

Characterization of $SU_q(\ell + 1)$ -equivariant spectral triples for the odd dimensional quantum spheres

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February 2, 2008

Dedicated to Prof. K. R. Parthasarathy on his seventieth birthday.

Abstract

The quantum group $SU_q(\ell + 1)$ has a canonical action on the odd dimensional sphere $S_q^{2\ell+1}$. All odd spectral triples acting on the L_2 space of $S_q^{2\ell+1}$ and equivariant under this action have been characterized. This characterization then leads to the construction of an optimum family of equivariant spectral triples having nontrivial K -homology class. These generalize the results of Chakraborty & Pal for $SU_q(2)$.

AMS Subject Classification No.: 58B34, 46L87, 19K33

Keywords. Spectral triples, noncommutative geometry, quantum group.

1 Introduction

Noncommutative differential geometry, which more commonly just goes by the name noncommutative geometry, is an extension of noncommutative topology and was initially developed in order to handle certain spaces like the leaf space of foliations or duals of groups whose topology or geometry are difficult to study using machinery available in classical geometry or topology. As the subject developed, more and more examples were found that are further away from classical spaces but can be handled by noncommutative geometric methods.

For quite sometime though, it was commonly believed that quantum groups or their homogeneous spaces, which are rather far removed from classical manifolds, are not covered by the formalism of noncommutative geometry. This notion changed with ([2]), where the authors treated the case of the quantum $SU(2)$ group and found a family of spectral triples acting on its L_2 -space that are equivariant with respect to its natural (co)action. This family is optimal, in the sense that given any nontrivial equivariant Dirac operator D acting on the L_2 space, there exists a Dirac operator \tilde{D} belonging to this family such that $\text{sign } D$ is a compact perturbation of $\text{sign } \tilde{D}$ and there exist reals a and b such that

$$|D| \leq a + b|\tilde{D}|.$$

Later Dabrowski et al [13] constructed another equivariant spectral triple for $SU_q(2)$ on two copies of the L_2 space, which was shown in [4] to be equivalent to a direct sum two spectral triples constructed in [2]. Equivariant triples for the two dimensional Podleś spheres were constructed in [11] and [12]. In a more recent paper ([10]), D’Andrea et al gave a construction of an equivariant spectral triple for the quantum four dimensional spheres.

Our aim in the present paper is to look for higher dimensional counterparts of the spectral triples found in [2]. We first formulate precisely what one means by an equivariant spectral triple in a general set up. We then use a combinatorial method, implicitly used in [2] and [3], to characterize completely all odd spectral triples acting on the L_2 space of the odd dimensional sphere $S_q^{2\ell+1}$ (see section 3 for the description) and equivariant under the action of the $SU_q(\ell+1)$ group **for all** $\ell > 1$. This also leads to the construction of an optimum family of equivariant nontrivial $(2\ell + 1)$ -summable odd spectral triples sharing all the properties of the triples for $SU_q(2)$ in [2].

One should mention in this context that the construction by Hawkins & Landi ([14]) does not deal with equivariance, and more importantly, they produce a (bounded) Fredholm module, not a spectral triple, which is essential for determining the smooth structure, giving a metric on the state space and also help in computing the index map through a local Chern character.

The paper is organised as follows. In the next section, we will describe the combinatorial method that was earlier used implicitly in [2] and [3]. We also formulate the notion of equivariance. This has been done using the quantum group at the function algebra level rather than passing on to the quantum universal enveloping algebra level. In section 3, we describe the C^* -algebra of continuous functions on the odd dimensional quantum spheres and state some of their relevant properties. In section 4, we briefly recall the quantum group $SU_q(\ell + 1)$ and its representation theory. In particular, we describe a nice basis for the L_2 space and study the Clebsch-Gordon coefficients. These are then used to describe the action by left multiplication on the L_2 space explicitly. In section 5, we give a description of the L_2 space of the sphere and give a natural covariant representation on it. In the last section, we give a precise characterization of the singular values and of the sign, which helps us to produce an optimal family of equivariant Dirac operators, extending the results of [2] in the present case.

2 Preliminaries

2.1 Equivariance

Suppose G is a compact group, quantum or classical, and \mathcal{A} is a unital C^* -algebra. Assume that G has an action on \mathcal{A} given by $\tau : \mathcal{A} \rightarrow \mathcal{A} \otimes C(G)$, so that $(\text{id} \otimes \Delta)\tau = (\tau \otimes \text{id})\tau$, Δ being the coproduct. In other words, we have a C^* -dynamical system (\mathcal{A}, G, τ) .

Definition 2.1 A covariant representation (π, u) of (\mathcal{A}, G, τ) consists of a unital $*$ -representation

$\pi : \mathcal{A} \rightarrow \mathcal{L}(\mathcal{H})$, a unitary representation u of G on \mathcal{H} , i.e. a unitary element of the multiplier algebra $M(\mathcal{K}(\mathcal{H}) \otimes C(G))$ such that they obey the condition $(\pi \otimes \text{id})\tau(a) = u(\pi(a) \otimes I)u^*$ for all $a \in \mathcal{A}$.

Definition 2.2 Suppose (\mathcal{A}, G, τ) is a C^* -dynamical system. An **odd G -equivariant spectral data** for (\mathcal{A}, G, τ) is a quadruple (π, u, \mathcal{H}, D) where

1. (π, u) is a covariant representation of (\mathcal{A}, G, τ) on \mathcal{H} ,
2. π faithful,
3. $u(D \otimes I)u^* = D \otimes I$,
4. (π, \mathcal{H}, D) is an odd spectral triple.

2.2 The general scheme

Let \mathcal{G} be a graph and (V_1, V_2) be a partition of the vertex set. We say that (V_1, V_2) admits an **infinite ladder** if there exist infinite number of disjoint paths each going from a point in V_1 to a point in V_2 . Here two paths are disjoint means that the set of vertices of one does not intersect the set of vertices of the other.

Suppose \mathcal{H} is a Hilbert space, and D is a self-adjoint operator on \mathcal{H} with compact resolvent. Then D admits a spectral resolution $\sum_{\gamma \in \Gamma} d_\gamma P_\gamma$, where the d_γ 's are all distinct and each P_γ is a finite dimensional projection. Assume now onward that all the d_γ 's are nonzero. Let c be a positive real. Let us define a graph \mathcal{G}_c as follows: take the vertex set V to be Γ . Connect two vertices γ and γ' by an edge if $|d_\gamma - d_{\gamma'}| < c$. Let $V^+ = \{\gamma \in V : d_\gamma > 0\}$ and $V^- = \{\gamma \in V : d_\gamma < 0\}$. This will give us a partition of V . This partition has the important property that (V^+, V^-) does not admit an infinite ladder. This is easy to see, because if there is a path from γ to δ and $d_\gamma > 0$, $d_\delta < 0$, then for some α on the path, one must have $d_\alpha \in [-c, c]$. Since the paths are disjoint, it would contradict the compact resolvent condition. We will call such a partition a **sign-determining** partition.

We will use this knowledge about the graph. We start with a self-adjoint operator with discrete spectrum. First choose a basis that diagonalizes the operator D . Next we use the action of the algebra elements on the basis elements of \mathcal{H} and the boundedness of their commutators with D . This gives certain growth restrictions on the d_γ 's. These will give us some information about the edges in the graph. We exploit this knowledge to characterize those partitions (V_1, V_2) of the vertex set that are sign-determining, i. e. do not admit any infinite ladder. The sign of the operator D must be of the form $\sum_{\gamma \in V_1} P_\gamma - \sum_{\gamma \in V_2} P_\gamma$ where (V_1, V_2) is a sign-determining partition. Of course, for a given c , the graph \mathcal{G}_c may have no edges, or too few edges (if the singular values of D happen to grow too fast), in which case, we will be left with too many

sign-determining partitions. Therefore, we would like to characterize those partitions that are sign-determining for all sufficiently large values of c .

In general the scheme outlined above will be extremely difficult to carry out, as the action of the algebra elements with respect to the basis that diagonalizes D may be quite complicated, and therefore using boundedness of commutator conditions will in general be very difficult. This is where equivariance plays an extremely crucial role. It gives us a *nice basis* that diagonalizes D , so that the boundedness of commutator conditions are simpler and the subsequent steps become much more tractable.

3 The odd dimensional quantum spheres

Let $q \in [0, 1]$. The C^* -algebra $A_\ell \equiv C(S_q^{2\ell+1})$ of the quantum sphere $S_q^{2\ell+1}$ is the universal C^* -algebra generated by elements $z_1, z_2, \dots, z_{\ell+1}$ satisfying the following relations (see [15]):

$$\begin{aligned} z_i z_j &= q z_j z_i, & 1 \leq j < i \leq \ell + 1, \\ z_i z_j^* &= q z_j^* z_i, & 1 \leq i \neq j \leq \ell + 1, \\ z_i z_i^* - z_i^* z_i + (1 - q^2) \sum_{k>i} z_k z_k^* &= 0, & 1 \leq i \leq \ell + 1, \\ \sum_{i=1}^{\ell+1} z_i z_i^* &= 1. \end{aligned}$$

The K -theory groups for these algebras were computed in [23] and [15].

Proposition 3.1 ([23],[15]) $K_0(A_\ell) = K_1(A_\ell) = \mathbb{Z}$.

The group $SU_q(\ell + 1)$ has an action on $S_q^{2\ell+1}$. Before we describe the action, let us recall the definition of the quantum group $SU_q(\ell + 1)$. The C^* -algebra $C(SU_q(\ell + 1))$ is the universal C^* -algebra generated by $\{u_{ij} : i, j = 1, \dots, \ell + 1\}$ obeying the relations:

$$\begin{aligned} \sum_k u_{ki}^* u_{kj} &= \delta_{ij} I, & \sum_k u_{ik} u_{jk}^* &= \delta_{ij} I \\ \sum_{k_i \text{'s distinct}} (-q)^{I(k_1, k_2, \dots, k_{\ell+1})} u_{j_1 k_1} \cdots u_{j_{\ell+1} k_{\ell+1}} &= \begin{cases} (-q)^{I(j_1, j_2, \dots, j_{\ell+1})} & j_i \text{'s distinct} \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

where $I(k_1, k_2, \dots, k_{\ell+1})$ is the number of inversions in $(k_1, k_2, \dots, k_{\ell+1})$. The group laws are given by the following maps:

$$\begin{aligned} \Delta(u_{ij}) &= \sum_k u_{ik} \otimes u_{kj} & (\text{Comultiplication}) \\ S(u_{ij}) &= u_{ji}^* & (\text{Antipode}) \\ \epsilon(u_{ij}) &= \delta_{ij} & (\text{Counit}) \end{aligned}$$

The map

$$\tau(z_i) = \sum_k z_k \otimes u_{ki}^*$$

extends to a *-homomorphism τ from A_ℓ into $A_\ell \otimes C(SU_q(\ell+1))$ and obeys $(\text{id} \otimes \Delta)\tau = (\tau \otimes \text{id})\tau$. In other words this gives an action of $SU_q(\ell+1)$ on A_ℓ .

4 Preliminaries on $SU_q(\ell+1)$

Our next job will be to get a description of the covariant representation of the system $(A_\ell, SU_q(\ell+1), \tau)$ on $L_2(S_q^{2\ell+1})$. For this we need a few facts on the representation theory of $SU_q(\ell+1)$. In the first subsection we describe an important indexing of the basis elements of the representation space of the irreducibles. Then we describe the Clebsch-Gordon coefficients and compute certain estimates. In the last subsection, we write down explicitly the left multiplication operator on $L_2(SU_q(\ell+1))$.

4.1 Gelfand-Tsetlin tableaux

Irreducible unitary representations of the group $SU_q(\ell+1)$ are indexed by Young tableaux $\lambda = (\lambda_1, \dots, \lambda_{\ell+1})$, where λ_i 's are nonnegative integers, $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_{\ell+1}$ (Theorem 1.5, [24]). Write \mathcal{H}_λ for the Hilbert space where the irreducible λ acts. There are various ways of indexing the basis elements of \mathcal{H}_λ . The one we will use is due to Gelfand and Tsetlin. According to their prescription, basis elements for \mathcal{H}_λ are parametrized by arrays of the form

$$\mathbf{r} = \begin{pmatrix} r_{11} & r_{12} & \cdots & r_{1,\ell} & r_{1,\ell+1} \\ r_{21} & r_{22} & \cdots & r_{2,\ell} & \\ & & \cdots & & \\ r_{\ell,1} & r_{\ell,2} & & & \\ r_{\ell+1,1} & & & & \end{pmatrix},$$

where r_{ij} 's are integers satisfying $r_{1j} = \lambda_j$ for $j = 1, \dots, \ell+1$, $r_{ij} \geq r_{i+1,j} \geq r_{i,j+1} \geq 0$ for all i, j . Such arrays are known as Gelfand-Tsetlin tableaux, to be abbreviated as GT tableaux for the rest of this section. For a GT tableaux \mathbf{r} , the symbol \mathbf{r}_i will denote its i th row. It is well-known that two representations indexed respectively by λ and λ' are equivalent if and only if $\lambda_j - \lambda'_j$ is independent of j ([24]). Thus one gets an equivalence relation on the set of Young tableaux $\{\lambda = (\lambda_1, \dots, \lambda_{\ell+1}) : \lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_{\ell+1}, \lambda_j \in \mathbb{N}\}$. This, in turn, induces an equivalence relation on the set of all GT tableaux $\Gamma = \{\mathbf{r} : r_{ij} \in \mathbb{N}, r_{ij} \geq r_{i+1,j} \geq r_{i,j+1}\}$: one says \mathbf{r} and \mathbf{s} are equivalent if $r_{ij} - s_{ij}$ is independent of i and j . By Γ we will mean the above set modulo this equivalence.

We will denote by u^λ the irreducible unitary indexed by λ , $\{e(\lambda, \mathbf{r}) : \mathbf{r}_1 = \lambda\}$ will denote an orthonormal basis for \mathcal{H}_λ and $u_{\mathbf{r}\mathbf{s}}^\lambda$ will stand for the matrix entries of u^λ in this basis. The

symbol $\mathbb{1}$ will denote the Young tableaux $(1, 0, \dots, 0)$. We will often omit the symbol $\mathbb{1}$ and just write u in order to denote $u^{\mathbb{1}}$. Notice that any GT tableaux \mathbf{r} with first row $\mathbb{1}$ must be, for some $i \in \{1, 2, \dots, \ell + 1\}$, of the form (r_{ab}) , where

$$r_{ab} = \begin{cases} 1 & \text{if } 1 \leq a \leq i \text{ and } b = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Thus such a GT tableaux is uniquely determined by the integer i . We will write just i for this GT tableaux \mathbf{r} . Thus for example, a typical matrix entry of $u^{\mathbb{1}}$ will be written simply as u_{ij} .

Let $\mathbf{r} = (r_{ab})$ be a GT tableaux. Let $H_{ab}(\mathbf{r}) := r_{a+1,b} - r_{a,b+1}$ and $V_{ab}(\mathbf{r}) := r_{ab} - r_{a+1,b}$. An element \mathbf{r} of Γ is completely specified by the following differences

$$\mathbf{D}(\mathbf{r}) = \begin{pmatrix} V_{11}(\mathbf{r}) & H_{11}(\mathbf{r}) & H_{12}(\mathbf{r}) & \cdots & H_{1,\ell-1}(\mathbf{r}) & H_{1,\ell}(\mathbf{r}) \\ V_{21}(\mathbf{r}) & H_{21}(\mathbf{r}) & H_{22}(\mathbf{r}) & \cdots & H_{2,\ell-1}(\mathbf{r}) & \\ \cdots & & & & & \\ V_{\ell,1}(\mathbf{r}) & H_{\ell,1}(\mathbf{r}) & & & & \end{pmatrix}.$$

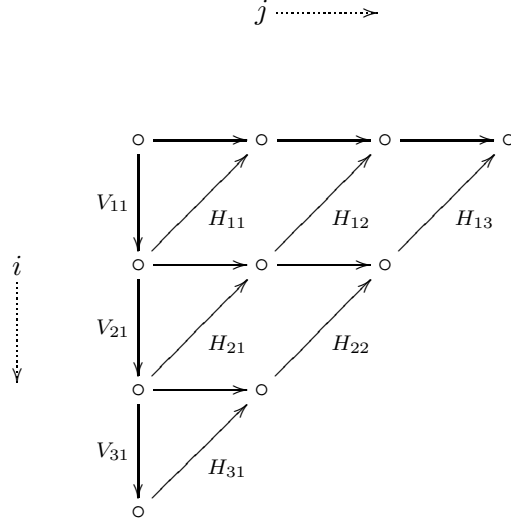
The differences satisfy the following inequalities

$$\sum_{k=0}^b H_{a-k,k+1}(\mathbf{r}) \leq V_{a+1,1}(\mathbf{r}) + \sum_{k=0}^b H_{a-k+1,k+1}(\mathbf{r}), \quad 1 \leq a \leq \ell, \quad 0 \leq b \leq a - 1. \quad (4.1)$$

Conversely, if one has an array of the form

$$\begin{pmatrix} V_{11} & H_{11} & H_{12} & \cdots & H_{1,\ell-1} & H_{1,\ell} \\ V_{21} & H_{21} & H_{22} & \cdots & H_{2,\ell-1} & \\ \cdots & & & & & \\ V_{\ell,1} & H_{\ell,1} & & & & \end{pmatrix},$$

where V_{ij} 's and H_{ij} 's are in \mathbb{N} and obey the inequalities (4.1), then the above array is of the form $\mathbf{D}(\mathbf{r})$ for some GT tableaux \mathbf{r} . Thus the quantities V_{a1} and H_{ab} give a coordinate system for elements in Γ . The following diagram explains this new coordinate system. The hollow circles stand for the r_{ij} 's. The entries are decreasing along the direction of the arrows, and the V_{ij} 's and the H_{ij} 's are the difference between the two endpoints of the corresponding arrows.



4.2 Clebsch-Gordon coefficients

In this subsection, we recall the Clebsch-Gordon coefficients for the group $SU_q(\ell+1)$. This will be important in writing down the natural representation of $C(S_q^{2\ell+1})$ on $L_2(S_q^{2\ell+1})$ explicitly.

Look at the representation $u^{\mathbb{1}} \otimes u^\lambda$ acting on $\mathcal{H}_{\mathbb{1}} \otimes \mathcal{H}_\lambda$. The representation decomposes as a direct sum $\oplus_\mu u^\mu$, i.e. one has a corresponding decomposition $\oplus_\mu \mathcal{H}_\mu$ of $\mathcal{H}_{\mathbb{1}} \otimes \mathcal{H}_\lambda$. Thus one has two orthonormal bases $\{e_s^\mu\}$ and $\{e_i^{\mathbb{1}} \otimes e_r^\lambda\}$. The Clebsch-Gordon coefficient $C_q(\mathbb{1}, \lambda, \mu; i, \mathbf{r}, \mathbf{s})$ is defined to be the inner product $\langle e_s^\mu, e_i^{\mathbb{1}} \otimes e_r^\lambda \rangle$. Since $\mathbb{1}$, λ and μ are just the first rows of i , \mathbf{r} and \mathbf{s} respectively, we will often denote the above quantity just by $C_q(i, \mathbf{r}, \mathbf{s})$.

Next, we will compute the quantities $C_q(i, \mathbf{r}, \mathbf{s})$. We will use the calculations given in ([16], pp. 220), keeping in mind that for our case (i.e. for $SU_q(\ell+1)$), the top right entry of the GT tableaux is zero.

Let $M = (m_1, m_2, \dots, m_i) \in \mathbb{N}^i$ be such that $1 \leq m_j \leq \ell+2-j$. Denote by $M(\mathbf{r})$ the tableaux \mathbf{s} defined by

$$s_{jk} = \begin{cases} r_{jk} + 1 & \text{if } k = m_j, 1 \leq j \leq i, \\ r_{jk} & \text{otherwise.} \end{cases} \quad (4.2)$$

With this notation, observe now that $C_q(i, \mathbf{r}, \mathbf{s})$ will be zero unless \mathbf{s} is $M(\mathbf{r})$ for some $M \in \mathbb{N}^i$. (One has to keep in mind though that not all tableaux of the form $M(\mathbf{r})$ is a valid GT tableaux)

From ([16], pp. 220), we have

$$C_q(i, \mathbf{r}, M(\mathbf{r})) = \prod_{a=1}^{i-1} \left\langle \begin{array}{cc|c} (1, \mathbf{0}) & \mathbf{r}_a & \mathbf{r}_a + e_{m_a} \\ (1, \mathbf{0}) & \mathbf{r}_{a+1} & \mathbf{r}_{a+1} + e_{m_{a+1}} \end{array} \right\rangle \times \left\langle \begin{array}{cc|c} (1, \mathbf{0}) & \mathbf{r}_i & \mathbf{r}_i + e_{m_i} \\ (0, \mathbf{0}) & \mathbf{r}_{i+1} & \mathbf{r}_{i+1} \end{array} \right\rangle, \quad (4.3)$$

where e_k stands for a vector (in the appropriate space) whose k^{th} coordinate is 1 and the rest

are all zero, \mathbf{r}_j stands for the j^{th} row of the tableaux \mathbf{r} , and

$$\begin{aligned} \left\langle \begin{array}{c|c} (1, \mathbf{0}) & \mathbf{r}_a \\ (1, \mathbf{0}) & \mathbf{r}_{a+1} \end{array} \middle| \begin{array}{c} \mathbf{r}_a + e_j \\ \mathbf{r}_{a+1} + e_k \end{array} \right\rangle^2 &= q^{-r_{aj} + r_{a+1,k} - k + j} \times \prod_{\substack{i=1 \\ i \neq j}}^{\ell+2-a} \frac{[r_{a,i} - r_{a+1,k} - i + k]_q}{[r_{a,i} - r_{a,j} - i + j]_q} \\ &\times \prod_{\substack{i=1 \\ i \neq k}}^{\ell+1-a} \frac{[r_{a+1,i} - r_{a,j} - i + j - 1]_q}{[r_{a+1,i} - r_{a+1,k} - i + k - 1]_q}, \end{aligned} \quad (4.4)$$

$$\begin{aligned} \left\langle \begin{array}{c|c} (1, \mathbf{0}) & \mathbf{r}_a \\ (0, \mathbf{0}) & \mathbf{r}_{a+1} \end{array} \middle| \begin{array}{c} \mathbf{r}_a + e_j \\ \mathbf{r}_{a+1} \end{array} \right\rangle^2 &= q^{\left(1 - j + \sum_{i=1}^{\ell+1-a} r_{a+1,i} - \sum_{\substack{i=1 \\ i \neq j}}^{\ell+2-a} r_{a,i}\right)} \\ &\times \left(\frac{\prod_{i=1}^{\ell+1-a} [r_{a+1,i} - r_{a,j} - i + j - 1]_q}{\prod_{\substack{i=1 \\ i \neq j}}^{\ell+2-a} [r_{a,i} - r_{a,j} - i + j]_q} \right), \end{aligned} \quad (4.5)$$

where for an integer n , $[n]_q$ denotes the q -number $(q^n - q^{-n})/(q - q^{-1})$. After some lengthy but straightforward computations, we get the following two relations:

$$\left| \left\langle \begin{array}{c|c} (1, \mathbf{0}) & \mathbf{r}_a \\ (1, \mathbf{0}) & \mathbf{r}_{a+1} \end{array} \middle| \begin{array}{c} \mathbf{r}_a + e_j \\ \mathbf{r}_{a+1} + e_k \end{array} \right\rangle \right| = A' q^A, \quad (4.6)$$

$$\left| \left\langle \begin{array}{c|c} (1, \mathbf{0}) & \mathbf{r}_a \\ (0, \mathbf{0}) & \mathbf{r}_{a+1} \end{array} \middle| \begin{array}{c} \mathbf{r}_a + e_j \\ \mathbf{r}_{a+1} \end{array} \right\rangle \right| = B' q^B, \quad (4.7)$$

where

$$\begin{aligned} A &= \begin{cases} \sum_{j \wedge k < b < j \vee k} (r_{a+1,b} - r_{a,b}) + (r_{a+1,j \wedge k} - r_{a,j \vee k}) & \text{if } j \neq k, \\ 0 & \text{if } j = k. \end{cases} \\ &= \sum_{j \wedge k \leq b < j \vee k} (r_{a+1,b} - r_{a,b+1}) + 2 \sum_{k < b < j} (r_{a,b} - r_{a+1,b}) \\ &= \sum_{j \wedge k \leq b < j \vee k} H_{ab}(\mathbf{r}) + 2 \sum_{k < b < j} V_{ab}(\mathbf{r}). \end{aligned} \quad (4.8)$$

$$B = \sum_{j \leq b < \ell+2-a} H_{ab}(\mathbf{r}), \quad (4.9)$$

and A' and B' both lie between two positive constants independent of \mathbf{r} , a , j and k (Here and elsewhere in this paper, an empty summation would always mean zero).

Combining these, one gets

$$C_q(i, \mathbf{r}, M(\mathbf{r})) = P \cdot q^{C(i, \mathbf{r}, M)}, \quad (4.10)$$

where

$$C(i, \mathbf{r}, M) = \sum_{a=1}^{i-1} \left(\sum_{m_a \wedge m_{a+1} \leq b < m_a \vee m_{a+1}} H_{ab}(\mathbf{r}) + 2 \sum_{m_{a+1} < b < m_a} V_{ab}(\mathbf{r}) \right) + \sum_{m_i \leq b < \ell+2-i} H_{ib}(\mathbf{r}), \quad (4.11)$$

and P lies between two positive constants that are independent of i , \mathbf{r} and M .

Remark 4.1 The formulae (4.4) and (4.5) are obtained from equations (45) and (46), page 220, [16] by replacing q with q^{-1} . Equation (45) is a special case of the more general formula (48), page 221, [16]. However, there is a small error in equation (48) there. The correct form can be found in equations (3.1, 3.2a, 3.2b) in [1]. That correction has been incorporated in equations (4.4) and (4.5) here.

4.3 Left multiplication operators

We next write down the representation of $C(SU_q(\ell+1))$ on $L_2(SU_q(\ell+1))$ by left multiplication. Later we will work with a certain restriction of this representation.

The matrix entries $u_{\mathbf{rs}}^\lambda$ form a complete orthogonal set of vectors in $L_2(SU_q(\ell+1))$. Write $e_{\mathbf{rs}}^\lambda$ for $\|u_{\mathbf{rs}}^\lambda\|^{-1}u_{\mathbf{rs}}^\lambda$. Then the $e_{\mathbf{rs}}^\lambda$'s form a complete orthonormal basis for $L_2(SU_q(\ell+1))$. Let π denote the representation of \mathcal{A} on $L_2(SU_q(\ell+1))$ by left multiplications. We will now derive an expression for $\pi(u_{ij})e_{\mathbf{rs}}^\lambda$.

From the definition of matrix entries and that of the CG coefficients, one gets

$$u^\rho e(\rho, \mathbf{t}) = \sum_{\mathbf{s}} u_{\mathbf{st}}^\rho e(\rho, \mathbf{s}), \quad (4.12)$$

$$e(\mu, \mathbf{n}) = \sum_{j, \mathbf{s}} C_q(j, \mathbf{s}, \mathbf{n}) e(\mathbb{1}, j) \otimes e(\lambda, \mathbf{s}). \quad (4.13)$$

Apply $u \otimes u^\lambda$ on both sides and note that $u \otimes u^\lambda$ acts on $e(\mu, \mathbf{n})$ as u^μ :

$$\sum_{\mathbf{m}} u_{\mathbf{mn}}^\mu e(\mu, \mathbf{m}) = \sum_{j, \mathbf{s}} \sum_{i, \mathbf{r}} C_q(j, \mathbf{s}, \mathbf{n}) u_{ij} u_{\mathbf{rs}}^\lambda e(\mathbb{1}, i) \otimes e(\lambda, \mathbf{r}). \quad (4.14)$$

Next, use (4.13) to expand $e(\mu, \mathbf{m})$ on the left hand side to get

$$\sum_{i, \mathbf{r}, \mathbf{m}} u_{\mathbf{mn}}^\mu C_q(i, \mathbf{r}, \mathbf{m}) e(\mathbb{1}, i) \otimes e(\lambda, \mathbf{r}) = \sum_{j, \mathbf{s}} \sum_{i, \mathbf{r}} C_q(j, \mathbf{s}, \mathbf{n}) u_{ij} u_{\mathbf{rs}}^\lambda e(\mathbb{1}, i) \otimes e(\lambda, \mathbf{r}). \quad (4.15)$$

Equating coefficients, one gets

$$\sum_{\mathbf{m}} C_q(i, \mathbf{r}, \mathbf{m}) u_{\mathbf{mn}}^\mu = \sum_{j, \mathbf{s}} C_q(j, \mathbf{s}, \mathbf{n}) u_{ij} u_{\mathbf{rs}}^\lambda. \quad (4.16)$$

Now using orthogonality of the matrix $((C_q(\mathbb{1}, \lambda, \mu; j, \mathbf{s}, \mathbf{n}))_{(\mu, \mathbf{n}), (j, \mathbf{s})})$, we obtain

$$u_{ij} u_{\mathbf{rs}}^\lambda = \sum_{\mu, \mathbf{m}, \mathbf{n}} C_q(i, \mathbf{r}, \mathbf{m}) C_q(j, \mathbf{s}, \mathbf{n}) u_{\mathbf{mn}}^\mu. \quad (4.17)$$

From ([16], pp. 441), one has $\|u_{\mathbf{rs}}^\lambda\| = d_\lambda^{-\frac{1}{2}} q^{-\psi(\mathbf{r})}$, where

$$\psi(\mathbf{r}) = -\frac{\ell}{2} \sum_{j=1}^{\ell+1} r_{1j} + \sum_{i=2}^{\ell+1} \sum_{j=1}^{\ell+2-i} r_{ij}, \quad d_\lambda = \sum_{\mathbf{r}: \mathbf{r}_1 = \lambda} q^{2\psi(\mathbf{r})}$$

Therefore

$$\pi(u_{ij})e_{\mathbf{rs}}^\lambda = \sum_{\mu, \mathbf{m}, \mathbf{n}} C_q(\mathbb{1}, \lambda, \mu; i, \mathbf{r}, \mathbf{m}) C_q(\mathbb{1}, \lambda, \mu; j, \mathbf{s}, \mathbf{n}) d_\lambda^{\frac{1}{2}} d_\mu^{-\frac{1}{2}} q^{\psi(\mathbf{r}) - \psi(\mathbf{m})} e_{\mathbf{mn}}^\mu. \quad (4.18)$$

Write

$$\kappa(\mathbf{r}, \mathbf{m}) = d_\lambda^{\frac{1}{2}} d_\mu^{-\frac{1}{2}} q^{\psi(\mathbf{r}) - \psi(\mathbf{m})}. \quad (4.19)$$

Lemma 4.2 *There exist constants $K_2 > K_1 > 0$ such that $K_1 < \kappa(\mathbf{r}, M(\mathbf{r})) < K_2$ for all \mathbf{r} .*

Proof: Observe that

$$\psi(\mathbf{r}) = (\rho, \lambda(\mathbf{r})) = -\frac{\ell}{2} \sum_{j=1}^{\ell+1} r_{1j} + \sum_{i=2}^{\ell+1} \sum_{j=1}^{\ell+2-i} r_{ij}.$$

Therefore

$$\min\{\psi(\mathbf{r}) : \mathbf{r}_1 = \lambda\} = -\frac{\ell}{2} \sum_1^\ell \lambda_i + \sum_{k=2}^\ell (k-1)\lambda_k.$$

This implies that

$$d_\lambda^{\frac{1}{2}} = q^{-\frac{\ell}{2} \sum_1^\ell \lambda_i + \sum_{k=2}^\ell (k-1)\lambda_k} (1 + o(q)),$$

which gives us

$$\left(\frac{d_\lambda}{d_{\lambda+e_k}} \right)^{\frac{1}{2}} = q^{\frac{\ell}{2} - M_1 + 1} (1 + o(q)).$$

Next,

$$q^{\psi(\mathbf{r}) - \psi(M(\mathbf{r}))} = q^{-\frac{\ell}{2} \sum_{j=1}^{\ell+1} r_{1j} + \sum_{i=2}^{\ell+1} \sum_{j=1}^{\ell+2-i} r_{ij} + \frac{\ell}{2} (\sum_{j=1}^{\ell+1} r_{1j} + 1) - (\sum_{i=2}^{\ell+1} \sum_{j=1}^{\ell+2-i} r_{ij} + i - 1)} = q^{\frac{\ell}{2} - i + 1}.$$

Thus

$$\kappa(\mathbf{r}, M(\mathbf{r})) = q^{\frac{\ell}{2} - i - M_1 + 2} (1 + o(q)).$$

Hence the conclusion follows. \square

5 Covariant representation

Let us write G for $SU_q(\ell+1)$ and H for $SU_q(\ell)$. H is a subgroup of G . This means that there is a C^* -epimorphism $\phi : C(G) \rightarrow C(H)$ obeying $\Delta_H \phi = (\phi \otimes \phi) \Delta_G$. In such a case, one defines the quotient space $G \setminus H$ by

$$C(G \setminus H) := \{a \in C(G) : (\phi \otimes id) \Delta(a) = I \otimes a\}.$$

The group G has a canonical right action $C(G \setminus H) \rightarrow C(G \setminus H) \otimes C(G)$ coming from the restriction of the comultiplication Δ to $C(G \setminus H)$. Let ρ denote the restriction of the Haar state

on $C(G)$ to $C(G \setminus H)$. Then clearly one has $(\rho \otimes id)\Delta(a) = \rho(a)I$, which means ρ is the invariant state for $C(G \setminus H)$. This also means that $L_2(G \setminus H) = L_2(\rho)$ is just the closure of $C(G \setminus H)$ in $L_2(G)$. (For a formulation of quotient spaces etc. in the context of compact quantum groups, see [20])

Now suppose we make the following explicit choice of ϕ . Let $u^{\mathbb{1}}$ denote the fundamental unitary for G , i. e. the irreducible unitary representation corresponding to the Young tableaux $\mathbb{1} = (1, 0, \dots, 0)$. Similarly write $v^{\mathbb{1}}$ for the fundamental unitary for H . Fix some bases for the corresponding representation spaces. Then $C(G)$ is the C^* -algebra generated by the matrix entries $\{u_{ij}^{\mathbb{1}}\}$ and $C(H)$ is the C^* -algebra generated by the matrix entries $\{v_{ij}^{\mathbb{1}}\}$. Now define ϕ by

$$\phi(u_{ij}^{\mathbb{1}}) = \begin{cases} I & \text{if } i = j = 1, \\ v_{i-1, j-1}^{\mathbb{1}} & \text{if } 2 \leq i, j \leq \ell + 1, \\ 0 & \text{otherwise.} \end{cases} \quad (5.1)$$

Then $C(G \setminus H)$ is the C^* -subalgebra of $C(G)$ generated by the entries $u_{1, j}$ for $1 \leq j \leq \ell + 1$. Define $\theta : A_\ell \rightarrow C(G \setminus H)$ by

$$\theta(z_i) = q^{-i+1}u_{1, i}^*.$$

This gives an isomorphism between $C(G \setminus H)$ and A_ℓ and the following diagram commutes:

$$\begin{array}{ccc} A_\ell & \xrightarrow{\tau} & A_\ell \otimes C(G) \\ \theta \downarrow & & \downarrow \theta \otimes id \\ C(G \setminus H) & \xrightarrow{\Delta} & C(G \setminus H) \otimes C(G) \end{array}$$

In other words, (A_ℓ, G, τ) is the quotient space $G \setminus H$. As we shall see shortly, this choice of ϕ will make $L_2(G \setminus H)$ a span of certain rows of the $e_{r, s}$'s and this in turn will help us make use of the calculations already done in the initial sections.

Proposition 5.1 *Assume $\ell > 1$. The right regular representation u of G keeps $L_2(G \setminus H)$ invariant, and the restriction of u to $L_2(G \setminus H)$ decomposes as a direct sum of exactly one copy of each of the irreducibles given by the young tableaux $\lambda_{n, k} := (n + k, k, k, \dots, k, 0)$, with $n, k \in \mathbb{N}$.*

Proof: Write σ for the composition $h_H \circ \phi$ where h_H is the Haar state for H . From the description of $C(G \setminus H)$, it follows that

$$\begin{aligned} C(G \setminus H) &= \{a \in C(G) : (\sigma \otimes id)\Delta(a) = a\} \\ &= \{(\sigma \otimes id)\Delta(a) : a \in C(G)\}. \end{aligned}$$

Now the map $a \mapsto \sigma * a := (\sigma \otimes id)\Delta(a)$ on $C(G)$ extends to a bounded linear operator L_σ on $L_2(G)$ (lemma 3.1, [19]), and it is easy to see that $L_\sigma^2 = L_\sigma$. It follows then that $L_2(G \setminus H) = \ker(L_\sigma - I) = \text{ran } L_\sigma$. From the discussion preceding theorem 3.3, [19], it now follows that u keeps $L_2(G \setminus H)$ invariant and in fact the restriction of u to $L_2(G \setminus H)$ is the representation induced by the trivial representation of H . From the analogue of Frobenius reciprocity theorem for compact quantum groups (theorem 3.3, [19]) it now follows that the multiplicity of any irreducible u^λ in it would be same as the multiplicity of the trivial representation of H in the restriction of u^λ to H . But from the representation theory of $SU_q(\ell + 1)$, we know that the restriction of u^λ to $SU_q(\ell)$ decomposes into a direct sum of one copy of each irreducible $\mu : (\mu_1 \geq \mu_2 \geq \dots \geq \mu_\ell)$ of $SU_q(\ell)$ for which

$$\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \mu_2 \geq \dots \geq \lambda_\ell \geq \mu_\ell \geq 0. \quad (5.2)$$

Now the trivial representation of $SU_q(\ell)$ is indexed by Young tableaux of the form $\mu : (k, k, \dots, k)$ where $k \in \mathbb{N}$. But such a μ will obey the restriction 5.2 above if and only if λ is of the form $(n + k, k, k, \dots, k, 0)$. \square

Remark 5.2 For the case $\ell = 1$, the restriction of the irreducible $(n, 0)$ to the trivial subgroup decomposes into $n + 1$ copies of the trivial representation. Therefore, in this case, $L_2(S_q^3)$ decomposes into a direct sum of $n + 1$ copies of each representation $(n, 0)$.

Proposition 5.3 Let Γ_0 be the set of all GT tableaux \mathbf{r}^{nk} given by

$$r_{ij}^{nk} = \begin{cases} n + k & \text{if } i = j = 1, \\ 0 & \text{if } i = 1, j = \ell + 1, \\ k & \text{otherwise,} \end{cases}$$

for some $n, k \in \mathbb{N}$. Let Γ_0^{nk} be the set of all GT tableaux with top row $(n + k, k, \dots, k, 0)$. Then the family of vectors

$$\{e_{\mathbf{r}^{nk}, \mathbf{s}} : n, k \in \mathbb{N}, \mathbf{s} \in \Gamma_0^{nk}\}$$

form a complete orthonormal basis for $L_2(G \setminus H)$.

Proof: Let A be the linear span of the elements $\{u_{\mathbf{r}^{n,k}, \mathbf{s}} : n, k \in \mathbb{N}, \mathbf{s} \in \Gamma_0^{n,k}\}$. Clearly the closure of A in $L_2(G)$ is the closed linear span of $\{e_{\mathbf{r}^{n,k}, \mathbf{s}} : n, k \in \mathbb{N}, \mathbf{s} \in \Gamma_0^{n,k}\}$. It is also immediate that the restriction of the right regular representation to the above subspace is a direct sum of one copy of each of the irreducibles $(n + k, k, k, \dots, k, 0)$.

We will next show that for any $a \in A$, $u_{1j}a$ and u_{1j}^*a are also in A . Take $a = u_{\mathbf{r}^{n,k}, \mathbf{s}}$. Use equation (4.17) to get

$$u_{1,j}u_{\mathbf{r}^{n,k}, \mathbf{s}} = \sum_{M, M'} C_q(1, \mathbf{r}^{n,k}, M(\mathbf{r}^{n,k}))C_q(j, \mathbf{s}, M'(\mathbf{s}))u_{M(\mathbf{r}^{n,k}), M'(\mathbf{s})}$$

$$\begin{aligned}
&= \sum_{M'} C_q(1, \mathbf{r}^{n,k}, M_{11}(\mathbf{r}^{n,k})) C_q(j, \mathbf{s}, M'(\mathbf{s})) u_{M_{11}(\mathbf{r}^{n,k}), M'(\mathbf{s})} \\
&\quad + \sum_{M''} C_q(1, \mathbf{r}^{n,k}, M_{\ell+1,1}(\mathbf{r}^{n,k})) C_q(j, \mathbf{s}, M''(\mathbf{s})) u_{M_{\ell+1,1}(\mathbf{r}^{n,k}), M''(\mathbf{s})} \\
&= \sum_{M'} C_q(1, \mathbf{r}^{n,k}, \mathbf{r}^{n+1,k}) C_q(j, \mathbf{s}, M'(\mathbf{s})) u_{\mathbf{r}^{n+1,k}, M'(\mathbf{s})} \\
&\quad + \sum_{M''} C_q(1, \mathbf{r}^{n,k}, \mathbf{r}^{n,k-1}) C_q(j, \mathbf{s}, M''(\mathbf{s})) u_{\mathbf{r}^{n,k-1}, M''(\mathbf{s})}, \tag{5.3}
\end{aligned}$$

where the first sum is over all moves $M' \in \mathbb{N}^j$ whose first coordinate is 1 and the second sum is over all moves $M'' \in \mathbb{N}^j$ whose first coordinate is $\ell + 1$. Thus $u_{1j}a \in A$.

Next, note that if $\langle u_{1j}^* e_{\mathbf{r}^{n,k}, \mathbf{s}}, e_{\mathbf{r}', \mathbf{s}'} \rangle \neq 0$, then one must have $\mathbf{r}' = \mathbf{r}^{n-1,k}$ or $\mathbf{r}' = \mathbf{r}^{n,k+1}$. Therefore it follows that $u_{1j}^* u_{\mathbf{r}^{n,k}, \mathbf{s}}$ is a linear combination of the $u_{\mathbf{r}^{n-1,k}, \mathbf{s}}$ $u_{\mathbf{r}^{n,k+1}, \mathbf{s}}$'s, and hence belongs to A . Since A contains the element $u_{\mathbf{0}, \mathbf{0}} = 1$, it contains u_{1j} and u_{ij}^* . Thus A contains the *-algebra B generated by the u_{1j} 's. But by the previous theorem, restriction of the right regular representation to the L_2 closure $L_2(G \setminus H)$ of B also decomposes as a direct sum of one copy of each of the irreducibles $(n+k, k, \dots, k, 0)$. So it follows that $L_2(G \setminus H)$ is equal to the subspace stated in the theorem. \square

Thus the right regular representation u restricts to the subspace $L_2(G \setminus H)$ and it also follows from the above discussion that the restriction of the left multiplication to $C(G \setminus H)$ keeps $L_2(G \setminus H)$ invariant. Let us denote the restriction of u to $L_2(G \setminus H)$ by \hat{u} and the restriction of π to $C(G \setminus H)$ viewed as a map on $L_2(G \setminus H)$ by $\hat{\pi}$. It is easy to check that $(\hat{\pi}, \hat{u})$ is a covariant representation for the system (A_ℓ, G, τ) .

6 Equivariant spectral triples

The following lemma is straightforward.

Lemma 6.1 *Let D be a self-adjoint operator with compact resolvent on $L_2(G \setminus H)$ that is equivariant with respect to the covariant representation $(\hat{\pi}, \hat{u})$ then it is of the form*

$$e_{\mathbf{r}, \mathbf{s}} \mapsto d(\mathbf{r}) e_{\mathbf{r}, \mathbf{s}}, \quad \mathbf{r} \in \Gamma_0.$$

For such a D , one can then write down the commutators with algebra elements:

$$[D, \pi(u_{ij})] e_{\mathbf{r}, \mathbf{s}}^\lambda = \sum (d(\mathbf{m}) - d(\mathbf{r})) C_q(\mathbb{1}, \lambda, \mu; i, \mathbf{r}, \mathbf{m}) C_q(\mathbb{1}, \lambda, \mu; j, \mathbf{s}, \mathbf{n}) \kappa(\mathbf{r}, \mathbf{m}) e_{\mathbf{m}, \mathbf{n}}^\mu. \tag{6.1}$$

Therefore the condition for boundedness of commutators reads as follows:

$$|(d(\mathbf{m}) - d(\mathbf{r})) C_q(\mathbb{1}, \lambda, \mu; i, \mathbf{r}, \mathbf{m}) C_q(\mathbb{1}, \lambda, \mu; j, \mathbf{s}, \mathbf{n}) \kappa(\mathbf{r}, \mathbf{m})| < c, \tag{6.2}$$

where c is independent of $i, j, \lambda, \mu, \mathbf{r}, \mathbf{s}, \mathbf{m}$ and \mathbf{n} .

Using lemma 4.2, we get

$$|(d(\mathbf{m}) - d(\mathbf{r}))C_q(\mathbb{1}, \lambda, \mu; i, \mathbf{r}, \mathbf{m})C_q(\mathbb{1}, \lambda, \mu; j, \mathbf{s}, \mathbf{n})| < c. \quad (6.3)$$

Choosing j , \mathbf{s} and \mathbf{n} suitably, one can ensure that (6.3) implies the following:

$$|(d(\mathbf{m}) - d(\mathbf{r}))C_q(\mathbb{1}, \lambda, \mu; i, \mathbf{r}, \mathbf{m})| < c. \quad (6.4)$$

It follows from (6.1) that this condition is also sufficient for the boundedness of the commutators $[D, u_{ij}]$.

From (4.10), one gets

$$|d(\mathbf{r}) - d(M(\mathbf{r}))| \leq cq^{-C(i, \mathbf{r}, M)}. \quad (6.5)$$

Next, let us look at the growth restrictions coming from the boundedness of commutators. In this case, one has the boundedness of only the operators $[D, \pi(u_{ij})]$. Which means, in effect, one will now have the condition (6.5) only for $i = 1$ and $\mathbf{r} \in \Gamma_0$:

$$|d(\mathbf{r}) - d(M(\mathbf{r}))| \leq cq^{-C(1, \mathbf{r}, M)}. \quad (6.6)$$

Observe that only allowed moves here are the moves $M = M_{1,1} \equiv (1)$ and $M = M_{\ell+1,1} \equiv (\ell+1)$. Looking at the corresponding quantity $C(1, \mathbf{r}, M)$, we find that there are two conditions:

$$|d(\mathbf{r}^{nk}) - d(\mathbf{r}^{n, k-1})| \leq c, \quad (6.7)$$

$$|d(\mathbf{r}^{nk}) - d(\mathbf{r}^{n+1, k})| \leq cq^{-\sum_{j=1}^{\ell} H_{1j}(\mathbf{r}^{nk})} = cq^{-k}. \quad (6.8)$$

We can now form a graph by taking Γ_0 to be the set of vertices, and by joining two vertices \mathbf{r} and \mathbf{s} by an edge if $|d(\mathbf{r}) - d(\mathbf{s})| \leq c$.

Lemma 6.2 *Let $\mathcal{F}_n = \{\mathbf{r}^{n, k} : k \in \mathbb{N}\}$, $n \in \mathbb{N}$. Then any two points in \mathcal{F}_n are connected by a path lying entirely in \mathcal{F}_n .*

If $n < n'$, then any point in \mathcal{F}_n is connected to any point in $\mathcal{F}_{n'}$ by a path such that $n \leq V_{1,1}(\mathbf{r}) \leq n'$ for every vertex \mathbf{r} lying on that path.

Proof: Take two points $\mathbf{r}^{n, j}$ and $\mathbf{r}^{n, k}$ in \mathcal{F}_n . Assume $j < k$. From the condition (6.7), it follows that any point \mathbf{r} is connected to $M_{\ell+1,1}(\mathbf{r})$ by an edge. Therefore the first conclusion follows from the observation that if we start at $\mathbf{r}^{n, k}$ and apply the move $M_{\ell+1,1}$ successively $k - j$ number of times, we reach the point $\mathbf{r}^{n, j}$, and the vertices on this path are the points $\mathbf{r}^{n, i}$ for $i = j, j + 1, \dots, k$. Observe also that throughout this path, $V_{1,1}(\mathbf{r})$ remains n .

For the second part, take a point $\mathbf{r}^{n, k}$ in \mathcal{F}_n and a point $\mathbf{r}^{n', j}$ in $\mathcal{F}_{n'}$. From what we have done above, there is a path from $\mathbf{r}^{n, k}$ to $\mathbf{r}^{n, 0}$ throughout which $V_{1,1}(\mathbf{r}) = n$. Similarly there is a path from $\mathbf{r}^{n', j}$ to $\mathbf{r}^{n', 0}$ throughout which $V_{1,1}(\mathbf{r}) = n'$. Next, note from (6.8) that for $p \in \mathbb{N}$, the points $\mathbf{r}^{p, 0}$ and $\mathbf{r}^{p+1, 0}$ are connected by an edge and $V_{1,1}(\mathbf{r}^{p, 0}) = p$, $V_{1,1}(\mathbf{r}^{p+1, 0}) = p + 1$. So start at $\mathbf{r}^{n, 0}$ and reach successively the points $\mathbf{r}^{n+1, 0}$, $\mathbf{r}^{n+2, 0}$ and so on to eventually reach the point $\mathbf{r}^{n', 0}$; also the coordinate $V_{1,1}(\cdot)$ remains between n and n' on this path. \square

Theorem 6.3 *Let D be an equivariant Dirac operator on $L_2(G \setminus H)$. Then*

1. D must be of the form

$$e_{\mathbf{r},\mathbf{s}} \mapsto d(\mathbf{r})e_{\mathbf{r},\mathbf{s}}, \quad \mathbf{r} \in \Gamma,$$

where the singular values obey $|d(\mathbf{r})| = O(r_{11})$, and

2. $\text{sign} D$ must be of the form $2P - I$ or $I - 2P$ where P is, up to a compact perturbation, the projection onto the closed span of $\{e_{\mathbf{r}^{nk},\mathbf{s}} : n \in F, k \in \mathbb{N}, \mathbf{s} \in \Gamma_0^{nk}\}$, for some finite subset F of \mathbb{N} .

Proof: Start with an equivariant self-adjoint operator D with compact resolvent, so that it is indeed of the form $e_{\mathbf{r},\mathbf{s}} \mapsto d(\mathbf{r})e_{\mathbf{r},\mathbf{s}}$. By applying a compact perturbation if necessary, make sure that $d(\mathbf{r}) \neq 0$ for all $\mathbf{r} \in \Gamma_0$. We have seen during the proof of the previous lemma that for any n and k in \mathbb{N} , the vertices \mathbf{r}^{nk} and $\mathbf{r}^{n,k+1}$ are connected by an edge, and for any $n \in \mathbb{N}$, the vertices $\mathbf{r}^{n,0}$ and $\mathbf{r}^{n+1,0}$ is connected by an edge. Thus any vertex \mathbf{r}^{nk} can be reached from the vertex \mathbf{r}^{00} by a path of length $n + k$. Therefore one gets the first assertion.

Next, define

$$\begin{aligned} \Gamma_0^+ &= \{\mathbf{r} \in \Gamma_0 : d(\mathbf{r}) > 0\}, \\ \Gamma_0^- &= \{\mathbf{r} \in \Gamma_0 : d(\mathbf{r}) < 0\}, \\ \mathcal{F}_n^+ &= \mathcal{F}_n \cap \Gamma_0^+, \\ \mathcal{F}_n^- &= \mathcal{F}_n \cap \Gamma_0^-. \end{aligned}$$

Observe that for the path produced in the proof of the forgoing lemma to connect two points $\mathbf{r}^{n,k}$ and $\mathbf{r}^{n,j}$ in \mathcal{F}_n , the coordinate $H_{1,\ell}(\cdot)$ remains between j and k . Now suppose for some n , both \mathcal{F}_n^+ and \mathcal{F}_n^- are infinite. Then there are points

$$0 \leq k_1 < k_2 < \dots$$

such that \mathbf{r}^{nk} is in \mathcal{F}_n^+ for $k = k_{2j}$ and \mathbf{r}^{nk} is in \mathcal{F}_n^- for $k = k_{2j+1}$. Using the above observation, we can then produce an infinite ladder by joining each $\mathbf{r}^{n,k_{2j-1}}$ to $\mathbf{r}^{n,k_{2j}}$. Thus for each $n \in \mathbb{N}$, exactly one of the sets \mathcal{F}_n^+ and \mathcal{F}_n^- is finite. Also, note that by the first part of the previous lemma, the set of all $n \in \mathbb{N}$ for which both \mathcal{F}_n^+ and \mathcal{F}_n^- are nonempty is finite. Therefore by applying a compact perturbation, we can ensure that for every n , either $\mathcal{F}_n^+ = \mathcal{F}_n$ or $\mathcal{F}_n^- = \mathcal{F}_n$.

Finally, if there are infinitely many n 's for which $\mathcal{F}_n^+ = \mathcal{F}_n$ and infinitely many n 's for which $\mathcal{F}_n^- = \mathcal{F}_n$, then one can choose a sequence of integers

$$0 \leq n_1 < n_2 < \dots$$

such that $\mathcal{F}_n^+ = \mathcal{F}_n$ for $n = n_{2j}$ and $\mathcal{F}_n^- = \mathcal{F}_n$ for $n = n_{2j+1}$. Now use the second part of the previous lemma to join each $\mathbf{r}^{n_{2j-1},0}$ to $\mathbf{r}^{n_{2j},0}$ to produce an infinite ladder.

Thus there is a finite subset F of \mathbb{N} such that exactly one of the following is true:

$$\mathcal{F}_n = \begin{cases} \mathcal{F}_n^+ & \text{if } n \in F, \\ \mathcal{F}_n^- & \text{if } n \notin F, \end{cases} \quad \text{or} \quad \mathcal{F}_n = \begin{cases} \mathcal{F}_n^- & \text{if } n \in F, \\ \mathcal{F}_n^+ & \text{if } n \notin F. \end{cases}$$

This is precisely what the second part of the theorem says. \square

Next, take the operator $D : e_{\mathbf{r},\mathbf{s}} \mapsto d(\mathbf{r})e_{\mathbf{r},\mathbf{s}}$ on $L_2(G \setminus H)$ where the $d(\mathbf{r})$'s are given by:

$$d(\mathbf{r}^{nk}) = \begin{cases} -k & \text{if } n = 0, \\ n + k & \text{if } n > 0. \end{cases} \quad (6.9)$$

Theorem 6.4 *The operator D is an equivariant $(2\ell + 1)$ -summable Dirac operator acting on $L_2(G \setminus H)$, that gives a nondegenerate pairing with $K_1(C(G \setminus H))$.*

The operator D is optimal, i. e. if D_0 is any equivariant Dirac operator on $L_2(G \setminus H)$, then there are positive reals a and b such that

$$|D_0| \leq a + b|D|.$$

Proof: Recall from equation (4.18) that the elements $u_{1,j}$ act on the basis elements $e_{\mathbf{r}^{n,k},\mathbf{s}}$ as follows:

$$\begin{aligned} u_{1,j}e_{\mathbf{r}^{n,k},\mathbf{s}} &= \sum_{M,M'} C_q(1, \mathbf{r}^{n,k}, M(\mathbf{r}^{n,k}))C_q(j, \mathbf{s}, M'(\mathbf{s}))\kappa(\mathbf{r}^{n,k}, \mathbf{s})e_{M(\mathbf{r}^{n,k}),M'(\mathbf{s})} \\ &= \sum_{M'} C_q(1, \mathbf{r}^{n,k}, M_{11}(\mathbf{r}^{n,k}))C_q(j, \mathbf{s}, M'(\mathbf{s}))\kappa(\mathbf{r}^{n,k}, \mathbf{s})e_{M_{11}(\mathbf{r}^{n,k}),M'(\mathbf{s})} \\ &\quad + \sum_{M''} C_q(1, \mathbf{r}^{n,k}, M_{\ell+1,1}(\mathbf{r}^{n,k}))C_q(j, \mathbf{s}, M''(\mathbf{s}))\kappa(\mathbf{r}^{n,k}, \mathbf{s})e_{M_{\ell+1,1}(\mathbf{r}^{n,k}),M''(\mathbf{s})} \\ &= \sum_{M'} C_q(1, \mathbf{r}^{n,k}, \mathbf{r}^{n+1,k})C_q(j, \mathbf{s}, M'(\mathbf{s}))\kappa(\mathbf{r}^{n,k}, \mathbf{s})e_{\mathbf{r}^{n+1,k},M'(\mathbf{s})} \\ &\quad + \sum_{M''} C_q(1, \mathbf{r}^{n,k}, \mathbf{r}^{n,k-1})C_q(j, \mathbf{s}, M''(\mathbf{s}))\kappa(\mathbf{r}^{n,k}, \mathbf{s})e_{\mathbf{r}^{n,k-1},M''(\mathbf{s})}, \end{aligned} \quad (6.10)$$

where the first sum is over all moves $M' \in \mathbb{N}^j$ whose first coordinate is 1 and the second sum is over all moves $M'' \in \mathbb{N}^j$ whose first coordinate is $\ell + 1$. If we now plug in the values of the Clebsch-Gordon coefficients from equations (4.10) and (4.11), we get

$$\begin{aligned} u_{1,j}e_{\mathbf{r}^{n,k},\mathbf{s}} &= \sum_{M'} P_1' P_2' q^{k+C(j,\mathbf{s},M')} \kappa(\mathbf{r}^{n,k}, \mathbf{s})e_{\mathbf{r}^{n+1,k},M'(\mathbf{s})} \\ &\quad + \sum_{M''} P_1'' P_2'' q^{C(j,\mathbf{s},M'')} \kappa(\mathbf{r}^{n,k}, \mathbf{s})e_{\mathbf{r}^{n,k-1},M''(\mathbf{s})}, \end{aligned} \quad (6.11)$$

where P_i' , P_j'' and $k(\mathbf{r}^{n,k}, \mathbf{s})$ all lie between two fixed positive numbers. Boundedness of the commutators $[D, u_{1,j}]$ now follow directly.

For summability, notice that the eigenspace of $|D|$ corresponding to the eigenvalue $n \in \mathbb{N}$ is the span of

$$\{e_{\mathbf{r}^k, n-k, \mathbf{s}} : 0 \leq k \leq n, \mathbf{s} \in \Gamma_0^{k, n-k}\}.$$

Now just count the number of elements in the above set to get summability.

Next, we will compute the pairing of the K -homology class of this D with a generator of the K_1 group. Write $\omega_q := q^{-\ell} u_{1, \ell+1}$. From the commutation relations, it follows that this element has spectrum

$$\{z \in \mathbb{C} : |z| = 0 \text{ or } q^n \text{ for some } n \in \mathbb{N}\}.$$

Then the element $\gamma_q := \chi_{\{1\}}(\omega_q^* \omega_q)(\omega_q - I) + I$ is unitary. We will show that the index of the operator $Q\gamma_q Q$ (viewed as an operator on $QL_2(G \setminus H)$) is 1, where $Q = \frac{I - \text{sign } D}{2}$, i. e. it is the projection onto the closed linear span of $\{e_{\mathbf{r}^0, k, \mathbf{s}} : k \in \mathbb{N}, \mathbf{s} \in \Gamma_0^{0, k}\}$. What we will actually do is compute the index of the operator $Q\gamma_0 Q$ and appeal to continuity of the index. From equation (6.10), we get

$$\begin{aligned} & u_{1, \ell+1} e_{\mathbf{r}^0, k, \mathbf{s}} \\ &= C_q(1, \mathbf{r}^{0, k}, M_{11}(\mathbf{r}^{0, k})) C_q(\ell + 1, \mathbf{s}, N_{1, 0}(\mathbf{s})) \kappa(\mathbf{r}^{0, k}, M_{11}(\mathbf{r}^{0, k})) e_{\mathbf{r}^{1, k}, N_{1, 0}(\mathbf{s})} \\ & \quad + C_q(1, \mathbf{r}^{0, k}, M_{\ell+1, 1}(\mathbf{r}^{0, k})) C_q(\ell + 1, \mathbf{s}, M_{\ell+1, \ell+1}(\mathbf{s})) \kappa(\mathbf{r}^{0, k}, M_{\ell+1, 1}(\mathbf{r}^{0, k})) e_{\mathbf{r}^{0, k-1}, M_{\ell+1, \ell+1}(\mathbf{s})}. \end{aligned} \tag{6.12}$$

Use the formula (4.3) for Clebsch-Gordon coefficients to get

$$C_q(1, \mathbf{r}^{0, k}, M_{11}(\mathbf{r}^{0, k})) = q^k (1 + o(q)), \tag{6.13}$$

$$C_q(1, \mathbf{r}^{0, k}, M_{\ell+1, 1}(\mathbf{r}^{0, k})) = 1 + o(q), \tag{6.14}$$

$$C_q(\ell + 1, \mathbf{s}, N_{1, 0}(\mathbf{s})) = 1 + o(q), \tag{6.15}$$

$$C_q(\ell + 1, \mathbf{s}, M_{\ell+1, \ell+1}(\mathbf{s})) = q^{s_{\ell+1, 1} + \ell} (1 + o(q)), \tag{6.16}$$

where $o(q)$ signifies a function of q that is continuous at $q = 0$ and $o(0) = 0$. We also have

$$\kappa(\mathbf{r}^{0, k}, M_{11}(\mathbf{r}^{0, k})) = q^\ell (1 + o(q)), \tag{6.17}$$

$$\kappa(\mathbf{r}^{0, k}, M_{\ell+1, 1}(\mathbf{r}^{0, k})) = 1 + o(q), \tag{6.18}$$

where $o(q)$ is as earlier. Plugging these values in (6.12) we get

$$\omega_q e_{\mathbf{r}^0, k, \mathbf{s}} = q^k (1 + o(q)) e_{\mathbf{r}^{1, k}, N_{1, 0}(\mathbf{s})} + q^{s_{\ell+1, 1} + \ell} (1 + o(q)) e_{\mathbf{r}^{0, k-1}, M_{\ell+1, \ell+1}(\mathbf{s})} \tag{6.19}$$

Putting $q = 0$, we get

$$\omega_0 e_{\mathbf{r}^0, k, \mathbf{s}} = \begin{cases} e_{\mathbf{r}^{0, k-1}, M_{\ell+1, \ell+1}(\mathbf{s})} & \text{if } k > 0 \text{ and } s_{\ell+1, 1} = 0, \\ e_{\mathbf{r}^{1, 0}, N_{1, 0}(\mathbf{s})} & \text{if } k = 0, \\ 0 & \text{otherwise.} \end{cases} \tag{6.20}$$

Thus $\omega_0^* \omega_0$ is the projection onto the span of $\{e_{\mathbf{r}^0, k, \mathbf{s}^k} : k \in \mathbb{N}\}$ where \mathbf{s}^k is the GT tableaux given by

$$s_{ij}^k = \begin{cases} 0 & \text{if } i = \ell + 2 - j, \\ k & \text{otherwise,} \end{cases}$$

which is uniquely determined by the conditions $s_{\ell+1,1} = 0$ and that $\mathbf{s} \in \Gamma_0^{0,k}$. Therefore the operator γ_0 is given by

$$\gamma_0 e_{\mathbf{r}^0, k, \mathbf{s}} = e_{\mathbf{r}^0, k, \mathbf{s}} - \chi_{\{\mathbf{s}=\mathbf{s}^k\}} e_{\mathbf{r}^0, k, \mathbf{s}} + \chi_{\{\mathbf{s}=\mathbf{s}^k\}} e_{\mathbf{r}^0, k-1, \mathbf{s}^{k-1}}.$$

It now follows that the index of $Q\gamma_0Q$ is 1.

Note that if D_0 is an equivariant Dirac operator with eigenvalues $d_0(\mathbf{r})$, then by theorem 6.3 there is a $b > 0$ such that

$$|d_0(\mathbf{r})| < br_{11} = b|d(\mathbf{r})|, \quad \mathbf{r} \neq \mathbf{0}.$$

Write $a = |d_0(\mathbf{0})|$. Then we have the required inequality. \square

Acknowledgement: The second author would like to thank the Isaac Newton Institute and in particular the organisers of the NCG semester for their hospitality.

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