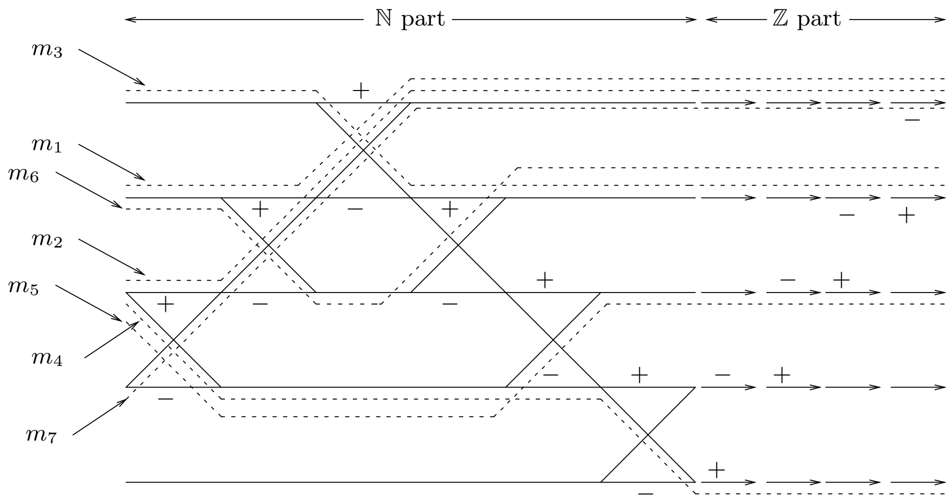
Algorithm A(r, n)**.** $(\min \Lambda_r \leq n \leq \max \Lambda_r)$
Remove all horizontal arrows labelled $+$ or $-$ from the s_i's in the strings s_{Λ_t}, $s_{\Lambda_{t-1}}$, $\ldots$, $s_{\Lambda_{r+1}}$ as well as from the s_i's corresponding to $i \in \Lambda_r$, $i > n$. What this will achieve is the following: any permissible move in the resulting diagram will not change the coordinates (j, i) where either $r + 1 \leq j \leq t$ or $j = r$ and $i > n$.
Apply the negative of the move whose r^{th} segment is $(n + 1, \max \Lambda_r + 1)$ for $\delta(r, n)$ number of times. This would kill the (r, n)-coordinate, i.e. will make it zero. Now remove the two horizontal lines labelled '$+$' and '$-$' corresponding to s_n appearing in the string s_{Λ_r}.

Step II.

Case I. $n > \min \Lambda_r$: keep r intact, reduce the value of n by 1 and go back to step I.
Case II. $r > 1$ and $n = \min \Lambda_r$: change n to $\max \Lambda_{r-1}$, then reduce the value of r by 1, and go back to step I.
Case III. $r = 1$ and $n = \min \Lambda_1$: proceed to step III.

Step III. All the (j, i)-coordinates for $j \geq 1$ are now zero. Next, apply moves ending at i for $i > i_0 + 1$ appropriate number of times starting from the top to kill the coordinates $(0, i)$ for $i > i_0$. Thus we have now reached an element δ for which $\delta(j, i) = 0$ whenever $j \geq 1, i \in \Lambda_j$ or $j = 0, i > i_0$. Therefore we now need to kill the coordinates $(0, i)$ for $i < i_0$. This is achieved as follows. Remove the horizontal arrows labelled $+$ or $-$ from all s_i's. Now apply the moves ending at i for $i < i_0$ appropriate number of times starting from the bottom. $\square$

The next diagram and the table that follows it will explain the proof in a simple case.



$\omega = (s_2 s_3 s_4)(s_3)(s_2)(s_1)$

$t = 4$, $i_{min} = 2$, $j_{min} = 2$, $j_{max} = 4$, $i_0 = 1$

The table below illustrates the sweepout procedure described in the proof of lemma 7.6. Starting from a point $\gamma \in \mathscr{C}$, it shows the successive moves applied and how the resulting element looks like at each stage. Observe that for any $\gamma \in \mathscr{C}$, one must have $\gamma(4,4) = 0 = \gamma(1,1)$.

coordinate	(4,2)	(4,3)	(4,4)	(3,3)	(2,2)	(1,1)	(0,1)	(0,2)	(0,3)	(0,4)
γ	$*$	$*$	0	$*$	a	0	b	$*$	$*$	$*$
move m_1	0	+1	0	0	0	0	0	0	0	-1
$\gamma_1 = -\gamma(4,3)m_1(\gamma)$	$*$	0	0	$*$	a	0	b	$*$	$*$	$*$
move m_2	+1	0	0	0	0	0	0	0	0	-1
$\gamma_2 = -\gamma(4,2)m_2(\gamma_1)$	0	0	0	$*$	a	0	b	$*$	$*$	$*$
move m_3	0	0	0	+1	0	0	0	0	-1	+1
$\gamma_3 = -\gamma(3,3)m_3(\gamma_2)$	0	0	0	0	a	0	b	$*$	$*$	$*$
move m_4	0	0	0	0	-1	0	+1	0	0	0
$\gamma_4 = \gamma(2,2)m_4(\gamma_3)$	0	0	0	0	0	0	$a+b$	$*$	$*$	$*$
move m_5	0	0	0	0	0	0	0	-1	+1	0
$\gamma_5 = \gamma_4(0,2)m_5(\gamma_4)$	0	0	0	0	0	0	$a+b$	0	$*$	$*$
move m_6	0	0	0	0	0	0	0	0	-1	+1
$\gamma_6 = \gamma_5(0,3)m_6(\gamma_5)$	0	0	0	0	0	0	$a+b$	0	0	$*$
move m_7	0	0	0	0	0	0	0	0	0	-1
$\gamma' = \gamma_6(0,4)m_7(\gamma_6)$	0	0	0	0	0	0	$a+b$	0	0	0

Lemma 7.7 *Both $C_0(\mathscr{C}^+)$ and $C_0(\mathscr{C}^-)$ can not be infinite.*

Proof: If both are infinite, there would exist elements $\gamma_n \in \mathscr{C}^+$ and $\delta_n \in \mathscr{C}^-$ such that

$$C_0(\gamma_1) < C_0(\delta_1) < C_0(\gamma_2) < C_0(\delta_2) < \ldots.$$

Let γ'_n and δ'_n be given by

$$\gamma'_n(j,i) = 0 \text{ for all } j \geq 1, \quad \gamma'_n(0,i) = \begin{cases} 0 & \text{if } i \neq i_0, \\ C_0(\gamma_n) & \text{if } i = i_0, \end{cases}$$

$$\delta'_n(j,i) = 0 \text{ for all } j \geq 1, \quad \delta'_n(0,i) = \begin{cases} 0 & \text{if } i \neq i_0, \\ C_0(\delta_n) & \text{if } i = i_0. \end{cases}$$

Use the earlier lemma to get paths between γ_n and γ'_n and between δ_n and δ'_n. Remove all the labelled arrows from all the s_i's. Let m_i be the move in the resulting diagram whose 0^{th} segment is (i,i), and let $m = \sum_{\overleftarrow{i=1}}^{i_0} m_i$. Apply this move $C_0(\delta_n) - C_0(\gamma_n)$ times to connect γ'_n and δ'_n. Thus there is a path p_n connecting γ_n and δ_n, and throughout this path, $C_0(\cdot)$ lies between $C_0(\gamma_n)$ and $C_0(\delta_n)$. Therefore the paths p_n are disjoint. □

For the next two lemmas, we will assume that $C_0(\mathscr{C}^-)$ is finite.

Lemma 7.8 *Assume $C_0(\mathscr{C}^-)$ is finite. Let C_1 be as defined prior to lemma 7.6, i.e. $C_1(\gamma) = \gamma(j_{max}, i_{min})$. Then the set $C_1(\mathscr{C}^-)$ is finite.*

Proof: Let $K \in \mathbb{N}$ be such that $C_0(\mathscr{C}^-) \subseteq [-K, K]$. If $C_1(\mathscr{C}^-)$ is not finite, there is a $\gamma_n \in \mathscr{C}^-$ such that

$$C_1(\gamma_1) < C_1(\gamma_2) < C_1(\gamma_3) < \ldots.$$

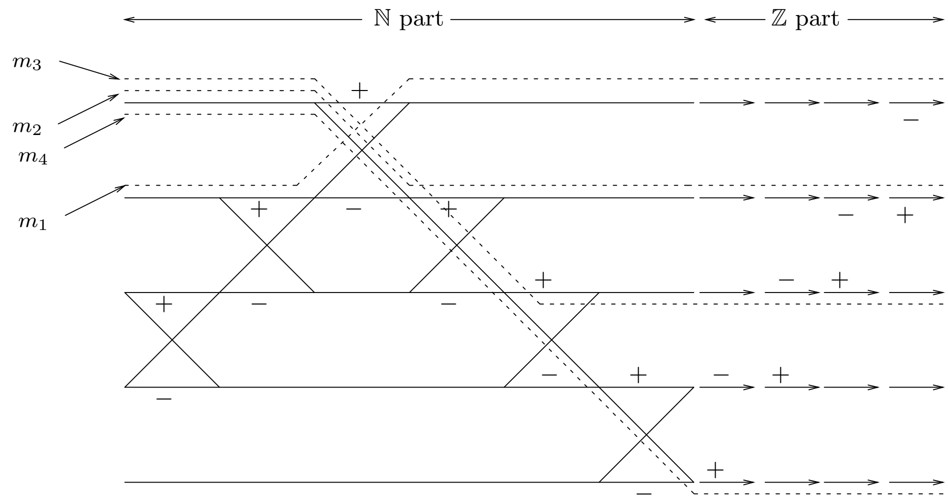
Now the idea is to get a path p_n joining γ_n to some δ_n such that $C_1(\cdot)$ remains constant throughout p_n, and $C_0(\delta_n) > K$, so that each $\delta_n \in \mathscr{C}^+$.

Start at γ_n. Apply algorithm $\mathbf{A}(r,n)$ for

$$\begin{array}{rcll} r & = & t, t-1, \ldots, j_{max}+1, & \min \Lambda_r \leq n \leq \max \Lambda_r, \\ r & = & j_{max}, & i_{min}+1 \leq n, \\ r & < & j_{max}, & \min \Lambda_r \leq n \leq \max \Lambda_r. \end{array}$$

Now apply the move m_{i_0}, where m_i's are the moves described in the proof of the previous lemma, $3K$ times. □

Again we give a diagram and a table to illustrate the above proof for the case $\omega = (s_2 s_3 s_4)(s_3)(s_2)(s_1)$.



$\omega = (s_2 s_3 s_4)(s_3)(s_2)(s_1)$

$t = 4,\ i_{min} = 2,\ j_{min} = 2,\ j_{max} = 4,\ i_0 = 1$

The next table illustrates the argument in the above proof. Starting from a point $\gamma \in \mathscr{C}^-$, it shows the successive moves applied and how the resulting element looks like at each stage.

coordinate	(4,2)	(4,3)	(4,4)	(3,3)	(2,2)	(1,1)	(0,1)	(0,2)	(0,3)	(0,4)
γ	a	$*$	0	$*$	$*$	0	b	$*$	$*$	$*$
move m_1	0	+1	0	0	0	0	0	0	0	−1
$\gamma_1 = -\gamma(4,3)m_1(\gamma)$	a	0	0	$*$	$*$	0	b	$*$	$*$	$*$
move m_2	0	0	0	+1	0	0	0	0	−1	+1
$\gamma_2 = -\gamma(3,3)m_2(\gamma_1)$	a	0	0	0	$*$	0	b	$*$	$*$	$*$
move m_3	0	0	0	0	+1	0	0	−1	+1	0
$\gamma_3 = -\gamma(2,2)m_3(\gamma_2)$	a	0	0	0	0	0	b	$*$	$*$	$*$
move m_4	0	0	0	0	0	0	+1	0	0	0
$\gamma_4 = 3Km_4(\gamma_3)$	a	0	0	0	0	0	$b+3K$	$*$	$*$	$*$

Lemma 7.9 *Assume $C_0(\mathscr{C}^-)$ is finite. Let $C \equiv (j,i)$ be any coordinate other than $C_1 \equiv (j_{max}, i_{min})$. Then $C(\mathscr{C}^-)$ is finite.*

Proof: By the previous lemma, $C_1(\mathscr{C}^-)$ is also bounded. Let $K \in \mathbb{N}$ be such that $C_0(\mathscr{C}^-) \subseteq [-K,K]$ and $C_1(\mathscr{C}^-) \subseteq [-K,K]$. The strategy would be the same as in the proof of the earlier lemma with a slight modification. If $C(\mathscr{C}^-)$ is infinite, we can choose $\gamma_n \in \mathscr{C}^-$ such that

$$C(\gamma_n) + K + 1 < C(\gamma_{n+1})$$

for every $n \in \mathbb{N}$. Now connect every γ_n to an element $\delta_n \in \mathscr{C}^+$ by a path p_n such that on p_n, the C_1 coordinate does not vary by more than K. This will ensure that the paths p_n are all disjoint.

For getting p_n as described above, start at γ_n and apply successively the moves

$$H_{\ell+1}, H_\ell, \ldots, H_{i_{min}+1},$$

each one $K+1$ times. This will increase the C_1-coordinate by $K+1$. Therefore the endpoint of the path will lie in $\mathscr{C}^+$. $\square$

Thus it now follows that if $C_0(\mathscr{C}^-)$ is finite, then $\mathscr{C}^-$ is finite. Similar argument would tell us that if $C_0(\mathscr{C}^+)$ is finite, then $\mathscr{C}^+$ is finite. Therefore by lemma 7.7, either $\mathscr{C}^+$ or $\mathscr{C}^-$ is finite. This, together with proposition 7.5 will give us the following theorem.

Theorem 7.10 *Let D be a Dirac operator on $L_2(\Gamma)$ that diagonalises with respect to the canonical orthonormal basis. Then up to a compact perturbation, $\operatorname{sign} D$ must be of the form $2P-I$ or $I-2P$ where P is a projection onto the closed linear span of $\{e_\gamma : \gamma \in \cup_{i=1}^k \mathscr{F}_{\gamma_i}\}$ for some finite collection $\gamma_1, \gamma_2, \ldots, \gamma_k$ in Γ.*

Proof: We first claim that there are finitely many free planes $\mathscr{F}_\gamma$ for which both $\mathscr{F}_\gamma^+$ and $\mathscr{F}_\gamma^-$ are nonempty. It follows from the proofs of lemma 7.4 and proposition 7.5 that any two points on a free plane can be connected by a path lying entirely on that plane. Therefore, if there are infinitely many distinct $\mathscr{F}_\gamma$'s for which both $\mathscr{F}_\gamma^+$ and $\mathscr{F}_\gamma^-$ are nonempty, one can easily produce an infinite ladder. Thus the claim is established.

Since for each γ, either $\mathscr{F}_\gamma^+$ or $\mathscr{F}_\gamma^-$ is finite, employing an appropriate compact perturbation, it is now possible to ensure that for every $\gamma \in \Gamma$, either $\mathscr{F}_\gamma \subseteq \Gamma^+$ or $\mathscr{F}_\gamma \subseteq \Gamma^-$. This, along with the fact that either $\mathscr{C}^+$ or $\mathscr{C}^-$ must be finite, imply the desired result. $\square$

We next show that under this restriction, compactness of the commutator $[\text{sign}\, D, u_{ij}]$, or, equivalently, that of $[P, u_{ij}]$'s will imply that sign D is trivial.

Let $\gamma_1, \gamma_2, \ldots, \gamma_k$ be elements in Γ and let P be the projection onto span$\{e_\gamma : \gamma \in \cup_i \mathscr{F}_{\gamma_i}\}$. Then for any operator T, we have

$$[P,T]e_\gamma = \begin{cases} PTe_\gamma & \text{if } \gamma \notin \cup_i \mathscr{F}_{\gamma_i}, \\ (P-I)Te_\gamma & \text{if } \gamma \in \cup_i \mathscr{F}_{\gamma_i}. \end{cases}$$

Now let $r = \max \Lambda_t$ and take $T = \pi_\omega(u_{r+1,r})$. Then

$$T(t,r) = q^N, \quad T(0,r-1) = S, \quad T(0,r) = S^*, \tag{7.2}$$

and $T(j,i) = I$ for all other pairs (j,i), except possibly $T(t-1,r-1)$, which is S^* if $t-1 \in J_{r-1}$, and I otherwise. It is easy to check that for $\gamma \in \mathscr{F}_{\gamma_i}$, $\gamma(t,r) + \gamma(0,r) = \gamma_i(t,r) + \gamma_i(0,r)$. Therefore the set $\{\gamma(t,r) + \gamma(0,r) : \gamma \in \cup_i \mathscr{F}_{\gamma_i}\}$ is bounded. Let $n \in \mathbb{N}$ be such that this set is contained in $[-n,n]$. Suppose $\gamma \in \cup_i \mathscr{F}_{\gamma_i}$ obey $\gamma(t,r) = 0$. Then it follows from (7.2) that $T^{2n+1}e_\gamma = e_{\gamma'}$, where

$$\gamma'(0,r) = \gamma(0,r) + 2n + 1, \quad \gamma'(0,r-1) = \gamma(0,r-1) - 2n - 1, \quad \gamma'(t,r) = \gamma(t,r).$$

It is clear from this that $\gamma' \notin \cup_i \mathscr{F}_{\gamma_i}$, so that $PT^{2n+1}e_\gamma = 0$. This means $[P, T^{2n+1}]e_\gamma = -e_{\gamma'}$ for all $\gamma \in \cup_i \mathscr{F}_{\gamma_i}$ with $\gamma(t, r) = 0$. Since there are infinitely many choices of such γ, it follows that $[P, T^{2n+1}]$ can not be compact.

We thus have the following theorem.

Theorem 7.11 *Let $\ell > 1$. Then there does not exist any Dirac operator on $L_2(\Gamma)$ that diagonalises with respect to the canonical orthonormal basis and has nontrivial sign.*

Remark 7.12 Let F be a subset of $\{1, 2, \ldots, \ell\}$. Define $\pi_{\omega,F}$ to be the representation obtained by integrating $\psi_\omega * \chi_{\mathbf{z}}$ with respect to those components z_i of $\mathbf{z}$ for which $i \in F$. If one looks at the representations $\pi_{\omega,F}$ instead of π_ω, a similar analysis will show that nontrivial spectral triples would exist only in the case where ω is of the form s_k (so that $\ell(\omega) = 1$), and $F = \{k\}$. The nontrivial triples in this case will essentially be those of $SU_q(2)$ obtained in [2] and will correspond to the 'k^{th} copy' of $SU_q(2)$ sitting inside $SU_q(\ell + 1)$ via the map

$$u_{ij} \mapsto \begin{cases} \alpha & \text{if } j = i = k, \\ \alpha^* & \text{if } j = i = k+1, \\ -q\beta^* & \text{if } j = k+1, i = k, \\ \beta & \text{if } j = k, i = k+1, \\ I & \text{if } i = j, \\ 0 & \text{otherwise.} \end{cases}$$

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