Equivariant spectral triples for $SU_q(\ell+1)$ and the odd dimensional quantum spheres

Partha Sarathi Chakraborty and Arupkumar Pal

February 1, 2008

Abstract

We formulate the notion of equivariance of an operator with respect to a covariant representation of a C^*-dynamical system. We then use a combinatorial technique used by the authors earlier in characterizing spectral triples for $SU_q(2)$ to investigate equivariant spectral triples for two classes of spaces: the quantum groups $SU_q(\ell+1)$ for $\ell > 1$, and the odd dimensional quantum spheres $S_q^{2\ell+1}$ of Vaksman & Soibelman. In the former case, a precise characterization of the sign and the singular values of an equivariant Dirac operator acting on the L_2 space is obtained. Using this, we then exhibit equivariant Dirac operators with nontrivial sign on direct sums of multiple copies of the L_2 space. In the latter case, viewing $S_q^{2\ell+1}$ as a homogeneous space for $SU_q(\ell+1)$, we give a complete characterization of equivariant Dirac operators, and also produce an optimal family of spectral triples with nontrivial K-homology class.

AMS Subject Classification No.: 58B34, 46L87, 19K33
Keywords. Spectral triples, noncommutative geometry, quantum group.

1 Introduction

Groups have always played a very crucial role in the study of geometry of a space, mainly as objects that govern the symmetry of the space. One would expect the same in noncommutative geometry also. Moreover, since one now deals with a larger class of spaces, mainly noncommutative ones, it is natural to expect that one would require a larger class, Hopf algebras or the quantum groups, to play a similar role. In the classical case, groups which govern symmetry are themselves nice geometric objects. Here we want to look at quantum groups from the same angle. In a previous paper ([3]), the authors treated the case of the quantum $SU(2)$ group and found a family of spectral triples acting on its L_2-space that are equivariant with respect to its natural (co)action. This family is optimal, in the sense that given any nontrivial equivariant Dirac operator D acting on the L_2 space, there exists a Dirac operator $\widetilde{D}$ belonging to this family such that $\operatorname{sign} D$ is a compact perturbation of $\operatorname{sign} \widetilde{D}$ and there exist reals a and b such that

$$|D| \leq a + b|\widetilde{D}|.$$

A generic triple from this family, that is also a generator of the K-homology group, was analysed by Connes in [8] where he used the general theory developed by him and Moscovici ([9]) to make elaborate computations and finally ended up with a local index formula. One beautiful and somewhat surprising observation in his paper was that the description of the cocycle given by the difference between the character of the triple and the cocycle for which index formula was given involved the Dedekind eta function. This gave further impetus to the construction of spectral triples for quantum groups and their homogeneous spaces ([10], [11], [12], [13], [17], [21]). It should perhaps be pointed out here that the construction by Krähmer ([17]) is algebraic in nature and does not address the crucial analytic issues involved in the definition of a spectral triple. The construction by Hawkins & Landi ([13]) on the other hand does not deal with equivariance; and more crucially, they restrict themselves to the construction of bounded Kasparov modules. But in Noncommutative geometry, spectral triples or the unbounded Kasparov modules are key ingredients, as they work as a looking glass allowing one to distinguish between continuous and smooth functions.

Our aim in the present paper is to look for higher dimensional counterparts of the spectral triples found in [3]. We first formulate precisely what one means by an equivariant spectral triple in a general set up (this is already implicit in [3]) and then study equivariant Dirac operators for two classes of spaces, both of which can be thaught of as higher dimensional analogues of $SU_q(2)$ which was worked out earlier. First, we analyse equivariant Dirac operators acting on the L_2-spaces of the groups $SU_q(\ell+1)$. We derive a precise expression for the singular values of an equivariant Dirac operator, and show that a Dirac operator with these singular values will have the correct summability property. We also show that for $\ell > 1$, an equivariant Dirac operator acting on $L_2(G)$ have to have trivial sign. Thus for $\ell > 1$, one would be forced to bring in multiplicity when looking for equivariant Dirac operators with nontrivial sign. Using this observation, we then exhibit a family of equivariant Dirac operators acting on direct sums of multiple copies of the L_2 space and having nontrivial sign. Whether these Dirac operators have nontrivial K-homology class is still not known. In the last section, we take up the odd dimensional quantum spheres $S_q^{2\ell+1}$. In this case, the outcome turns out to be more satisfactory. After characterizing the sign and the singular values of Dirac operators on $L_2(S_q^{2\ell+1})$ equivariant under the action of the group $SU_q(\ell+1)$, we produce, just like in the $SU_q(2)$ case, an optimum family of nontrivial equivariant Dirac operators that are $(2\ell+1)$-summable.

The paper is organised as follows. In the next section, we will recall from [2] the combinatorial method that was earlier used implicitly in [3] and [4]. In section 3, we formulate the notion of equivariance. This has been done using the quantum group at the function algebra level rather than passing on to the quantum universal envelopping algebra level. In section 4, we briefly recall the quantum group $SU_q(\ell+1)$ and its representation theory. In particular, we describe a nice basis for the L_2 space and study the Clebsch-Gordon coefficients. These are used in section 5 to describe the action by left multiplication on the L_2 space explicitly. In section 6, we write down the conditions coming from the boundedness of commutators with D. In sections 7 and 8, we analyze the equivariant Dirac operators for $SU_q(\ell+1)$. First we give a precise characterization of the singular values in section 7, and then a characterization of the sign in section 8. In section 9, we deal with the odd dimensional quantum spheres.

2 The general scheme

Let us recall the combinatorial set up from [2].

Suppose $\mathcal{H}$ is a Hilbert space, and D is a self-adjoint operator on $\mathcal{H}$ with compact resolvent. Then D admits a spectral resolution $\sum_{\gamma\in\Gamma} d_\gamma P_\gamma$, where the d_γ's are all distinct and each P_γ is a finite dimensional projection. Assume now onward that all the d_γ's are nonzero. Let c be a positive real. Let us define a graph $\mathcal{G}_c$ as follows: take the vertex set V to be Γ. Connect two vertices γ and γ' by an edge if $|d_\gamma - d_{\gamma'}| < c$. Let $V^+ = \{\gamma \in V : d_\gamma > 0\}$ and $V^- = \{\gamma \in V : d_\gamma < 0\}$. This will give us a partition of V. This partition has the following important property: there does not exist infinite number of disjoint paths each going from a point in V^+ to a point in V^-. Here disjoint paths mean paths for which the set of vertices of one does not intersect the set of vertices of the other. This is easy to see, because if there is a path from γ to δ and $d_\gamma > 0$, $d_\delta < 0$, then for some α on the path, one must have $d_\alpha \in [-c, c]$. Since the paths are disjoint, it would contradict the compact resolvent condition. We will call such a partition a sign-determining partition.

We will use this knowledge about the graph. We start with an equivariant operator that is self-adjoint and has discrete spectrum. Equivariance will give us an idea about the spectral resolution $\sum_{\gamma\in\Gamma} d_\gamma P_\gamma$. Next we use the action of the algebra elements on the basis elements of $\mathcal{H}$ and the boundedness of their commutators with D. This gives certain growth restrictions on the d_γ's. These will give us some information about the edges in the graph. We exploit this knowledge to characterize those partitions (V_1, V_2) of the vertex set that are sign-determining, i. e. do not admit any infinite ladder. The sign of the operator D must be of the form $\sum_{\gamma\in V_1} P_\gamma - \sum_{\gamma\in V_2} P_\gamma$ where (V_1, V_2) is a sign-determining partition. Of course, for a given c, the graph $\mathcal{G}_c$ may have no edges, or too few edges (if the singular values of D happen to grow too fast), in which case, we will be left with too many sign-determining partitions. Fortunately, the operators we are interested in are meant to be the Dirac operators of some commutative/noncommutative manifold. Therefore the singular values of D will grow at the rate of $O(n^{1/d})$ for some $d \geq 1$. So one can choose a large enough c and work with the graph $\mathcal{G}_c$. In other words, we would like to characterize those partitions that are sign-determining for all sufficiently large values of c.

3 Equivariance

Suppose G is a compact group, quantum or classical, and $\mathcal{A}$ is a unital C^*-algebra. Assume that G has an action on $\mathcal{A}$ given by $\tau : \mathcal{A} \to \mathcal{A} \otimes C(G)$, so that $(\mathrm{id} \otimes \Delta)\tau = (\tau \otimes \mathrm{id})\tau$, Δ being the coproduct. In other words, we have a C^*-dynamical system $(\mathcal{A}, G, \tau)$. Our goal is to study spectral triples for $\mathcal{A}$ equivariant under this action. Let us first say what we mean by 'equivariant' here.

A covariant representation (π, u) of $(\mathcal{A}, G, \tau)$ consists of a unital *-representation $\pi : \mathcal{A} \to \mathcal{L}(\mathcal{H})$, a unitary representation u of G on $\mathcal{H}$, i.e. a unitary element of the multiplier algebra $M(\mathcal{K}(\mathcal{H}) \otimes C(G))$ such that they obey the condition $(\pi \otimes \mathrm{id})\tau(a) = u(\pi(a) \otimes I)u^*$ for all $a \in \mathcal{A}$.

Definition 3.1 Suppose $(\mathcal{A}, G, \tau)$ is a C^*-dynamical system. An operator D acting on a Hilbert space $\mathcal{H}$ is said to be **equivariant** with respect to a covariant representation (π, u) of the system if $D \otimes I$ commutes with u.

Since the operator D is self-adjoint with compact resolvent, it will admit a spectral resolution $\sum_\lambda d_\lambda P_\lambda$, where the d_λ's are distinct and each P_λ is finite dimensional. Also, D has been assumed to be equivariant — so that the P_λ's commute with u (to be precise, the $(P_\lambda \otimes I)$'s do), i.e. u keeps each $P_\lambda\mathcal{H}$ invariant. As G is compact, each $P_\lambda\mathcal{H}$ will decompose further as $\oplus_\mu P_{\lambda\mu}\mathcal{H}$ such that the restriction of u to each $P_{\lambda\mu}$ is irreducible. In other words, one can now write D in the form $\sum_{\gamma\in\Gamma} d_\gamma P_\gamma$ for some index set Γ and a family of finite dimensional projections P_γ such that each P_γ commutes with u and the restriction of u to each P_γ is irreducible.

In this paper, we will deal with two cases, the group in question in both cases will be $G = SU_q(\ell+1)$. The C^*-algebra $\mathcal{A}$ on which the group acts will be $C(SU_q(\ell+1))$ in one case and $C(S_q^{2\ell+1})$ in the other. Let us discuss the first case a little here. The action τ here will be the natural action coming from the coproduct, $\mathcal{H}$ is $L_2(G)$, π is the representation of $\mathcal{A} = C(SU_q(\ell+1))$ on $\mathcal{H}$ by left multiplication, and u is the right regular representation. Structure of the regular representation of a compact (quantum) group along with the remarks made above tell us the following. Let Λ be the set of unitary irreducible representation-types for G. Then $\mathcal{H}$ decomposes as $\oplus_{\lambda\in\Lambda}\mathcal{H}_\lambda$, where the restriction of u to $\mathcal{H}_\lambda$ is equivalent to $\dim\lambda$ copies of the irreducible λ, and also that D respects this decomposition. Further, restriction of D to $\mathcal{H}_\lambda$ is of the form $\sum_\mu d_{\lambda\mu}P_{\lambda\mu}$, u commutes with each of these $P_{\lambda\mu}$'s, and the restriction of u to $P_{\lambda\mu}\mathcal{H}$ is equivalent to λ. Let N_λ be any set with $|N_\lambda| = \dim\lambda$. One can then choose an orthonormal basis $\{e^\lambda_{ij} : i, j \in N_\lambda\}$ such that the spaces $P_{\lambda\mu}\mathcal{H}$ are precisely span $\{e^\lambda_{ij} : j \in N_\lambda\}$ for distinct values of $i \in N_\lambda$. Since D is of the form $\sum_\lambda\sum_\mu d_{\lambda\mu}P_{\lambda\mu}$, in this system of bases, D will look like $e^\lambda_{ij} \mapsto d(\lambda, i)e^\lambda_{ij}$. In what follows, we will make a special choice of N_λ, which will make the combinatorial analysis very convenient.

4 Preliminaries on $SU_q(\ell+1)$

Let $\mathfrak{g}$ be a complex simple Lie algebra of rank ℓ. let $((a_{ij}))$ be the associated Cartan matrix, q be a real number lying in the interval $(0,1)$ and let $q_i = q^{(\alpha_i,\alpha_i)/2}$, where α_i's are the simple roots of $\mathfrak{g}$. Then the quantised universal envelopping algebra (QUEA) $U_q(\mathfrak{g})$ is the algebra generated by E_i, F_i, K_i and K_i^{-1}, $i=1,\ldots,\ell$, satisfying the following relations

$$K_iK_j = K_jK_i, \quad K_iK_i^{-1} = K_i^{-1}K_i = 1,$$

$$K_iE_jK_i^{-1} = q_i^{\frac{1}{2}a_{ij}}E_j, \quad K_iF_jK_i^{-1} = q_i^{-\frac{1}{2}a_{ij}}F_j,$$

$$E_iF_j - F_jE_i = \delta_{ij}\frac{K_i^2 - K_i^{-2}}{q_i - q_i^{-1}},$$

$$\sum_{r=0}^{1-a_{ij}} (-1)^r \binom{1-a_{ij}}{r}_{q_i} E_i^{1-a_{ij}-r}E_jE_i^r = 0 \quad \forall\, i \neq j,$$

$$\sum_{r=0}^{1-a_{ij}} (-1)^r \binom{1-a_{ij}}{r}_{q_i} F_i^{1-a_{ij}-r}F_jF_i^r = 0 \quad \forall\, i \neq j,$$

where $\binom{n}{r}_q$ denote the q-binomial coefficients. Hopf *-structure comes from the following maps:

$$\Delta(K_i) = K_i \otimes K_i, \quad \Delta(K_i^{-1}) = K_i^{-1} \otimes K_i^{-1},$$

$$\Delta(E_i) = E_i \otimes K_i + K_i^{-1} \otimes E_i, \quad \Delta(F_i) = F_i \otimes K_i + K_i^{-1} \otimes F_i,$$

$$\epsilon(K_i) = 1, \quad \epsilon(E_i) = 0 = \epsilon(F_i),$$

$$S((K_i) = K_i^{-1}, \quad S(E_i) = -q_iE_i, \quad S(F_i) = -q_i^{-1}F_i,$$

$$K_i^* = K_i, \quad E_i^* = -q_i^{-1}F_i, \quad F_i^* = -q_iE_i.$$

In the type A case, the associated Cartan matrix is given by

$$a_{ij} = \begin{cases} 2 & \text{if } i = j, \\ -1 & \text{if } i = j \pm 1, \\ 0 & \text{otherwise,} \end{cases}$$

and $(\alpha_i,\alpha_i) = 2$ so that $q_i = q$ for all i. The QUEA in this case is denoted by $u_q(su(\ell+1))$.

Take the collection of matrix entries of all finite-dimensional unitarizable $u_q(su(\ell+1))$-modules. The algebra generated by these gets a natural Hopf*-structure as the dual of $u_q(su(\ell+1))$. One can also put a natural C^*-norm on this. Upon completion with respect to this norm, one gets a unital C^*-algebra that plays the role of the algebra of continuous functions on $SU_q(\ell+1)$. For a detailed account of this, refer to chapter 3, [16]. In [23], Woronowicz gave a different description of this C^*-algebra. which was later shown by Rosso ([20]) to be equivalent to the earlier one.

For remainder of this article, we will take G to be $SU_q(\ell+1)$ and $\mathcal{A}$ will be the C^*-algebra of continuous functions on G.

Gelfand-Tsetlin tableaux. Irreducible unitary representations of the group $SU_q(\ell+1)$ are indexed by Young tableaux $\lambda = (\lambda_1, \ldots, \lambda_{\ell+1})$, where λ_i's are nonnegative integers, $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_{\ell+1}$ (Theorem 1.5, [23]). Write $\mathcal{H}_\lambda$ for the Hilbert space where the irreducible λ acts. There are various ways of indexing the basis elements of $\mathcal{H}_\lambda$. The one we will use is due to Gelfand and Tsetlin. According to their prescription, basis elements for $\mathcal{H}_\lambda$ are parametrized by arrays of the form

$$\mathbf{r} = \begin{pmatrix} r_{11} & r_{12} & \cdots & r_{1,\ell} & r_{1,\ell+1} \\ r_{21} & r_{22} & \cdots & r_{2,\ell} & \\ & \cdots & & & \\ r_{\ell,1} & r_{\ell,2} & & & \\ r_{\ell+1,1} & & & & \end{pmatrix},$$

where r_{ij}'s are integers satisfying $r_{1j} = \lambda_j$ for $j = 1, \ldots, \ell+1$, $r_{ij} \geq r_{i+1,j} \geq r_{i,j+1} \geq 0$ for all i, j. Such arrays are known as Gelfand-Tsetlin tableaux, to be abreviated as GT tableaux for the rest of this section. For a GT tableaux $\mathbf{r}$, the symbol $\mathbf{r}_{i\cdot}$ will denote its i^{th} row. It is well-known that two representations indexed respectively by λ and λ' are equivalent if and only if $\lambda_j - \lambda'_j$ is independent of j ([23]). Thus one gets an equivalence relation on the set of Young tableaux $\{\lambda = (\lambda_1, \ldots, \lambda_{\ell+1}) : \lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_{\ell+1}, \lambda_j \in \mathbb{N}\}$. This, in turn, induces an equivalence relation on the set of all GT tableaux $\Gamma = \{\mathbf{r} : r_{ij} \in \mathbb{N}, r_{ij} \geq r_{i+1,j} \geq r_{i,j+1}\}$: one says $\mathbf{r}$ and $\mathbf{s}$ are equivalent if $r_{ij} - s_{ij}$ is independent of i and j. By Γ we will mean the above set modulo this equivalence.

We will denote by u^λ the irreducible unitary indexed by λ, $\{e(\lambda, \mathbf{r}) : \mathbf{r}_{1\cdot} = \lambda\}$ will denote an orthonormal basis for $\mathcal{H}_\lambda$ and $u^\lambda_{\mathbf{rs}}$ will stand for the matrix entries of u^λ in this basis. The symbol $\mathbb{1}$ will denote the Young tableaux $(1, 0, \ldots, 0)$. We will often omit the symbol $\mathbb{1}$ and just write u in order to denote $u^{\mathbb{1}}$. Notice that any GT tableaux $\mathbf{r}$ with first row $\mathbb{1}$ must be, for some $i \in \{1, 2, \ldots, \ell+1\}$, of the form (r_{ab}), where

$$r_{ab} = \begin{cases} 1 & \text{if } 1 \leq a \leq i \text{ and } b = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Thus such a GT tableaux is uniquely determined by the integer i. We will write just i for this GT tableaux $\mathbf{r}$. Thus for example, a typical matrix entry of $u^{\mathbb{1}}$ will be written simply as u_{ij}.

Let $\mathbf{r} = (r_{ab})$ be a GT tableaux. Let $H_{ab}(\mathbf{r}) := r_{a+1,b} - r_{a,b+1}$ and $V_{ab}(\mathbf{r}) := r_{ab} - r_{a+1,b}$. An element $\mathbf{r}$ of Γ is completely specified by the following differences

$$\mathbf{D}(\mathbf{r}) = \begin{pmatrix} V_{11}(\mathbf{r}) & H_{11}(\mathbf{r}) & H_{12}(\mathbf{r}) & \cdots & H_{1,\ell-1}(\mathbf{r}) & H_{1,\ell}(\mathbf{r}) \\ V_{21}(\mathbf{r}) & H_{21}(\mathbf{r}) & H_{22}(\mathbf{r}) & \cdots & H_{2,\ell-1}(\mathbf{r}) & \\ & \cdots & & & & \\ V_{\ell,1}(\mathbf{r}) & H_{\ell,1}(\mathbf{r}) & & & & \end{pmatrix}.$$

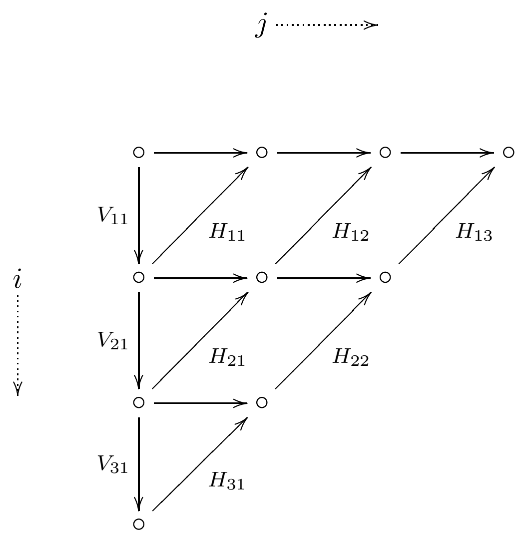
The differences satisfy the following inequalities

$$\sum_{k=0}^{b} H_{a-k,k+1}(\mathbf{r}) \leq V_{a+1,1}(\mathbf{r}) + \sum_{k=0}^{b} H_{a-k+1,k+1}(\mathbf{r}), \quad 1 \leq a \leq \ell, \ \ 0 \leq b \leq a-1. \tag{4.1}$$

Conversely, if one has an array of the form

$$\begin{pmatrix} V_{11} & H_{11} & H_{12} & \cdots & H_{1,\ell-1} & H_{1,\ell} \\ V_{21} & H_{21} & H_{22} & \cdots & H_{2,\ell-1} & \\ & \cdots & & & & \\ V_{\ell,1} & H_{\ell,1} & & & & \end{pmatrix},$$

where V_{ij}'s and H_{ij}'s are in $\mathbb{N}$ and obey the inequalities (4.1), then the above array is of the form $\mathbf{D}(\mathbf{r})$ for some GT tableaux $\mathbf{r}$. Thus the quantities V_{a1} and H_{ab} give a coordinate system for elements in Γ. The following diagram explains this new coordinate system. The hollow circles stand for the r_{ij}'s. The entries are decreasing along the direction of the arrows, and the V_{ij}'s and the H_{ij}'s are the difference between the two endpoints of the corresponding arrows.



Clebsch-Gordon coefficients. Look at the representation $u^{\mathbb{1}} \otimes u^{\lambda}$ acting on $\mathcal{H}_{\mathbb{1}} \otimes \mathcal{H}_{\lambda}$. The representation decomposes as a direct sum $\oplus_{\mu} u^{\mu}$, i.e. one has a corresponding decomposition $\oplus_{\mu} \mathcal{H}_{\mu}$ of $\mathcal{H}_{\mathbb{1}} \otimes \mathcal{H}_{\lambda}$. Thus one has two orthonormal bases $\{e^{\mu}_{\mathbf{s}}\}$ and $\{e^{\mathbb{1}}_{i} \otimes e^{\lambda}_{\mathbf{r}}\}$. The Clebsch-Gordon coefficient $C_q(\mathbb{1}, \lambda, \mu; i, \mathbf{r}, \mathbf{s})$ is defined to be the inner product $\langle e^{\mu}_{\mathbf{s}}, e^{\mathbb{1}}_{i} \otimes e^{\lambda}_{\mathbf{r}} \rangle$. Since $\mathbb{1}$, λ and μ are just the first rows of i, $\mathbf{r}$ and $\mathbf{s}$ respectively, we will often denote the above quantity just by $C_q(i, \mathbf{r}, \mathbf{s})$.

Next, we will compute the quantities $C_q(i, \mathbf{r}, \mathbf{s})$. We will use the calculations given in ([15], pp. 220), keeping in mind that for our case (i.e. for $SU_q(\ell+1)$), the top right entry of the GT tableaux is zero.

Let $M = (m_1, m_2, \ldots, m_i) \in \mathbb{N}^i$ be such that $1 \le m_j \le \ell + 2 - j$. Denote by $M(\mathbf{r})$ the tableaux $\mathbf{s}$ defined by

$$s_{jk} = \begin{cases} r_{jk} + 1 & \text{if } k = m_j,\ 1 \le j \le i, \\ r_{jk} & \text{otherwise.} \end{cases} \tag{4.2}$$

With this notation, observe now that $C_q(i,\mathbf{r},\mathbf{s})$ will be zero unless $\mathbf{s}$ is $M(\mathbf{r})$ for some $M\in\mathbb{N}^i$. (One has to keep in mind though that not all tableaux of the form $M(\mathbf{r})$ is a valid GT tableaux)

From ([15], pp. 220), we have

$$C_q(i,\mathbf{r},M(\mathbf{r})) = \prod_{a=1}^{i-1}\left\langle \begin{array}{cc|c} (1,\mathbf{0}) & \mathbf{r}_{a\cdot} & \mathbf{r}_{a\cdot}+e_{m_a} \\ (1,\mathbf{0}) & \mathbf{r}_{a+1\cdot} & \mathbf{r}_{a+1\cdot}+e_{m_{a+1}} \end{array}\right\rangle \times \left\langle \begin{array}{cc|c} (1,\mathbf{0}) & \mathbf{r}_{i\cdot} & \mathbf{r}_{i\cdot}+e_{m_i} \\ (0,\mathbf{0}) & \mathbf{r}_{i+1\cdot} & \mathbf{r}_{i+1\cdot} \end{array}\right\rangle, \tag{4.3}$$

where e_k stands for a vector (in the appropriate space) whose k^{th} coordinate is 1 and the rest are all zero, and

$$\begin{aligned} \left\langle \begin{array}{cc|c} (1,\mathbf{0}) & \mathbf{r}_{a\cdot} & \mathbf{r}_{a\cdot}+e_j \\ (1,\mathbf{0}) & \mathbf{r}_{a+1\cdot} & \mathbf{r}_{a+1\cdot}+e_k \end{array}\right\rangle^2 &= q^{-r_{aj}+r_{a+1,k}-k+j}\times\prod_{\substack{i=1\\ i\neq j}}^{\ell+2-a}\frac{[r_{a,i}-r_{a+1,k}-i+k]_q}{[r_{a,i}-r_{a,j}-i+j]_q} \\ &\quad\times\prod_{\substack{i=1\\ i\neq k}}^{\ell+1-a}\frac{[r_{a+1,i}-r_{a,j}-i+j-1]_q}{[r_{a+1,i}-r_{a+1,k}-i+k-1]_q}, \end{aligned}\tag{4.4}$$

$$\begin{aligned} \left\langle \begin{array}{cc|c} (1,\mathbf{0}) & \mathbf{r}_{a\cdot} & \mathbf{r}_{a\cdot}+e_j \\ (0,\mathbf{0}) & \mathbf{r}_{a+1\cdot} & \mathbf{r}_{a+1\cdot} \end{array}\right\rangle^2 &= q^{\left(1-j+\sum_{i=1}^{\ell+1-a}r_{a+1,i}-\sum_{\substack{i=1\\ i\neq j}}^{\ell+2-a}r_{a,i}\right)} \\ &\quad\times\left(\frac{\prod_{i=1}^{\ell+1-a}[r_{a+1,i}-r_{aj}-i+j-1]_q}{\prod_{\substack{i=1\\ i\neq j}}^{\ell+2-a}[r_{a,i}-r_{aj}-i+j]_q}\right), \end{aligned}\tag{4.5}$$

where for an integer n, $[n]_q$ denotes the q-number $(q^n-q^{-n})/(q-q^{-1})$. After some lengthy but straightforward computations, we get the following two relations:

$$\left|\left\langle \begin{array}{cc|c} (1,\mathbf{0}) & \mathbf{r}_{a\cdot} & \mathbf{r}_{a\cdot}+e_j \\ (1,\mathbf{0}) & \mathbf{r}_{a+1\cdot} & \mathbf{r}_{a+1\cdot}+e_k \end{array}\right\rangle\right| = A'q^A, \tag{4.6}$$

$$\left|\left\langle \begin{array}{cc|c} (1,\mathbf{0}) & \mathbf{r}_{a\cdot} & \mathbf{r}_{a\cdot}+e_j \\ (0,\mathbf{0}) & \mathbf{r}_{a+1\cdot} & \mathbf{r}_{a+1\cdot} \end{array}\right\rangle\right| = B'q^B, \tag{4.7}$$

where

$$\begin{aligned} A &= \begin{cases} \sum_{j\wedge k<b<j\vee k}(r_{a+1,b}-r_{a,b})+(r_{a+1,j\wedge k}-r_{a,j\vee k}) & \text{if } j\neq k, \\ 0 & \text{if } j=k. \end{cases} \\ &= \sum_{j\wedge k\leq b<j\vee k}(r_{a+1,b}-r_{a,b+1})+2\sum_{k<b<j}(r_{a,b}-r_{a+1,b}) \\ &= \sum_{j\wedge k\leq b<j\vee k}H_{ab}(\mathbf{r})+2\sum_{k<b<j}V_{ab}(\mathbf{r}). \end{aligned}\tag{4.8}$$

$$B = \sum_{j\leq b<\ell+2-a}H_{ab}(\mathbf{r}), \tag{4.9}$$

and A' and B' both lie between two positive constants independent of $\mathbf{r}$, a, j and k (Here and elsewhere in this paper, an empty summation would always mean zero).

Combining these, one gets

$$C_q(i,\mathbf{r},M(\mathbf{r})) = P\cdot q^{C(i,\mathbf{r},M)}, \tag{4.10}$$

where

$$C(i,\mathbf{r},M) = \sum_{a=1}^{i-1}\left(\sum_{m_a\wedge m_{a+1}\le b<m_a\vee m_{a+1}} H_{ab}(\mathbf{r}) + 2\sum_{m_{a+1}<b<m_a} V_{ab}(\mathbf{r})\right) + \sum_{m_i\le b<\ell+2-i} H_{ib}(\mathbf{r}), \tag{4.11}$$

and P lies between two positive constants that are independent of i, $\mathbf{r}$ and M.

Remark 4.1 The formulae (4.4) and (4.5) are obtained from equations (45) and (46), page 220, [15] by replacing q with q^{-1}. Equation (45) is a special case of the more general formula (48), page 221, [15]. However, there is a small error in equation (48) there. The correct form can be found in equations (3.1, 3.2a, 3.2b) in [1]. That correction has been incorporated in equations (4.4) and (4.5) here.

5 Left multiplication operators

The matrix entries $u^\lambda_{\mathbf{rs}}$ form a complete orthogonal set of vectors in $L_2(G)$. Write $e^\lambda_{\mathbf{rs}}$ for $\|u^\lambda_{\mathbf{rs}}\|^{-1}u^\lambda_{\mathbf{rs}}$. Then the $e^\lambda_{\mathbf{rs}}$'s form a complete orthonormal basis for $L_2(G)$. Let π denote the representation of $\mathcal{A}$ on $L_2(G)$ by left multiplications. We will now derive an expression for $\pi(u_{ij})e^\lambda_{\mathbf{rs}}$.

From the definition of matrix entries and that of the CG coefficients, one gets

$$u^\rho e(\rho,\mathbf{t}) = \sum_{\mathbf{s}} u^\rho_{\mathbf{st}} e(\rho,\mathbf{s}), \tag{5.1}$$

$$e(\mu,\mathbf{n}) = \sum_{j,\mathbf{s}} C_q(j,\mathbf{s},\mathbf{n}) e(\mathbb{1},j)\otimes e(\lambda,\mathbf{s}). \tag{5.2}$$

Apply $u\otimes u^\lambda$ on both sides and note that $u\otimes u^\lambda$ acts on $e(\mu,\mathbf{n})$ as u^μ:

$$\sum_{\mathbf{m}} u^\mu_{\mathbf{mn}} e(\mu,\mathbf{m}) = \sum_{j,\mathbf{s}}\sum_{i,\mathbf{r}} C_q(j,\mathbf{s},\mathbf{n}) u_{ij} u^\lambda_{\mathbf{rs}} e(\mathbb{1},i)\otimes e(\lambda,\mathbf{r}). \tag{5.3}$$

Next, use (5.2) to expand $e(\mu,\mathbf{m})$ on the left hand side to get

$$\sum_{i,\mathbf{r},\mathbf{m}} u^\mu_{\mathbf{mn}} C_q(i,\mathbf{r},\mathbf{m}) e(\mathbb{1},i)\otimes e(\lambda,\mathbf{r}) = \sum_{j,\mathbf{s}}\sum_{i,\mathbf{r}} C_q(j,\mathbf{s},\mathbf{n}) u_{ij} u^\lambda_{\mathbf{rs}} e(\mathbb{1},i)\otimes e(\lambda,\mathbf{r}). \tag{5.4}$$

Equating coefficients, one gets

$$\sum_{\mathbf{m}} C_q(i,\mathbf{r},\mathbf{m}) u^\mu_{\mathbf{mn}} = \sum_{j,\mathbf{s}} C_q(j,\mathbf{s},\mathbf{n}) u_{ij} u^\lambda_{\mathbf{rs}}. \tag{5.5}$$

Now using orthogonality of the matrix $((C_q(\mathbb{1},\lambda,\mu;j,\mathbf{s},\mathbf{n})))_{(\mu,\mathbf{n}),(j,\mathbf{s})}$, we obtain

$$u_{ij}u^{\lambda}_{\mathbf{rs}} = \sum_{\mu,\mathbf{m},\mathbf{n}} C_q(i,\mathbf{r},\mathbf{m})C_q(j,\mathbf{s},\mathbf{n})u^{\mu}_{\mathbf{mn}}. \tag{5.6}$$

From ([15], pp. 441), one has $\|u^{\lambda}_{\mathbf{rs}}\| = d_{\lambda}^{-\frac{1}{2}}q^{-\psi(\mathbf{r})}$, where

$$\psi(\mathbf{r}) = -\frac{\ell}{2}\sum_{j=1}^{\ell+1} r_{1j} + \sum_{i=2}^{\ell+1}\sum_{j=1}^{\ell+2-i} r_{ij}, \qquad d_\lambda = \sum_{\mathbf{r}:\mathbf{r}_1=\lambda} q^{2\psi(\mathbf{r})}$$

Therefore

$$\pi(u_{ij})e^{\lambda}_{\mathbf{rs}} = \sum_{\mu,\mathbf{m},\mathbf{n}} C_q(\mathbb{1},\lambda,\mu;i,\mathbf{r},\mathbf{m})C_q(\mathbb{1},\lambda,\mu;j,\mathbf{s},\mathbf{n})d_{\lambda}^{\frac{1}{2}}d_{\mu}^{-\frac{1}{2}}q^{\psi(\mathbf{r})-\psi(\mathbf{m})}e^{\mu}_{\mathbf{mn}}. \tag{5.7}$$

Write

$$\kappa(\mathbf{r},\mathbf{m}) = d_{\lambda}^{\frac{1}{2}}d_{\mu}^{-\frac{1}{2}}q^{\psi(\mathbf{r})-\psi(\mathbf{m})}. \tag{5.8}$$

Lemma 5.1 *There exist constants $K_2 > K_1 > 0$ such that $K_1 < \kappa(\mathbf{r}, M(\mathbf{r})) < K_2$ for all $\mathbf{r}$.*

Proof: Observe that ([5], pp-365)

$$d_\lambda = \prod_{1\le i\le j\le \ell+1} \frac{[\lambda_i-\lambda_j+j-i]_q}{[j-i]_q}.$$

Therefore one gets

$$\frac{d_\lambda}{d_{\lambda+e_k}} = \prod_{j:k<j}\frac{[\lambda_k-\lambda_j+j-k]_q}{[\lambda_k-\lambda_j+j-k+1]_q} \times \prod_{i:i<k}\frac{[\lambda_i-\lambda_k+k-i]_q}{[\lambda_i-\lambda_k+k-i-1]_q}.$$

There are ℓ terms in the above product, and each term lies between two positive quantities that depend just on q. Next, we have

$$\psi(\mathbf{r}) = -\frac{\ell}{2}\sum_{j=1}^{\ell+1} r_{1j} + \sum_{i=2}^{\ell+1}\sum_{j=1}^{\ell+2-i} r_{ij}.$$

It follows from this that $\psi(\mathbf{r})-\psi(\mathbf{m})$ is bounded. Therefore the result follows. $\square$

6 Boundedness of commutators

Let D be an equivariant Dirac operator acting on $L_2(G)$. It follows from the discussion in section 3 that D must be of the form

$$e^{\lambda}_{\mathbf{rs}} \mapsto d(\mathbf{r})e^{\lambda}_{\mathbf{rs}}, \tag{6.1}$$

(Here, for a Young tableaux λ, N_λ is the set of all GT tableaux, modulo the appropriate equivalence relation, with top row λ). Then we have

$$[D,\pi(u_{ij})]e^{\lambda}_{\mathbf{rs}} = \sum (d(\mathbf{m}) - d(\mathbf{r}))C_q(\mathbb{1},\lambda,\mu;i,\mathbf{r},\mathbf{m})C_q(\mathbb{1},\lambda,\mu;j,\mathbf{s},\mathbf{n})\kappa(\mathbf{r},\mathbf{m})e^{\mu}_{\mathbf{mn}}. \qquad (6.2)$$

Therefore the condition for boundedness of commutators reads as follows:

$$|(d(\mathbf{m}) - d(\mathbf{r}))C_q(\mathbb{1},\lambda,\mu;i,\mathbf{r},\mathbf{m})C_q(\mathbb{1},\lambda,\mu;j,\mathbf{s},\mathbf{n})\kappa(\mathbf{r},\mathbf{m})| < c, \qquad (6.3)$$

where c is independent of i, j, λ, μ, $\mathbf{r}$, $\mathbf{s}$, $\mathbf{m}$ and $\mathbf{n}$.

Using lemma 5.1, we get

$$|(d(\mathbf{m}) - d(\mathbf{r}))C_q(\mathbb{1},\lambda,\mu;i,\mathbf{r},\mathbf{m})C_q(\mathbb{1},\lambda,\mu;j,\mathbf{s},\mathbf{n})| < c. \qquad (6.4)$$

Choosing j, $\mathbf{s}$ and $\mathbf{n}$ suitably, one can ensure that (6.4) implies the following:

$$|(d(\mathbf{m}) - d(\mathbf{r}))C_q(\mathbb{1},\lambda,\mu;i,\mathbf{r},\mathbf{m})| < c. \qquad (6.5)$$

It follows from (6.2) that this condition is also sufficient for the boundedness of the commutators $[D,u_{ij}]$.

From (4.10), one gets

$$|d(\mathbf{r}) - d(M(\mathbf{r}))| \leq cq^{-C(i,\mathbf{r},M)}. \qquad (6.6)$$

Let us next form a graph $\mathcal{G}_c$ as described in section 1 by connecting two elements $\mathbf{r}$ and $\mathbf{r}'$ if $|d(\mathbf{r}) - d(\mathbf{r}')| < c$. We will assume the existence of a partition (Γ^+,Γ^-) that does not admit any infinite ladder. For any subset F of Γ, we will denote by $F^\pm$ the sets $F\cap\Gamma^\pm$. Our next job is to study this graph in more detail using the boundedness conditions above. Let us start with a few definitions and notations. By an **elementary move**, we will mean a map M from some subset of Γ to Γ such that γ and $M(\gamma)$ are connected by an edge. A **move** will mean a composition of a finite number of elementary moves. If M_1 and M_2 are two moves, M_1M_2 and M_2M_1 will in general be different. For a family of moves $M_1, M_2, \ldots, M_r$, we will denote by $\sum_{j=1}^r M_j$ the move $M_1M_2\ldots M_r$, and by $\sum_{j=1}^r M_{r+1-j}$ the move $M_r\ldots M_2M_1$. For a nonnegative integer n and a move M, we will denote by nM the move obtained by applying M successively n times. Of special interest to us will be moves of the form $M:\mathbf{r}\mapsto\mathbf{s}$, where $\mathbf{s}$ is given by (4.2). We will use the vector $(m_1,\ldots,m_k)$ to denote M. The following families of moves will be particularly useful to us:

$$M_{ik} = (i, i-1,\ldots,i-k+1)\in\mathbb{N}^k, \quad N_{ik} = (\underbrace{i+1,\ldots,i+1}_{k}, i, i, \ldots, i)\in\mathbb{N}^{\ell+2-i}.$$

For describing a path in our graph, we will often use phrases like 'apply the move $\sum_{j=1}^k M_j$ to go from $\mathbf{r}$ to $\mathbf{s}$'. This will refer to the path given by

$$\Big(\mathbf{r},\ M_k(\mathbf{r}), M_{k-1}M_k(\mathbf{r}),\ \ldots,\ M_1M_2\ldots M_k(\mathbf{r}) = \mathbf{s}\Big).$$

The following lemma will be very useful in the next two sections.

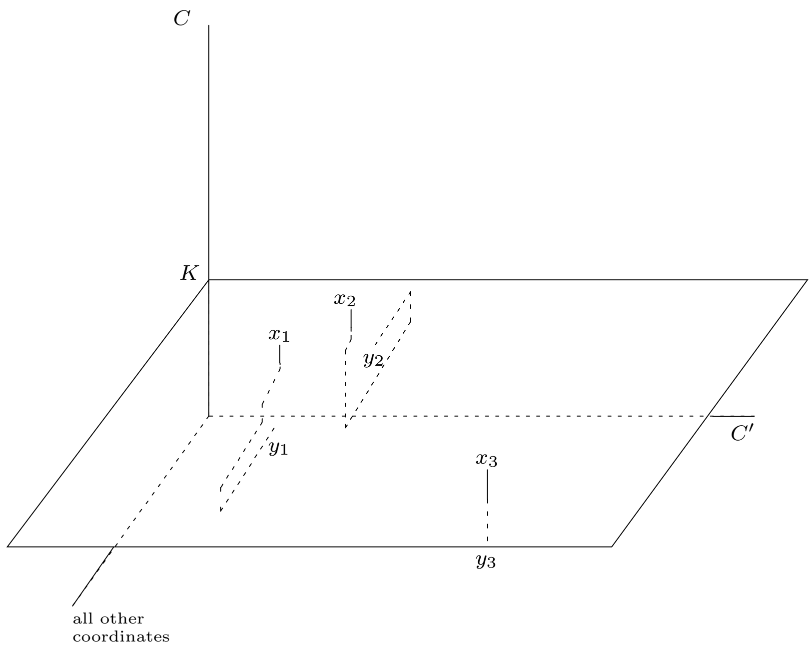
Lemma 6.1 *Let N_{jk} and M_{ik} be the moves defined above. Then*

1. $|d(\mathbf{r}) - d(N_{j0}(\mathbf{r}))| \leq c$,
2. $|d(\mathbf{r}) - d(M_{ik}(\mathbf{r}))| \leq cq^{-\sum_{a=1}^{k-1} H_{a,i+1-a} - \sum_{b=i}^{\ell} H_{k,b+k-1}}$. *In particular, if* $H_{a,i+1-a}(\mathbf{r}) = 0$ *for* $1 \leq a \leq k-1$ *and* $H_{k,b+k-1}(\mathbf{r}) = 0$ *for* $i \leq b \leq \ell$, *then* $|d(\mathbf{r}) - d(M_{ik}(\mathbf{r}))| \leq c$.

Proof: Direct consequence of (6.6). □

7 Characterization of $|D|$

In this section and the next, we will use lemma 6.1 to prove a characterization theorem for the sign of the operator D. Along the way, we will also give a very precise description of the singular values of D. The main ingredients in the proof are the finiteness of exactly one of the sets F^+ and F^- for appropriately chosen subsets F of Γ. General form of the argument for proving this will be as follows: for a carefully chosen coordinate C (in the present case, C would be one of the V_{a1}'s or H_{ab}'s), a sweepout argument will show that any γ can be connected by a path, throughout which $C(\cdot)$ remains constant, to another point γ' for which $C(\gamma') = C(\gamma)$ and all other coordinates of γ' are zero. This would help connect any two points γ and δ by a path such that $C(\cdot)$ would lie between $C(\gamma)$ and $C(\delta)$ on the path. This would finally result in the finiteness of at least one (and hence exactly one) of $C(F^+)$ and $C(F^-)$. Next, assuming one of these, say $C(F^-)$ is finite, one shows that for any other coordinate C', $C'(F^-)$ is also finite. This is done as follows. If $C'(F^-)$ is infinite, one chooses elements $y_n \in F^-$ with $C'(y_n) < C'(y_{n+1})$ for all n. Now starting at each y_n, produce paths keeping the C'-coordinate constant and taking the C-coordinate above the plane $C(\cdot) = K$, where $C(F^-) \subseteq [-K, K]$. This will produce an infinite ladder. The argument is explained in the following diagram.



Our next job is to define an important class of subsets of Γ. Observe that lemma 6.1 tells us that for any $\mathbf{r}$ and any j, the points $\mathbf{r}$ and $N_{j0}(\mathbf{r})$ are connected by an edge, whenever $N_{j0}(\mathbf{r})$ is a GT tableaux. Let $\mathbf{r}$ be an element of Γ. Define the **free plane passing through r** to be the minimal subset of Γ that contains $\mathbf{r}$ and is closed under application of the moves N_{j0}. We will denote this set by $\mathscr{F}_{\mathbf{r}}$. The following is an easy consequence of this definition.

Lemma 7.1 *Let* $\mathbf{r}$ *and* $\mathbf{s}$ *be two GT tableaux. Then* $\mathbf{s} \in \mathscr{F}_{\mathbf{r}}$ *if and only if* $V_{a,1}(\mathbf{r}) = V_{a,1}(\mathbf{s})$ *for all* a *and for each* b*, the difference* $H_{a,b}(\mathbf{r}) - H_{a,b}(\mathbf{s})$ *is independent of* a*.*

Corollary 7.2 *Let* $\mathbf{r}, \mathbf{s} \in \Gamma$*. Then either* $\mathscr{F}_{\mathbf{r}} = \mathscr{F}_{\mathbf{s}}$ *or* $\mathscr{F}_{\mathbf{r}} \cap \mathscr{F}_{\mathbf{s}} = \phi$*.*

Let $\mathbf{r} \in \Gamma$. For $1 \leq j \leq \ell + 1$, define a_j to be an integer such that $H_{a_j,j}(\mathbf{r}) = \min_i H_{ij}(\mathbf{r})$. Note three things here:
1. definition of a_j depends on $\mathbf{r}$,
2. for a given j and given $\mathbf{r}$, a_j need not be unique, and
3. if $\mathbf{s} \in \mathscr{F}_{\mathbf{r}}$, then for each j, the set of k's for which $H_{kj}(\mathbf{s}) = \min_i H_{ij}(\mathbf{s})$ is same as the set of all k's for which $H_{kj}(\mathbf{r}) = \min_i H_{ij}(\mathbf{r})$. Therefore, the a_j's can be chosen in a manner such that they remain the same for all elements lying on a given free plane.

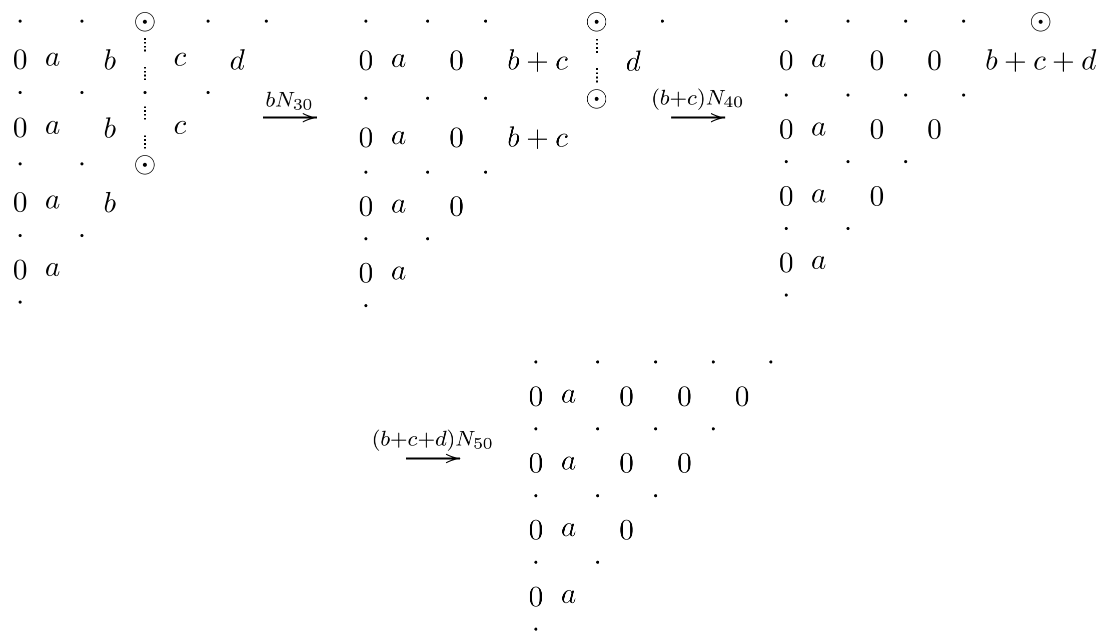
Lemma 7.3 *Let* $\mathbf{s} \in \mathscr{F}_{\mathbf{r}}$*. Let* $\mathbf{s}'$ *be another GT tableaux given by*

$$V_{a1}(\mathbf{s}') = V_{a1}(\mathbf{s}) \text{ and } H_{a1}(\mathbf{s}') = H_{a1}(\mathbf{s}) \text{ for all } a, \quad H_{a_b,b}(\mathbf{s}') = 0 \text{ for all } b > 1,$$

where the a_j*'s are as defined above. Then there is a path in* $\mathscr{F}_{\mathbf{r}}$ *from* $\mathbf{s}$ *to* $\mathbf{s}'$ *such that* $H_{11}(\cdot)$ *remains constant throughout this path.*

Proof: Apply the move $\sum_{b=2}^{\ell}\left(\sum_{j=2}^{\ell+2-b} H_{a_j,j}(\mathbf{s})\right) N_{\ell+3-b,0}$. $\square$

The following diagram will help explain the steps involved in the above proof in the case where $\mathbf{r}$ is the constant tableaux.



A dotted line joining two circled dots signifies a move that increases the r_{ij}'s lying on the dotted line by one. Where there is one circled dot and no dotted line, it means one applies the move that raises the r_{ij} corresponding to the circled dot by one.

Proposition 7.4 *Let $\mathbf{r}$ be a GT tableaux. Then either $\mathscr{F}_{\mathbf{r}}^{+}$ is finite or $\mathscr{F}_{\mathbf{r}}^{-}$ is finite.*

Proof: Suppose, if possible, both $H_{11}(\mathscr{F}_{\mathbf{r}}^{+})$ and $H_{11}(\mathscr{F}_{\mathbf{r}}^{-})$ are infinite. Then there exist two sequences of elements $\mathbf{r}_n$ and $\mathbf{s}_n$ with $\mathbf{r}_n \in \mathscr{F}_{\mathbf{r}}^{+}$ and $\mathbf{s}_n \in \mathscr{F}_{\mathbf{r}}^{-}$, such that

$$H_{11}(\mathbf{r}_1) < H_{11}(\mathbf{s}_1) < H_{11}(\mathbf{r}_2) < H_{11}(\mathbf{s}_2) < \cdots.$$

Now starting from $\mathbf{r}_n$, employ the forgoing lemma to reach a point $\mathbf{r}'_n \in \mathscr{F}_{\mathbf{r}}$ for which

$$V_{a1}(\mathbf{r}'_n) = V_{a1}(\mathbf{r}_n) \text{ and } H_{a1}(\mathbf{r}'_n) = H_{a1}(\mathbf{r}_n) \text{ for all } a, \quad H_{a_b,b}(\mathbf{r}'_n) = 0 \text{ for all } b > 1.$$

Similarly, start at $\mathbf{s}_n$ and go to a point $\mathbf{s}'_n \in \mathscr{F}_{\mathbf{r}}$ for which

$$V_{a1}(\mathbf{s}'_n) = V_{a1}(\mathbf{s}_n) \text{ and } H_{a1}(\mathbf{s}'_n) = H_{a1}(\mathbf{s}_n) \text{ for all } a, \quad H_{a_b,b}(\mathbf{s}'_n) = 0 \text{ for all } b > 1.$$

Now use the move N_{10} to get to $\mathbf{s}'_n$ from $\mathbf{r}'_n$. The paths thus constructed are all disjoint, because for the path from $\mathbf{r}_n$ to $\mathbf{s}_n$, the H_{11} coordinate lies between $H_{11}(\mathbf{r}_n)$ and $H_{11}(\mathbf{s}_n)$. This means $(\mathscr{F}_{\mathbf{r}}^{+}, \mathscr{F}_{\mathbf{r}}^{-})$ admits an infinite ladder. So one of the sets $H_{11}(\mathscr{F}_{\mathbf{r}}^{+})$ and $H_{11}(\mathscr{F}_{\mathbf{r}}^{-})$ must be finite. Let us assume that $H_{11}(\mathscr{F}_{\mathbf{r}}^{-})$ is finite.

Let us next show that for any $b > 1$, $H_{ab}(\mathscr{F}_{\mathbf{r}}^-)$ is finite. Let K be an integer such that $H_{11}(\mathbf{s}) < K$ for all $\mathbf{s} \in \mathscr{F}_{\mathbf{r}}^-$. If $H_{ab}(\mathscr{F}_{\mathbf{r}}^-)$ was infinite, there would exist elements $\mathbf{r}_n \in \mathscr{F}_{\mathbf{r}}^-$ such that

$$H_{ab}(\mathbf{r}_1) < H_{ab}(\mathbf{r}_2) < \cdots .$$

Now start at $\mathbf{r}_n$ and employ the move N_{10} successively K times to reach a point in $\mathscr{F}_{\mathbf{r}}^+ = \mathscr{F}_{\mathbf{r}} \backslash \mathscr{F}_{\mathbf{r}}^-$. These paths will all be disjoint, as throughout the path, H_{ab} remains fixed.

Since the coordinates $(H_{11}, H_{12}, \ldots, H_{1,\ell})$ completely specify a point in $\mathscr{F}_{\mathbf{r}}$, it follows that $\mathscr{F}_{\mathbf{r}}^-$ is finite. $\square$

Next we need a set that can be used for a proper indexing of the free planes. Such a set will be called a complementary axis.

Definition 7.5 A subset $\mathscr{C}$ of Γ is called a **complementary axis** if

1. $\cup_{\mathbf{r}\in\mathscr{C}} \mathscr{F}_{\mathbf{r}} = \Gamma$,
2. if $\mathbf{r}, \mathbf{s} \in \mathscr{C}$, and $\mathbf{r} \neq \mathbf{s}$, then $\mathscr{F}_{\mathbf{r}}$ and $\mathscr{F}_{\mathbf{s}}$ are disjoint.

Let us next give a choice of a complementary axis.

Theorem 7.6 *Define*

$$\mathscr{C} = \{\mathbf{r} \in \Gamma : \Pi_{a=1}^{\ell+1-b} H_{ab}(\mathbf{r}) = 0 \text{ for } 1 \leq b \leq \ell\}.$$

The set $\mathscr{C}$ defined above is a complementary axis.

Proof: Let $\mathbf{s} \in \Gamma$. A sweepout argument almost identical to that used in lemma 7.3 (application of the move $\sum_{b=1}^{\ell} \left(\sum_{j=1}^{\ell+1-b} H_{a_j,j}(\mathbf{s})\right) N_{\ell+2-b,0}$) will connect $\mathbf{s}$ to another element $\mathbf{s}'$ for which $H_{a_b,b}(\mathbf{s}') = 0$ for $1 \leq b \leq \ell$ by a path that lies entirely on $\mathscr{F}_{\mathbf{s}}$. Clearly, $\mathbf{s}' \in \mathscr{C}$. Since $\mathbf{s}' \in \mathscr{F}_{\mathbf{s}}$, by corollary 7.2, $\mathbf{s} \in \mathscr{F}_{\mathbf{s}'}$.

It remains to show that if $\mathbf{r}$ and $\mathbf{s}$ are two distinct elements of $\mathscr{C}$, then $\mathbf{s} \notin \mathscr{F}_{\mathbf{r}}$. Since $\mathbf{r} \neq \mathbf{s}$, there exist two integers a and b, $1 \leq b \leq \ell$ and $1 \leq a \leq \ell + 2 - b$, such that $H_{ab}(\mathbf{r}) \neq H_{ab}(\mathbf{s})$. Observe that $H_{1\ell}(\cdot)$ must be zero for both, as they are members of $\mathscr{C}$. So b can not be ℓ here. Next we will produce two integers i and j such that the differences $H_{ib}(\mathbf{r}) - H_{ib}(\mathbf{s})$ and $H_{jb}(\mathbf{r}) - H_{jb}(\mathbf{s})$ are distinct. If there is an integer k for which $H_{kb}(\mathbf{r}) = H_{kb}(\mathbf{s}) = 0$, then take $i = a$, $j = k$. If not, there would exist two integers i and j such that $H_{ib}(\mathbf{r}) = 0$, $H_{ib}(\mathbf{s}) > 0$ and $H_{jb}(\mathbf{r}) > 0$, $H_{jb}(\mathbf{s}) = 0$. Take these i and j. Since $H_{ib}(\mathbf{r}) - H_{ib}(\mathbf{s})$ and $H_{jb}(\mathbf{r}) - H_{jb}(\mathbf{s})$ are distinct, by lemma 7.1, $\mathbf{r}$ and $\mathbf{s}$ can not lie on the same free plane. $\square$

Lemma 7.7 *Let $\mathbf{r}$ be a GT tableaux. Let $\mathbf{s}$ be the GT tableaux defined by the prescription*

$$V_{a1}(\mathbf{s}) = V_{a1}(\mathbf{r}) \text{ for all } a, \quad H_{ab}(\mathbf{s}) = H_{ab}(\mathbf{r}) \text{ for all } a \geq 2, \text{ for all } b, \quad H_{1,b}(\mathbf{s}) = 0 \text{ for all } b.$$

Then there is a path from $\mathbf{r}$ to $\mathbf{s}$ such that $V_{a1}(\cdot)$ remains constant throughout the path.

Proof: Apply the move $\displaystyle\sum_{b=1}^{\ell} H_{1,b}(\mathbf{r})M_{b+1,1}$. $\square$

The above lemma is actually the first step in the following slightly more general sweepout algorithm.

Lemma 7.8 *Let* $\mathbf{r}$ *be a GT tableaux. Let* $\mathbf{s}$ *be the GT tableaux defined by the prescription*

$$V_{11}(\mathbf{s}) = V_{11}(\mathbf{r}), \quad V_{a1}(\mathbf{s}) = 0 \text{ for all } a > 1, \quad H_{ab}(\mathbf{s}) = 0 \text{ for all } a, b.$$

Then there is a path from $\mathbf{r}$ *to* $\mathbf{s}$ *such that* $V_{11}(\cdot)$ *remains constant throughout the path.*

Proof: Apply successively the moves

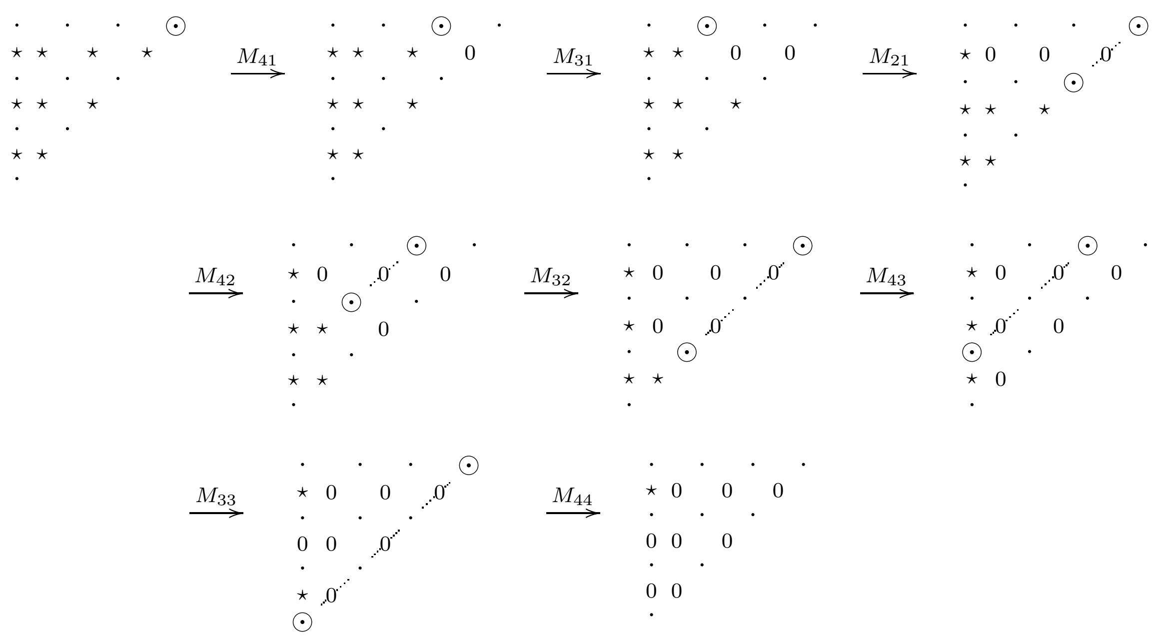
$$\sum_{b=1}^{\ell} H_{1,b}(\mathbf{r})M_{b+1,1}, \quad \sum_{b=1}^{\ell-1} H_{2,b}(\mathbf{r})M_{b+2,2}, \quad \ldots, \quad H_{\ell,1}(\mathbf{r})M_{\ell+1,\ell},$$

followed by

$$V_{21}(\mathbf{r})M_{33}, \quad (V_{21}(\mathbf{r}) + V_{31}(\mathbf{r}))M_{44}, \quad \ldots, \quad \left(\sum_{a=2}^{\ell} V_{a1}(\mathbf{r})\right) M_{\ell+1,\ell+1}. \tag{7.1}$$

$\square$

The following diagram will help explain the procedure described above in a simple case.



Corollary 7.9 $|d(\mathbf{r})| = O(r_{11})$.

Proof: If one employs the sequence of moves

$$V_{11}(\mathbf{r})M_{22}, \quad (V_{11}(\mathbf{r})+V_{21}(\mathbf{r}))M_{33}, \quad \ldots, \quad \left(\sum_{a=1}^{\ell} V_{a1}(\mathbf{r})\right) M_{\ell+1,\ell+1}$$

instead of the sequence given in (7.1), one would reach the constant (or zero) tableaux. Total length of this path from $\mathbf{r}$ to the zero tableaux is

$$\sum_{a=1}^{\ell}\sum_{b=1}^{\ell+1-a} H_{ab}(\mathbf{r}) + \sum_{b=1}^{\ell}\sum_{a=1}^{b} V_{a1}(\mathbf{r}),$$

which can easily be shown to be bounded by ℓr_{11}. □

Theorem 7.10 *Let $\widetilde{D}$ be the following operator:*

$$\widetilde{D} : e^{\lambda}_{\mathbf{r},\mathbf{s}} \mapsto r_{11} e^{\lambda}_{\mathbf{r},\mathbf{s}} \tag{7.2}$$

Then $(\mathcal{A}, \mathcal{H}, \widetilde{D})$ is an equivariant $\ell(\ell+2)$-summable odd spectral triple.

Moreover, if D is any equivariant Dirac operator acting on the L_2 space of $SU_q(\ell+1)$, then there exist positive reals a and b such that $|D| \leq a + b\widetilde{D}$. In particular, D cannot be p-summable for $p < \ell(\ell+2)$.

Proof: Boundedness of commutators with algebra elements follow from the observation that $|d(\mathbf{r}) - d(M(\mathbf{r})| \leq 1$ and hence equation (6.5) is satisfied.

Observe that the number of Young tableux $\lambda = (\lambda_1, \ldots, \lambda_\ell, \lambda_{\ell+1})$ with $n = \lambda_1 \geq \lambda_2 \geq \ldots \lambda_\ell \geq \lambda_{\ell+1} = 0$ is

$$\sum_{i_1=0}^{n}\sum_{i_2=0}^{i_1}\cdots\sum_{i_{\ell-1}=0}^{i_{\ell-2}} 1 = \text{polynomial in } n \text{ of degree } \ell-1.$$

Thus the number of such Young tableaux is $O(n^{\ell-1})$.

Next, let $\lambda : n = \lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_\ell \geq 0$ be an Young tableaux, and let V_λ be the space carrying the irreducible representation parametrized by λ. Then

$$\begin{aligned}
\dim V_\lambda &= \prod_{1\leq i<j\leq \ell+1} \frac{(\lambda_i - \lambda_{i+1}) + \ldots (\lambda_{j-1} - \lambda_j) + j - i}{j-i} \\
&= \prod_{1\leq i<j\leq \ell+1} \frac{\lambda_i - \lambda_j + j - i}{j-i} \\
&\leq (n+1)^{\frac{\ell(\ell+1)}{2}}.
\end{aligned}$$

Thus the dimension of an irreducible representation corresponding to a Young tableaux

$$n = \lambda_1 \geq \lambda_2 \geq \ldots \lambda_\ell \geq \lambda_{\ell+1} = 0$$

is $O(n^{\frac{1}{2}\ell(\ell+1)})$.

Using the two observations above, one can now show that the summability of $\widetilde{D}$ is $\ell(\ell+2)$. Optimality of $\widetilde{D}$ follows from corollary 7.9. $\square$

One should note, however, that the $\widetilde{D}$ defined above has trivial sign, and consequently trivial K-homology class.

Lemma 7.11 *Let D be an equivariant Dirac operator on $L_2(G)\otimes\mathbb{C}^m$. Then there are positive reals a, b such that $|D| \leq a + b|\widetilde{D}\otimes I|$.*

Proof: Let D be an equivariant Dirac operator on $L_2(G)\otimes\mathbb{C}^m$. Then D must be of the form $e_{\mathbf{r},\mathbf{s}}\otimes v \mapsto e_{\mathbf{r},\mathbf{s}}\otimes T(\mathbf{r})v$ where $T(\mathbf{r})$ are self-adjoint operators acting on $\mathbb{C}^m$. The growth conditions coming out of the boundedness of the commutators will now be exactly as in (6.6), with the scalars $d(\cdot)$ replaced by operators $T(\cdot)$ and absolute value replaced by operator norm. If we now form a graph by joining two vertices $\mathbf{r}$ and $\mathbf{s}$ whenever $\|T(\mathbf{r}) - T(\mathbf{s})\| \leq c$, then exactly as in the proof of corollary 7.9, one can show that any point $\mathbf{r}$ can be connected to the zero tableaux by a path of length $O(r_{11})$. This implies that there are positive reals a and b such that $|T(\mathbf{r})| \leq a + br_{11}$. The assertion in the lemma now follows from this. $\square$

8 Characterization of sign D

We continue our analysis of the growth conditions on the $d(\mathbf{r})$'s in this section in order to come up with a complete characterization of the sign of D.

Lemma 8.1 *The sets $V_{11}(\Gamma^+)$ and $V_{11}(\Gamma^-)$ can not both be infinite.*

Proof: If both the sets are infinite, then one can choose two sequences of points $\mathbf{r}_n$ and $\mathbf{s}_n$ such that $\mathbf{r}_n \in \Gamma^+$, $\mathbf{s}_n \in \Gamma^-$ and

$$V_{11}(\mathbf{r}_1) < V_{11}(\mathbf{s}_1) < V_{11}(\mathbf{r}_2) < V_{11}(\mathbf{s}_2) < \ldots.$$

Start at $\mathbf{r}_n$ and use lemma 7.8 above to reach a point $\mathbf{r}'_n$ for which $V_{11}(\mathbf{r}'_n) = V_{11}(\mathbf{r}_n)$ and all other coordinates are zero through a path where the V_{11} coordinate remains constant. Similarly, from $\mathbf{s}_n$, go to a point $\mathbf{s}'_n$ for which $V_{11}(\mathbf{s}'_n) = V_{11}(\mathbf{s}_n)$ and all other coordinates are zero. Now apply the move $(V_{11}(\mathbf{s}_n) - V_{11}(\mathbf{r}_n))M_{11}$ to go from $\mathbf{r}'_n$ to $\mathbf{s}'_n$. This will give us a path p_n from $\mathbf{r}_n$ to $\mathbf{s}_n$ on which $V_{11}(\cdot)$ remains between $V_{11}(\mathbf{r}_n)$ and $V_{11}(\mathbf{s}_n)$. Therefore all the paths p_n are disjoint. Thus (Γ^+,Γ^-) admits an infinite ladder. So at least one of $V_{11}(\Gamma^+)$ and $V_{11}(\Gamma^-)$ must be finite. $\square$

Lemma 8.2 *Let C be any of the coordinates V_{a1} or H_{ab} where $a > 1$. If $V_{11}(\Gamma^-)$ is finite, then $C(\Gamma^-)$ is also finite.*

Proof: Assume K is a positive integer such that $V_{11}(\Gamma^-) \subseteq [0, K]$. Now suppose, if possible, that $C(\Gamma^-)$ is infinite. Let $\mathbf{r}_n$ be a sequence of points in Γ^- such that

$$C(\mathbf{r}_1) < C(\mathbf{r}_2) < \ldots.$$

Start at $\mathbf{r}_n$, and use lemma 7.7 to reach a point $\mathbf{r}'_n$ and then apply M_{11} for $K+1$ times to get to a point $\mathbf{s}_n$ for which $V_{11}(\mathbf{s}_n) > K$. Throughout this path, $C(\cdot)$ is constant, so that the paths are all disjoint. Since $V_{11}(\mathbf{s}_n) > K$, we have $\mathbf{s}_n \in \Gamma^+$. Thus this gives us an infinite ladder for (Γ^+, Γ^-), which is impossible. $\square$

Lemma 8.3 *Suppose $H_{1\ell}(F)$ is bounded. If $V_{11}(\Gamma^-)$ is finite, then F^- is finite.*

Proof: The previous lemma, along with the assumption here tells us that the sets $V_{a1}(F^-)$ and $H_{a,\ell+1-a}(F^-)$ are all bounded for $1 \le a \le \ell$. Since for an $\mathbf{r} \in V$, one has $r_{11} = \sum_{a=1}^{\ell} V_{a1}(\mathbf{r}) + \sum_{a=1}^{\ell} H_{a,\ell+1-a}(\mathbf{r})$, the set $\{r_{11} : \mathbf{r} \in F^-\}$ is bounded. It follows that F^- is finite. $\square$

Corollary 8.4 *If $V_{11}(\Gamma^-)$ is finite, then $\mathscr{C}^-$ is finite.*

Proof: Follows from the observation that $H_{1\ell}(\mathbf{r}) = 0$ for all $\mathbf{r} \in \mathscr{C}$. $\square$

A similar argument will tell us that if $V_{11}(\Gamma^+)$ is finite, then $\mathscr{C}^+$ is finite. Thus from lemma 8.1, it follows that either $\mathscr{C}^+$ or $\mathscr{C}^-$ is finite.

Theorem 8.5 *Let D be an equivariant Dirac operator on $L_2(SU_q(\ell+1))$. Then $\operatorname{sign} D$ must be of the form $2P - I$ or $I - 2P$ where P is, up to a compact perturbation, the projection onto the closed span of $\{e^{\lambda}_{\mathbf{r},\mathbf{s}} : \mathbf{r} \in \mathscr{F}_{\mathbf{r}_i}$ for some $i\}$, with $\mathbf{r}_1, \ldots, \mathbf{r}_k$ being a finite collection of GT-tableaux.*

Proof: Let $\mathscr{C}' = \{\mathbf{r} \in \mathscr{C} : \mathscr{F}^+_{\mathbf{r}} \ne \phi \ne \mathscr{F}^-_{\mathbf{r}}\}$. Let us first show that $\mathscr{C}'$ is finite, i.e. except for finitely many $\mathbf{r}$'s in $\mathscr{C}$, one has either $\mathscr{F}_{\mathbf{r}} \subseteq \Gamma^+$ or $\mathscr{F}_{\mathbf{r}} \subseteq \Gamma^-$. It follows from the argument used in the proof of theorem 7.6 that any two points on a free plane can be connected by a path lying entirely on the plane. If $\mathscr{C}'$ is infinite, one can easily produce an infinite ladder using this fact.

Thus there are only finitely many free planes $\mathscr{F}_{\mathbf{r}}$ for which both $\mathscr{F}^+_{\mathbf{r}}$ and $\mathscr{F}^-_{\mathbf{r}}$ are nonempty. Since we already know that for every $\mathbf{r}$, either $\mathscr{F}^+_{\mathbf{r}}$ or $\mathscr{F}^-_{\mathbf{r}}$ is finite, it follows that by applying a compact perturbation, one can ensure that for every $\mathbf{r}$, exactly one of the sets $\mathscr{F}^+_{\mathbf{r}}$ and $\mathscr{F}^-_{\mathbf{r}}$ is empty. This, along with the observations that $\mathscr{C} \cap \mathscr{F}_{\mathbf{r}} = \{\mathbf{r}\}$ and that either $\mathscr{C}^+$ or $\mathscr{C}^-$ is finite gives us the required conclusion. $\square$

As a consequence of this sign characterization, we now get the following theorem.

Theorem 8.6 *Let $\ell > 1$. Let D be an equivariant Dirac operator acting on $L_2(G)$. Then D must have trivial sign.*

Proof: We will show that if P is as in the earlier theorem, then the commutators $[P, \pi(u_{ij})]$ can not all be compact.

Let us first prove it in the case when P is the projection onto the span of $\{e_{\mathbf{rs}} : \mathbf{r} \in \mathscr{F}_0\}$, where $\mathscr{F}_0$ is the free plane passing through the constant tableaux. We have

$$[P, \pi(u_{ij})]e_{\mathbf{rs}} = \begin{cases} P\pi(u_{ij})e_{\mathbf{rs}} & \text{if } \mathbf{r} \notin \mathscr{F}_0, \\ (P-I)\pi(u_{ij})e_{\mathbf{rs}} & \text{if } \mathbf{r} \in \mathscr{F}_0 \end{cases}.$$

Recall (section 5) the expression for $\pi(u_{ij})e_{\mathbf{rs}}$:

$$\pi(u_{ij})e_{\mathbf{rs}} = \sum_{\substack{R\in\mathbb{N}^i, S\in\mathbb{N}^j \\ R(1)=S(1)}} C_q(i, \mathbf{r}, R(\mathbf{r}))C_q(j, \mathbf{s}, S(\mathbf{s}))k(\mathbf{r}, R(\mathbf{r}))e_{R(\mathbf{r})S(\mathbf{s})}.$$

Hence for $\mathbf{r} \in \mathscr{F}_0$,

$$\begin{aligned} [P, \pi(u_{ij})]e_{\mathbf{rs}} &= (P-I)\pi(u_{ij})e_{\mathbf{rs}} \\ &= -\sum_{\substack{R\in\mathbb{N}^i, S\in\mathbb{N}^j \\ R(1)=S(1), R\neq N_{i0}}} C_q(i, \mathbf{r}, R(\mathbf{r}))C_q(j, \mathbf{s}, S(\mathbf{s}))k(\mathbf{r}, R(\mathbf{r}))e_{R(\mathbf{r}),S(\mathbf{s})}. \end{aligned}$$

In particular, for $i = j = 1$, one gets

$$[P, \pi(u_{11})]e_{\mathbf{rs}} = -\sum_{k=1}^{\ell} C_q(1, \mathbf{r}, M_{k1}(\mathbf{r}))C_q(1, \mathbf{s}, M_{k1}(\mathbf{s}))k(\mathbf{r}, M_{k1}(\mathbf{r}))e_{M_{k1}(\mathbf{r}),M_{k1}(\mathbf{s})}.$$

Now suppose $\mathbf{r} \in \mathscr{F}_0$ satisfies

$$r_{1,\ell} = 0 = r_{2,\ell} = r_{1,\ell+1}. \tag{8.1}$$

Then

$$\langle e_{M_{\ell1}(\mathbf{r}),M_{\ell1}(\mathbf{r})}, [P, \pi(u_{11})]e_{\mathbf{rr}}\rangle = -C_q(1, \mathbf{r}, M_{\ell1}(\mathbf{r}))^2 k(\mathbf{r}, M_{\ell1}(\mathbf{r})).$$

It follows from (4.10) and (4.11) that $C_q(1, \mathbf{r}, M_{\ell1}(\mathbf{r}))$ is bounded away from zero, so long as $\mathbf{r}$ obeys (8.1). We have also seen (lemma 5.1) that $k(\mathbf{r}, M_{\ell1}(\mathbf{r}))$ is bounded away from zero. Now it is easy to see that if $\ell > 1$, then there are infinitely many choices of $\mathbf{r}$ satisfying (8.1) such that they all lie in $\mathscr{F}_0$. Therefore $[P, \pi(u_{11})]$ is not compact.

For more general P (as in the previous theorem), the idea would be similar, but this time one has to get hold of a positive integer n such that for any $\mathbf{r} \in \cup_{i=1}^k \mathscr{F}_{\mathbf{r}_i}$, $nM_{\ell1}(\mathbf{r}) \notin \cup_{i=1}^k \mathscr{F}_{\mathbf{r}_i}$, and then compute $\langle e_{nM_{\ell1}(\mathbf{r}),nM_{\ell1}(\mathbf{r})}, (P-I)\pi(u_{11})^n e_{\mathbf{rr}}\rangle$. □

As mentioned in the introduction, the above theorem in particular says that in order to get equivariant Dirac operators with nontrivial sign for for $\ell > 1$, one needs to bring in multiplicities. We will see below that if one takes the tensor product of $L_2(G)$ with a suitable space, it is possible to produce such operators.

Theorem 8.7 *Let $\widetilde{D}$ be as in theorem 7.10 and let N_i be the following operators on $L_2(G)$:*

$$N_i e_{\mathbf{r},\mathbf{s}} = f_i(\mathbf{r}) e_{\mathbf{r},\mathbf{s}},$$

where $f_i(\mathbf{r}) = \min\{H_{ai}(\mathbf{r}) : 1 \leq a \leq \ell+1-i\}$. Let $\gamma_1, \gamma_2, \ldots, \gamma_{\ell+1}$ be $\ell+1$ spin matrices acting on $\mathbb{C}^m$. Define an operator D on $L_2(G) \otimes \mathbb{C}^m$ as follows:

$$D = \sum_{i=1}^{\ell} N_i \otimes \gamma_i + \widetilde{D} \otimes \gamma_{\ell+1}.$$

Then $(L_2(G) \otimes \mathbb{C}^m, \pi \otimes I, D)$ is an equivariant $\ell(\ell+2)$-summable spectral triple.

Moreover, the operator D is optimal, in the following sense: given any equivariant Dirac operator D' on $L_2(G) \otimes \mathbb{C}^m$ there are positive reals a, b such that $|D'| \leq a + b|D|$.

Proof: Compact resolvent condition and summability of D follow from the fact that the operator $|D|$ is given by $|D| e_{\mathbf{r},\mathbf{s}} = \lambda_{\mathbf{r}} e_{\mathbf{r},\mathbf{s}}$, where the singular values $\lambda_{\mathbf{r}}$ obey the inequality

$$r_{11} \leq \lambda_{\mathbf{r}} \leq K r_{11}$$

for some constant K that depends only on ℓ. Boundedness of commutators follow from the boundedness of commutators of the N_i's and $\widetilde{D}$ with the algebra elements, which is clear from condition (6.6).

Observe that $\widetilde{D} \otimes I \leq |D|$. Therefore optimality follows from lemma 7.11. □

Remark 8.8 Let $\widehat{V}_{i1}$ and $\widehat{H}_{ij}$ denote the following operators on $L_2(G)$:

$$\widehat{V}_{i1} e_{\mathbf{r},\mathbf{s}} = V_{i1}(\mathbf{r}) e_{\mathbf{r},\mathbf{s}}, \quad \widehat{H}_{ij} e_{\mathbf{r},\mathbf{s}} = H_{ij}(\mathbf{r}) e_{\mathbf{r},\mathbf{s}}, \quad i + j \leq \ell + 1.$$

Suppose now that $\gamma_1, \gamma_2, \ldots, \gamma_{\ell(\ell+3)/2}$ be spin matrices acting on some space $\mathbb{C}^m$, and D_k for $1 \leq k \leq \frac{\ell(\ell+3)}{2}$ are the operators $\widehat{V}_{i1}$ and $\widehat{H}_{ij}$ in some order. Now define D on $L_2(G) \otimes \mathbb{C}^m$ to be the operator

$$D = \sum D_k \otimes \gamma_k.$$

Then this operator D also enjoys all the features described in the above theorem.

9 The odd dimensional quantum spheres

In this section, we will use the combinatorial technique and the calculations done in the earlier sections to investigate equivariant Dirac operators for all the odd dimensional quantum spheres $S_q^{2\ell+1}$ of Vaksman & Soibelman ([22]). In what follows, we will write G for $SU_q(\ell+1)$ and H for $SU_q(\ell)$.

The C^*-algebra $C(S_q^{2\ell+1})$ of the quantum sphere $S_q^{2\ell+1}$ is the universal C^*-algebra generated by elements $z_1, z_2, \ldots, z_{\ell+1}$ satisfying the following relations (see [14]):

$$\begin{aligned} z_i z_j &= q z_j z_i, & 1 \leq j < i \leq \ell+1, \\ z_i z_j^* &= q z_j^* z_i, & 1 \leq i \neq j \leq \ell+1, \\ z_i z_i^* - z_i^* z_i + (1-q^2) \sum_{k>i} z_k z_k^* &= 0, & 1 \leq i \leq \ell+1, \\ \sum_{i=1}^{\ell+1} z_i z_i^* &= 1. \end{aligned}$$

Just like their classical counterparts, these spheres can be viewed as quotient spaces of the quantum groups $SU_q(\ell+1)$, i. e.

$$C(S_q^{2\ell+1}) \cong C(G \backslash H) = \{a \in C(G) : (\phi \otimes id)\Delta(a) = I \otimes a\}, \tag{9.2}$$

where ϕ is a C^*-homomorphism from $C(G)$ onto $C(H)$ that preserves the comultiplication, that is, it satisfies $\Delta\phi = (\phi \otimes \phi)\Delta$, where the Δ on the right hand side is the comultiplication for G and the Δ on the left hand side stands for the comultiplication for H. (For a formulation of quotient spaces etc. in the context of compact quantum groups, see [19])

The group G has a canonical right action $\tau : C(G\backslash H) \to C(G\backslash H) \otimes C(G)$ coming from the comultiplication Δ (i. e. τ is just the restriction of Δ to $C(G\backslash H)$). Let ρ denote the restriction of the Haar state on $C(G)$ to $C(G\backslash H)$. Then clearly one has $(\rho \otimes id)\tau(a) = \rho(a)I$, which means ρ is the invariant state for $C(G\backslash H)$. This also means that $L_2(G\backslash H) = L_2(\rho)$ is just the closure of $C(G\backslash H)$ in $L_2(G)$.

Proposition 9.1 *Assume $\ell > 1$. The right regular representation u of G keeps $L_2(G\backslash H)$ invariant, and the restriction of u to $L_2(G\backslash H)$ decomposes as a direct sum of exactly one copy of each of the irreducibles given by the young tableaux $\lambda_{n,k} := (n+k, k, k, \ldots, k, 0)$, with $n, k \in \mathbb{N}$.*

Proof: Write σ for the composition $h_H \circ \phi$ where h_H is the Haar state for H. From the description of $C(G\backslash H)$ above, it follows that

$$\begin{aligned} C(G\backslash H) &= \{a \in C(G) : (\sigma \otimes id)\Delta(a) = a\} \\ &= \{(\sigma \otimes id)\Delta(a) : a \in C(G)\}. \end{aligned}$$

Now the map $a \mapsto \sigma * a := (\sigma \otimes id)\Delta(a)$ on $C(G)$ extends to a bounded linear operator L_σ on $L_2(G)$ (lemma 3.1, [18]), and it is easy to see that $L_\sigma^2 = L_\sigma$. It follows then that $L_2(G\backslash H) = \ker(L_\sigma - I) = \operatorname{ran} L_\sigma$. From the discussion preceeding theorem 3.3, [18], it now follows that u keeps $L_2(G\backslash H)$ invariant and in fact the restriction of u to $L_2(G\backslash H)$ is the representation induced by the trivial repersentation of H. From the analogue of Frobenius reciprocity theorem for compact quantum groups (theorem 3.3, [18]) it now follows that the multiplicity of any irreducible u^λ in it would be same as the multiplicity of the trivial representation of H in the restriction of u^λ to H. But from the representation theory of $SU_q(\ell+1)$, we know that the restriction of u^λ to $SU_q(\ell)$ decomposes into a direct sum of one copy of each irreducible $\mu : (\mu_1 \geq \mu_2 \geq \ldots \geq \mu_\ell)$ of $SU_q(\ell)$ for which

$$\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \mu_2 \geq \ldots \geq \lambda_\ell \geq \mu_\ell \geq 0. \tag{9.3}$$

Now the trivial representation of $SU_q(\ell)$ is indexed by Young tableaux of the form $\mu : (k, k, \ldots, k)$ where $k \in \mathbb{N}$. But such a μ will obey the restriction 9.3 above if and only if λ is of the form $(n+k, k, k, \ldots, k, 0)$. □

Remark 9.2 For the case $\ell = 1$, the restriction of the irreducible $(n, 0)$ to the trivial subgroup decomposes into $n+1$ copies of the trivial representation. Therefore, in this case, $L_2(S_q^3)$ decomposes into a direct sum of $n+1$ copies of each representation $(n, 0)$.

Next, we will make an explicit choice of ϕ that would help us make use of the calculations already done in the initial sections for analyzing Dirac operators acting on $L_2(G\backslash H)$. More specifically, we will choose our ϕ in such a manner that $L_2(G\backslash H)$ turns out to be the span of certain rows of the $e_{\mathbf{r},\mathbf{s}}$'s. Let $u^{\mathbb{1}}$ denote the fundamental unitary for G, i. e. the irreducible unitary representation corresponding to the Young tableaux $\mathbb{1} = (1, 0, \ldots, 0)$. Similarly write $v^{\mathbb{1}}$ for the fundamental unitary for H. Fix some bases for the corresponding representation spaces. Then $C(G)$ is the C^*-algebra generated by the matrix entries $\{u_{ij}^{\mathbb{1}}\}$ and $C(H)$ is the C^*-algebra generated by the matrix entries $\{v_{ij}^{\mathbb{1}}\}$. Now define ϕ by

$$\phi(u_{ij}^{\mathbb{1}}) = \begin{cases} I & \text{if } i = j = 1, \\ v_{i-1,j-1}^{\mathbb{1}} & \text{if } 2 \leq i, j \leq \ell+1, \\ 0 & \text{otherwise.} \end{cases} \tag{9.4}$$

Then $C(G\backslash H)$ is the C^*-subalgebra of $C(G)$ generated by the entries $u_{1,j}$ for $1 \leq j \leq \ell+1$ (one recovers the relations for the generators of $C(S_q^{2\ell+1})$ if one sets $z_i = q^{-i+1}u_{1,i}^*$).

Proposition 9.3 *Let Γ_0 be the set of all GT tableaux $\mathbf{r}^{nk}$ given by*

$$r_{ij}^{nk} = \begin{cases} n+k & \text{if } i = j = 1, \\ 0 & \text{if } i = 1,\ j = \ell+1, \\ k & \text{otherwise,} \end{cases}$$

for some $n, k \in \mathbb{N}$. Let Γ_0^{nk} be the set of all GT tableaux with top row $(n+k, k, \ldots, k, 0)$. Then the family of vectors

$$\{e_{\mathbf{r}^{nk},\mathbf{s}} : n, k \in \mathbb{N},\ \mathbf{s} \in \Gamma_0^{nk}\}$$

form a complete orthonormal basis for $L_2(G\backslash H)$.

Proof: Let A be the linear span of the elements $\{u_{\mathbf{r}^{n,k},\mathbf{s}} : n,k \in \mathbb{N}, \mathbf{s} \in \Gamma_0^{n,k}\}$. Clearly the closure of A in $L_2(G)$ is the closed linear span of $\{e_{\mathbf{r}^{nk},\mathbf{s}} : n,k \in \mathbb{N},\ \mathbf{s} \in \Gamma_0^{nk}\}$. It is also immdiate that the restriction of the right regular representation to the above subspace is a direct sum of one copy of each of the irreducibles $(n+k,k,k,\ldots,k,0)$.

We will next show that for any $a \in A$, $u_{1j}a$ and u_{1j}^* a are also in A. Take $a = u_{\mathbf{r}^{n,k},\mathbf{s}}$. Use equation (5.6) to get

$$\begin{aligned}
u_{1,j}u_{\mathbf{r}^{n,k},\mathbf{s}} &= \sum_{M,M'} C_q(1,\mathbf{r}^{n,k},M(\mathbf{r}^{n,k}))C_q(j,\mathbf{s},M'(\mathbf{s}))u_{M(\mathbf{r}^{n,k}),M'(\mathbf{s})}\\
&= \sum_{M'} C_q(1,\mathbf{r}^{n,k},M_{11}(\mathbf{r}^{n,k}))C_q(j,\mathbf{s},M'(\mathbf{s}))u_{M_{11}(\mathbf{r}^{n,k}),M'(\mathbf{s})}\\
&\quad +\sum_{M''} C_q(1,\mathbf{r}^{n,k},M_{\ell+1,1}(\mathbf{r}^{n,k}))C_q(j,\mathbf{s},M''(\mathbf{s}))u_{M_{\ell+1,1}(\mathbf{r}^{n,k}),M''(\mathbf{s})}\\
&= \sum_{M'} C_q(1,\mathbf{r}^{n,k},\mathbf{r}^{n+1,k})C_q(j,\mathbf{s},M'(\mathbf{s}))u_{\mathbf{r}^{n+1,k},M'(\mathbf{s})}\\
&\quad +\sum_{M''} C_q(1,\mathbf{r}^{n,k},\mathbf{r}^{n,k-1}))C_q(j,\mathbf{s},M''(\mathbf{s}))u_{\mathbf{r}^{n,k-1},M''(\mathbf{s})},
\end{aligned} \tag{9.5}$$

where the first sum is over all moves $M' \in \mathbb{N}^j$ whose first coordinate is 1 and the second sum is over all moves $M'' \in \mathbb{N}^j$ whose first coordinate is $\ell+1$. Thus $u_{1j}a \in A$.

Next, note that if $\langle u_{1j}^* e_{\mathbf{r}^{n,k},\mathbf{s}}, e_{\mathbf{r}',\mathbf{s}'}\rangle \neq 0$, then one must have $\mathbf{r}' = \mathbf{r}^{n-1,k}$ or $\mathbf{r}' = \mathbf{r}^{n,k+1}$. Therefore it follows that $u_{1j}^* u_{\mathbf{r}^{n,k},\mathbf{s}}$ is a linear combination of the $u_{\mathbf{r}^{n-1,k},\mathbf{s}}$ $u_{\mathbf{r}^{n,k+1},\mathbf{s}}$'s, and hence belongs to A. Since A contains the element $u_{\mathbf{0},\mathbf{0}} = 1$, it contains u_{1j} and u_{ij}^*. Thus A contains the $*$-algebra B generated by the u_{1j}'s. But by the previous theorem, restriction of the right regular representation to the L_2 closure $L_2(G\backslash H)$ of B also decomposes as a direct sum of one copy of each of the irreducibles $(n+k,k,\ldots,k,0)$. So it follows that $L_2(G\backslash H)$ is equal to the subspace stated in the theorem. $\square$

A self-adjoint operator with compact resolvent on $L_2(G\backslash H)$ that commutes with the restriction of u there would be of the form

$$e_{\mathbf{r},\mathbf{s}} \mapsto d(\mathbf{r})e_{\mathbf{r},\mathbf{s}}, \quad \mathbf{r} \in \Gamma_0.$$

Next, let us look at the growth restrictions coming from the boundedness of commutators. In this case, one has the boundedness of only the operators $[D, \pi(u_{ij})]$. Which means, in effect, one will now have the condition (6.6) only for $i = 1$ and $\mathbf{r} \in \Gamma_0$:

$$|d(\mathbf{r}) - d(M(\mathbf{r}))| \leq cq^{-C(1,\mathbf{r},M)}. \tag{9.6}$$

Observe that only allowed moves here are the moves $M = M_{1,1} \equiv (1)$ and $M = M_{\ell+1,1} \equiv (\ell+1)$. Looking at the corresponding quantity $C(1,\mathbf{r},M)$, we find that there are two conditions:

$$|d(\mathbf{r}^{nk}) - d(\mathbf{r}^{n,k-1})| \leq c, \tag{9.7}$$

$$|d(\mathbf{r}^{nk}) - d(\mathbf{r}^{n+1,k})| \leq cq^{-\sum_{j=1}^{\ell} H_{1j}(\mathbf{r}^{nk})} = cq^{-k}. \tag{9.8}$$

As in the earlier sections, we can now form a graph by taking Γ_0 to be the set of vertices, and by joining two vertices $\mathbf{r}$ and $\mathbf{s}$ by an edge if $|d(\mathbf{r}) - d(\mathbf{s})| \leq c$.

Lemma 9.4 *Let $\mathscr{F}_n = \{\mathbf{r}^{n,k} : k \in \mathbb{N}\}$, $n \in \mathbb{N}$. Then any two points in $\mathscr{F}_n$ are connected by a path lying entirely in $\mathscr{F}_n$.*

If $n < n'$, then any point in $\mathscr{F}_n$ is connected to any point in $\mathscr{F}_{n'}$ by a path such that $n \leq V_{1,1}(\mathbf{r}) \leq n'$ for every vertex $\mathbf{r}$ lying on that path.

Proof: Take two points $\mathbf{r}^{n,j}$ and $\mathbf{r}^{n,k}$ in $\mathscr{F}_n$. Assume $j < k$. From the condition (9.7), it follows that any point $\mathbf{r}$ is connected to $M_{\ell+1,1}(\mathbf{r})$ by an edge. Therefore the first conclusion follows from the observation that if we start at $\mathbf{r}^{n,k}$ and apply the move $M_{\ell+1,1}$ successively $k - j$ number of times, we reach the point $\mathbf{r}^{n,j}$, and the vertices on this path are the points $\mathbf{r}^{n,i}$ for $i = j, j+1, \ldots, k$. Observe also that throughout this path, $V_{1,1}(\mathbf{r})$ remains n.

For the second part, take a point $\mathbf{r}^{n,k}$ in $\mathscr{F}_n$ and a point $\mathbf{r}^{n',j}$ in $\mathscr{F}_{n'}$. From what we have done above, there is a path from $\mathbf{r}^{n,k}$ to $\mathbf{r}^{n,0}$ throughout which $V_{1,1}(\mathbf{r}) = n$. Similarly there is a path from $\mathbf{r}^{n',j}$ to $\mathbf{r}^{n',0}$ throughout which $V_{1,1}(\mathbf{r}) = n'$. Next, note from (9.8) that for $p \in \mathbb{N}$, the points $\mathbf{r}^{p,0}$ and $\mathbf{r}^{p+1,0}$ are connected by an edge and $V_{1,1}(\mathbf{r}^{p,0}) = p$, $V_{1,1}(\mathbf{r}^{p+1,0}) = p+1$. So start at $\mathbf{r}^{n,0}$ and reach successively the points $\mathbf{r}^{n+1,0}$, $\mathbf{r}^{n+2,0}$ and so on to eventually reach the point $\mathbf{r}^{n',0}$; also the coordinate $V_{1,1}(\cdot)$ remains between n and n' on this path. $\Box$

Theorem 9.5 *Let D be an equivariant Dirac operator on $L_2(G\backslash H)$. Then*

1. *D must be of the form*
$$e_{\mathbf{r},\mathbf{s}} \mapsto d(\mathbf{r}) e_{\mathbf{r},\mathbf{s}}, \quad \mathbf{r} \in \Gamma,$$
where the singular values obey $|d(\mathbf{r})| = O(r_{11})$, and
2. *$\operatorname{sign} D$ must be of the form $2P - I$ or $I - 2P$ where P is, up to a compact perturbation, the projection onto the closed span of $\{e_{\mathbf{r}^{nk},\mathbf{s}} : n \in F, k \in \mathbb{N}, \mathbf{s} \in \Gamma_0^{nk}\}$, for some finite subset F of $\mathbb{N}$.*

Proof: Start with an equivariant self-adjoint operator D with compact resolvent, so that it is indeed of the form $e_{\mathbf{r},\mathbf{s}} \mapsto d(\mathbf{r}) e_{\mathbf{r},\mathbf{s}}$. By applying a compact perturbation if necessary, make sure that $d(\mathbf{r}) \neq 0$ for all $\mathbf{r} \in \Gamma_0$. We have seen during the proof of the previous lemma that for any n and k in $\mathbb{N}$, the vertices $\mathbf{r}^{nk}$ and $\mathbf{r}^{n,k+1}$ are connected by an edge, and for any $n \in \mathbb{N}$, the vertices $\mathbf{r}^{n,0}$ and $\mathbf{r}^{n+1,0}$ is connected by an edge. Thus any vertex $\mathbf{r}^{nk}$ can be reached from the vertex $\mathbf{r}^{00}$ by a path of length $n + k$. Therefore one gets the first assertion.

Next, define

$$\begin{aligned}
\Gamma_0^+ &= \{\mathbf{r} \in \Gamma_0 : d(\mathbf{r}) > 0\}, \\
\Gamma_0^- &= \{\mathbf{r} \in \Gamma_0 : d(\mathbf{r}) < 0\},
\end{aligned}$$

$$\mathscr{F}_n^+ = \mathscr{F}_n \cap \Gamma_0^+,$$
$$\mathscr{F}_n^- = \mathscr{F}_n \cap \Gamma_0^-.$$

Observe that for the path produced in the proof of the forgoing lemma to connect two points $\mathbf{r}^{n,k}$ and $\mathbf{r}^{n,j}$ in $\mathscr{F}_n$, the coordinate $H_{1,\ell}(\cdot)$ remains between j and k. Now suppose for some n, both $\mathscr{F}_n^+$ and $\mathscr{F}_n^-$ are infinite. Then there are points

$$0 \leq k_1 < k_2 < \dots$$

such that $\mathbf{r}^{nk}$ is in $\mathscr{F}_n^+$ for $k = k_{2j}$ and $\mathbf{r}^{nk}$ is in $\mathscr{F}_n^-$ for $k = k_{2j+1}$. Using the above observation, we can then produce an infinite ladder by joining each $\mathbf{r}^{n,k_{2j-1}}$ to $\mathbf{r}^{n,k_{2j}}$. Thus for each $n \in \mathbb{N}$, exactly one of the sets $\mathscr{F}_n^+$ and $\mathscr{F}_n^-$ is finite. Also, note that by the first part of the previous lemma, the set of all $n \in \mathbb{N}$ for which both $\mathscr{F}_n^+$ and $\mathscr{F}_n^-$ are nonempty is finite. Therefore by applying a compact perturbation, we can ensure that for every n, either $\mathscr{F}_n^+ = \mathscr{F}_n$ or $\mathscr{F}_n^- = \mathscr{F}_n$.

Finally, if there are infinitely many n's for which $\mathscr{F}_n^+ = \mathscr{F}_n$ and infinitely many n's for which $\mathscr{F}_n^- = \mathscr{F}_n$, then one can choose a sequence of integers

$$0 \leq n_1 < n_2 < \dots$$

such that $\mathscr{F}_n^+ = \mathscr{F}_n$ for $n = n_{2j}$ and $\mathscr{F}_n^- = \mathscr{F}_n$ for $n = n_{2j+1}$. Now use the second part of the previous lemma to join each $\mathbf{r}^{n_{2j-1},0}$ to $\mathbf{r}^{n_{2j},0}$ to produce an infinite ladder.

Thus there is a finite subset F of $\mathbb{N}$ such that exactly one of the following is true:

$$\mathscr{F}_n = \begin{cases} \mathscr{F}_n^+ & \text{if } n \in F, \\ \mathscr{F}_n^- & \text{if } n \notin F, \end{cases} \qquad \text{or} \qquad \mathscr{F}_n = \begin{cases} \mathscr{F}_n^- & \text{if } n \in F, \\ \mathscr{F}_n^+ & \text{if } n \notin F. \end{cases}$$

This is precisely what the second part of the theorem says. □

Next, take the operator $D : e_{\mathbf{r},\mathbf{s}} \mapsto d(\mathbf{r})e_{\mathbf{r},\mathbf{s}}$ on $L_2(G\backslash H)$ where the $d(\mathbf{r})$'s are given by:

$$d(\mathbf{r}^{nk}) = \begin{cases} -k & \text{if } n = 0, \\ n + k & \text{if } n > 0. \end{cases} \tag{9.9}$$

Theorem 9.6 *The operator D is an equivariant $(2\ell + 1)$-summable Dirac operator acting on $L_2(G\backslash H)$, that gives a nondegenerate pairing with $K_1(C(G\backslash H))$.*

The operator D is optimal, i. e. if D_0 is any equivariant Dirac operator on $L_2(G\backslash H)$, then there are positive reals a and b such that

$$|D_0| \leq a + b|D|.$$

Proof: Recall from equation (5.7) that the elements $u_{1,j}$ act on the basis elements $e_{\mathbf{r}^{n,k},\mathbf{s}}$ as follows:

$$\begin{aligned}
u_{1,j}e_{\mathbf{r}^{n,k},\mathbf{s}} &= \sum_{M,M'} C_q(1,\mathbf{r}^{n,k},M(\mathbf{r}^{n,k}))C_q(j,\mathbf{s},M'(\mathbf{s}))\kappa(\mathbf{r}^{n,k},\mathbf{s})e_{M(\mathbf{r}^{n,k}),M'(\mathbf{s})}\\
&= \sum_{M'} C_q(1,\mathbf{r}^{n,k},M_{11}(\mathbf{r}^{n,k}))C_q(j,\mathbf{s},M'(\mathbf{s}))\kappa(\mathbf{r}^{n,k},\mathbf{s})e_{M_{11}(\mathbf{r}^{n,k}),M'(\mathbf{s})}\\
&\quad +\sum_{M''} C_q(1,\mathbf{r}^{n,k},M_{\ell+1,1}(\mathbf{r}^{n,k}))C_q(j,\mathbf{s},M''(\mathbf{s}))\kappa(\mathbf{r}^{n,k},\mathbf{s})e_{M_{\ell+1,1}(\mathbf{r}^{n,k}),M''(\mathbf{s})}\\
&= \sum_{M'} C_q(1,\mathbf{r}^{n,k},\mathbf{r}^{n+1,k})C_q(j,\mathbf{s},M'(\mathbf{s}))\kappa(\mathbf{r}^{n,k},\mathbf{s})e_{\mathbf{r}^{n+1,k},M'(\mathbf{s})}\\
&\quad +\sum_{M''} C_q(1,\mathbf{r}^{n,k},\mathbf{r}^{n,k-1}))C_q(j,\mathbf{s},M''(\mathbf{s}))\kappa(\mathbf{r}^{n,k},\mathbf{s})e_{\mathbf{r}^{n,k-1},M''(\mathbf{s})},
\end{aligned}\tag{9.10}$$

where the first sum is over all moves $M' \in \mathbb{N}^j$ whose first coordinate is 1 and the second sum is over all moves $M'' \in \mathbb{N}^j$ whose first coordinate is $\ell+1$. If we now plug in the values of the Clebsch-Gordon coefficients from equations (4.10) and (4.11), we get

$$\begin{aligned}
u_{1,j}e_{\mathbf{r}^{n,k},\mathbf{s}} &= \sum_{M'} P_1'P_2'q^{k+C(j,\mathbf{s},M')}\kappa(\mathbf{r}^{n,k},\mathbf{s})e_{\mathbf{r}^{n+1,k},M'(\mathbf{s})}\\
&\quad +\sum_{M''} P_1''P_2''q^{C(j,\mathbf{s},M'')}\kappa(\mathbf{r}^{n,k},\mathbf{s})e_{\mathbf{r}^{n,k-1},M''(\mathbf{s})},
\end{aligned}\tag{9.11}$$

where P_i', P_j'' and $k(\mathbf{r}^{n,k},\mathbf{s})$ all lie between two fixed positive numbers. Boundedness of the commutators $[D,u_{1,j}]$ now follow directly.

For summability, notice that the eigenspace of $|D|$ corresponding to the eigenvalue $n\in\mathbb{N}$ is the span of

$$\{e_{\mathbf{r}^{k,n-k},\mathbf{s}} : 0\le k\le n, \mathbf{s}\in\Gamma_0^{k,n-k}\}.$$

Now just count the number of elements in the above set to get summability.

Next, we will compute the pairing of the K-homology class of this D with a generator of the K_1 group. Write $\omega_q := q^{-\ell}u_{1,\ell+1}$. From the commutation relations, it follows that this element has spectrum

$$\{z\in\mathbb{C} : |z| = 0 \text{ or } q^n \text{ for some } n\in\mathbb{N}\}.$$

Then the element $\gamma_q := \chi_{\{1\}}(\omega_q^*\omega_q)(\omega_q - I) + I$ is unitary. We will show that the index of the operator $Q\gamma_q Q$ (viewed as an operator on $QL_2(G\backslash H)$) is 1, where $Q = \frac{I-\operatorname{sign} D}{2}$, i. e. it is the projection onto the closed linear span of $\{e_{\mathbf{r}^{0,k},\mathbf{s}} : k\in\mathbb{N}, \mathbf{s}\in\Gamma_0^{0,k}\}$. What we will actually do is compute the index of the operator $Q\gamma_0 Q$ and appeal to continuity of the index. From equation (9.10), we get

$$\begin{aligned}
&u_{1,\ell+1}e_{\mathbf{r}^{0,k},\mathbf{s}}\\
&\quad = C_q(1,\mathbf{r}^{0,k},M_{11}(\mathbf{r}^{0,k}))C_q(\ell+1,\mathbf{s},N_{1,0}(\mathbf{s}))\kappa(\mathbf{r}^{0,k},M_{11}(\mathbf{r}^{0,k}))e_{\mathbf{r}^{1,k},N_{1,0}(\mathbf{s})}\\
&\qquad +C_q(1,\mathbf{r}^{0,k},M_{\ell+1,1}(\mathbf{r}^{0,k}))C_q(\ell+1,\mathbf{s},M_{\ell+1,\ell+1}(\mathbf{s}))\kappa(\mathbf{r}^{0,k},M_{\ell+1,1}(\mathbf{r}^{0,k}))e_{\mathbf{r}^{0,k-1},M_{\ell+1,\ell+1}(\mathbf{s})}.
\end{aligned}$$

(9.12)

Use the formula (4.3) for Clebsch-Gordon coefficients to get

$$\begin{aligned} C_q(1,\mathbf{r}^{0,k},M_{11}(\mathbf{r}^{0,k})) &= q^k(1+o(q)), && (9.13)\\ C_q(1,\mathbf{r}^{0,k},M_{\ell+1,1}(\mathbf{r}^{0,k}) &= 1+o(q), && (9.14)\\ C_q(\ell+1,\mathbf{s},N_{1,0}(\mathbf{s})) &= 1+o(q), && (9.15)\\ C_q(\ell+1,\mathbf{s},M_{\ell+1,\ell+1}(\mathbf{s})) &= q^{s_{\ell+1,1}+\ell}(1+o_4(q)), && (9.16) \end{aligned}$$

where $o(q)$ signifies a function of q that is continuous at $q=0$ and $o(0)=0$. We also have

$$\begin{aligned} \kappa(\mathbf{r}^{0,k},M_{11}(\mathbf{r}^{0,k})) &= q^\ell(1+o(q)), && (9.17)\\ \kappa(\mathbf{r}^{0,k},M_{\ell+1,1}(\mathbf{r}^{0,k})) &= 1+o(q), && (9.18) \end{aligned}$$

where $o(q)$ is as earlier. Plugging these values in (9.12) we get

$$\omega_q e_{\mathbf{r}^{0,k},\mathbf{s}} = q^k(1+o(q))e_{\mathbf{r}^{1,k},N_{1,0}(\mathbf{s})} + q^{s_{\ell+1,1}}(1+o(q))e_{\mathbf{r}^{0,k-1},M_{\ell+1,\ell+1}(\mathbf{s})} \tag{9.19}$$

Putting $q=0$, we get

$$\omega_0 e_{\mathbf{r}^{0,k},\mathbf{s}} = \begin{cases} e_{\mathbf{r}^{0,k-1},M_{\ell+1,\ell+1}(\mathbf{s})} & \text{if } k>0 \text{ and } s_{\ell+1,1}=0,\\ e_{\mathbf{r}^{1,0},N_{1,0}(\mathbf{s})} & \text{if } k=0,\\ 0 & \text{otherwise.} \end{cases} \tag{9.20}$$

Thus $\omega_0^*\omega_0$ is the projection onto the span of $\{e_{\mathbf{r}^{0,k},\mathbf{s}^k} : k\in\mathbb{N}\}$ where $\mathbf{s}^k$ is the GT tableaux given by

$$s_{ij}^k = \begin{cases} 0 & \text{if } i=\ell+2-j,\\ k & \text{otherwise,} \end{cases}$$

which is uniquely determined by the conditions $s_{\ell+1,1}=0$ and that $\mathbf{s}\in\Gamma_0^{0,k}$. Therefore the operator γ_0 is given by

$$\gamma_0 e_{\mathbf{r}^{0,k},\mathbf{s}} = e_{\mathbf{r}^{0,k},\mathbf{s}} - \chi_{\{\mathbf{s}=\mathbf{s}^k\}} e_{\mathbf{r}^{0,k},\mathbf{s}} + \chi_{\{\mathbf{s}=\mathbf{s}^k\}} e_{\mathbf{r}^{0,k-1},\mathbf{s}^{k-1}}.$$

It now follows that the index of $Q\gamma_0 Q$ is 1.

Optimality follows from part 1 of the previous theorem. □

References

[1] Alisauskas, S.; Smirnov, Yu. F. : Multiplicity-free $u_q(n)$ coupling coefficients. *J. Phys.* A 27 (1994), no. 17, 5925–5939.

[2] Chakraborty, P. S. ; Pal, A. : Characterization of spectral triples: A combinatorial approach, arXiv:math.OA/0305157.

[3] Chakraborty, P. S. ; Pal, A. : Equivariant spectral triples on the quantum $SU(2)$ group, arXiv:math.KT/0201004, *K-Theory*, 28(2003), No. 2, 107-126.

[4] Chakraborty, P. S. ; Pal, A. : Spectral triples and associated Connes-de Rham complex for the quantum $SU(2)$ and the quantum sphere, arXiv:math.QA/0210049, *Commun. Math. Phys.*, 240(2003), No. 3, 447-456.

[5] Chari, Vyjayanthi ; Pressley, Andrew: *A guide to quantum groups*, Cambridge University Press, Cambridge, 1995.

[6] Connes, A. : *Noncommutative Geometry*, Academic Press, 1994.

[7] Connes, A. : Gravity coupled with matter and the foundation of non-commutative geometry, *Comm. Math. Phys.*, 182 (1996), no. 1, 155–176.

[8] Connes, A. : Cyclic cohomology, quantum group symmetries and the local index formula for $SU_q(2)$, *J. Inst. Math. Jussieu* 3 (2004), no. 1, 17–68, arXiv:math.QA/0209142.

[9] Connes, A.; Moscovici, H. : The local index formula in noncommutative geometry, *Geom. Funct. Anal.* 5 (1995), no. 2, 174–243.

[10] Dabrowski, Ludwik; Sitarz, Andrzej : Dirac operator on the standard Podleś quantum sphere. *Noncommutative geometry and quantum groups* (Warsaw, 2001), 49–58, Banach Center Publ., 61, Polish Acad. Sci., Warsaw, 2003.

[11] Dabrowski, Ludwik; Landi, Giovanni ; Paschke, Mario ; Sitarz, Andrzej : The Spectral Geometry of the Equatorial Podles Sphere, arXiv:math.QA/0408034.

[12] Dabrowski, Ludwik; Landi, Giovanni; Sitarz, Andrzej; van Suijlekom, Walter; Varilly, Joseph C. : The Dirac operator on $SU_q(2)$, arXiv:math.QA/0411609.

[13] Hawkins, Eli; Landi, Giovanni : Fredholm modules for quantum Euclidean spheres. *J. Geom. Phys.* 49 (2004), no. 3-4, 272–293.

[14] Hong, Jeong Hee; Szymański, Wojciech : Quantum spheres and projective spaces as graph algebras. *Comm. Math. Phys.* 232 (2002), no. 1, 157–188.

[15] Klimyk, A. ; Schmuedgen, K. : *Quantum Groups and their Representations*, Springer, New York, 1998.

[16] Korogodski, Leonid I.; Soibelman, Yan S. : *Algebras of functions on quantum groups. Part I.* Mathematical Surveys and Monographs, 56. American Mathematical Society, Providence, RI, 1998.

[17] Krähmer, Ulrich : Dirac operators on quantum flag manifolds, arXiv:math.QA/0305071, *Lett. Math. Phys.*, 67 (2004), no. 1, 49–59.

[18] Pal, A. : Induced representation and Frobenius reciprocity for compact quantum groups. *Proc. of the Indian Acad. of Sc.*, No. 2, 105(1995), 157-167.

[19] Podleś, Piotr : Symmetries of quantum spaces. Subgroups and quotient spaces of quantum SU(2) and SO(3) groups, *Comm. Math. Phys.* 170 (1995), no. 1, 1–20.

[20] Rosso, Marc : Algèbres enveloppantes quantifis, groupes quantiques compacts de matrices et calcul diffentiel non commutatif. *Duke Math. J.* 61 (1990), no. 1, 11–40.

[21] van Suijlekom, Walter; Dabrowski, Ludwik; Landi, Giovanni; Sitarz, Andrzej; Varilly, Joseph C. : Local index formula for $SU_q(2)$, arXiv:math.QA/0501287.

[22] Vaksman, L. L.; Soibelman, Yan S. : Algebra of functions on the quantum group SU(n+1), and odd-dimensional quantum spheres. (Russian) *Algebra i Analiz* 2 (1990), no. 5, 101–120; translation in *Leningrad Math. J.* 2 (1991), no. 5, 1023–1042.

[23] Woronowicz, S. L. : Tannaka-Kreĭn duality for compact matrix pseudogroups. Twisted SU(N) groups, *Invent. Math.* 93 (1988), no. 1, 35–76.

Partha Sarathi Chakraborty (chakrabortyps@cf.ac.uk)
School of Mathematics, Cardiff University, Senghennydd Road, Cardiff, UK

Arupkumar Pal (arup@isid.ac.in)
Indian Statistical Institute, 7, SJSS Marg, New Delhi–110 016, INDIA