ON DISCRETENESS OF COMMENSURATORS

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ABSTRACT. We begin by showing that commensurators of Zariski dense sub-groups of isometry groups of symmetric spaces of non-compact type are discrete provided that the limit set on the Furstenberg boundary is not invariant under the action of a (virtual) simple factor. In particular for rank one or simple Lie groups, Zariski dense subgroups with non-empty domain of discontinuity have discrete commensurators. This generalizes a Theorem of Greenberg for Kleinian groups. We then prove that for all finitely generated, Zariski dense, infinite covolume discrete subgroups of $\mathrm{Isom}(\mathbb{H}^3)$, commensurators are discrete. Together these prove discreteness of commensurators for all known examples of finitely presented, Zariski dense, infinite covolume discrete subgroups of $\mathrm{Isom}(X)$ for X an irreducible symmetric space of non-compact type.

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CONTENTS

1. INTRODUCTION

Motivated by Margulis' celebrated characterization of arithmeticity of irreducible lattices in semi-simple Lie groups in terms of density of the commensurator [Mar90], Shalom asked (cf. [LLR09]) for a description of commensurators of Zariski dense subgroups of semi-simple Lie groups. We start with the observation that for Zariski dense subgroups of isometry groups of rank one symmetric spaces, commensurators are discrete provided that the domain of discontinuity is non-empty.

Proposition 1.1 and Corollary 1.2: *Suppose* Γ *is a Zariski-dense subgroup of a semi-simple Lie group* $L = Isom(X)$ *for* X *a rank one symmetric space of non-compact type. Then the commensurator* $\mathrm{Comm}(\Gamma)$ *of* Γ *is discrete if the limit set* Λ_Γ *is not all of* ∂X.

When $X = \mathbb{H}^3$, the above is due to Greenberg [Gre74], [Gre77].

Using the theory of limit sets (in the Furstenberg boundary $Y = G/B$) of Zariski-dense subgroups of semi-simple Lie groups of higher rank developed by Benoist [Ben97] we generalize Proposition 1.1 to arbitrary semi-simple Lie groups.

Proposition 1.7: *Suppose* Γ *is a Zariski-dense subgroup of a semi-simple Lie group* $L = Isom(X)$ *for* X *a symmetric space of non-compact type. Further suppose that the limit set* $\Lambda_\Gamma \subset Y$ *is not invariant under any non-trivial semi-simple (virtual) factor subgroup* L_0. *Then the commensurator* $\mathrm{Comm}(\Gamma)$ *of* Γ *is discrete in* L.

We then specialize to the case that Γ is abstractly a (relative) hyperbolic group in the sense of Gromov and X is of rank one.

Infinite covolume discrete subgroups of semi-simple Lie groups have received little attention in the context of commensurators until recently. However, the theory of Kleinian groups deals primarily with infinite covolume discrete subgroups of $\mathrm{PSL}_2(\mathbb{C})$. Commensurators of some non-free Kleinian groups have been investigated by Leininger, Long and Reid [LLR09]. In this paper we reprove and extend their results to all finitely generated Kleinian groups.

Theorem 3.7: *Let* G *be a finitely generated, Zariski dense Kleinian group such that* $\mathbb{H}^3/G$ *has infinite volume; then the commensurator* $\mathrm{Comm}(G)$ *of* G *is discrete in* $\mathrm{PSL}_2(\mathbb{C})$.

The above statement was proven by Leininger, Long and Reid [LLR09] when G is non-free and without parabolics. As in [LLR09] Theorem 3.7 can be strengthened as follows.

Theorem 3.8: *Let* G *be a finitely generated Kleinian group such that* G *is Zariski dense in* $\mathrm{PSL}_2(\mathbb{C})$ *and* $\mathbb{H}^3/G$ *has infinite volume. Then* $[\mathrm{Comm}(G) : G] < \infty$ *unless* G *is virtually a fiber subgroup, in which case* $\mathrm{Comm}(G)$ *is the fundamental group of a virtually fibered finite volume hyperbolic 3-manifold.*

Together, Proposition 1.7 and Theorem 3.7 prove discreteness of commensurators for all known examples of finitely presented, Zariski dense, infinite covolume discrete subgroups of $\mathrm{Isom}(X)$ for X a symmetric space of non-compact type.

We also provide an example of a Zariski dense subgroup $G \subset Isom(\mathbb{H}^4)$ such that $\mathrm{Comm}(G)$ is discrete, $[\mathrm{Comm}(G) : G] = \infty$ but the limit set $\Lambda_G \neq \partial\mathbb{H}^4$. We also justify a statement of Gromov [Gro00] by showing that negatively curved 4-manifolds cannot fiber over the circle.

1.1. **Zariski-dense subgroups of semi-simple Lie groups.** Our first observation is that a version of Margulis' dichotomy holds for all Zariski dense subgroups. (Some version of this is probably well-known to experts in the theory of Lie groups and Algebraic Groups.)

Proposition 1.1. *Suppose* Γ *is a Zariski-dense subgroup of a semi-simple Lie group* $L = Isom(X)$ *for* X *a rank one symmetric space of non-compact type. Then the commensurator* $\mathrm{Comm}(\Gamma)$ *of* Γ *is either discrete or dense in* L.

Proof: Let $\overline{Comm(\Gamma)}$ be the closure of $Comm(\Gamma)$ in the Lie group L. If the connected component of the identity L_0 of $\overline{Comm(\Gamma)}$ is trivial, then $\overline{Comm(\Gamma)}$ and hence $Comm(\Gamma)$ is totally disconnected and therefore discrete.

Else, let $\mathfrak{l}_0$ denote the Lie algebra of L_0 and $\mathfrak{l}$ denote the Lie algebra of L. Since L_0 is the connected component of the identity of $\overline{Comm(\Gamma)}$, L_0 is normal in $\overline{Comm(\Gamma)}$. Then $\mathfrak{l}_0$ is an invariant subspace of $\mathfrak{l}$ under the adjoint action Ad_Γ of Γ. Hence $\mathfrak{l}_0$ is an invariant subspace of $\mathfrak{l}$ under the adjoint action of $ZC(\Gamma)$, the Zariski closure of Γ. (This follows from the fact that Ad_G acts on the Grassmannian of $dim(\mathfrak{l}_0)$ - planes in $\mathfrak{l}$ and Ad_Γ fixes the plane $\mathfrak{l}_0$.) Since Γ is Zariski-dense, it follows that $\mathfrak{l}_0$ is an ideal in $\mathfrak{l}$. Let $\mathfrak{l} = \mathfrak{g}_1 \oplus \mathfrak{g}_2 \oplus \cdots \oplus \mathfrak{g}_k$ be the decomposition of $\mathfrak{l}$ into its simple Lie algebra summands. Hence $\mathfrak{l}_\mathfrak{o} = \mathfrak{g}_{i_1} \oplus \mathfrak{g}_{i_2} \oplus \cdots \oplus \mathfrak{g}_{i_s}$ for $i_1, \cdots, i_s$ distinct elements of $\{1, \cdots, k\}$.

2 cases arise:

Case 1: L_0 is non-compact. Let $K \subset L$ be the maximal compact subgroup of L and $X = L/K$ be the associated symmetric space. Then $X_0 = L_0/(L_0 \cap K)$ is the symmetric subspace of L associated to the subgroup L_0. Further, L_0 is normal in L, since $\mathfrak{l}_0$ is an ideal in $\mathfrak{l}$ and L_0 is connected. Since L_0 is non-compact, its limit set is non-empty and since L_0 is normal, its limit set is all of ∂X (This is essentially the only place where we really use the rank one assumption).

Using the associated Killing form [Hel01] for instance, X_0 is a totally geodesic subspace of X. But from the previous paragraph, the limit set of L_0 is all of ∂X. Hence $X_0 = X$ and $\overline{Comm(\Gamma)} = L = Isom(X)$.

Case 2: L_0 is compact. Then L_0 fixes some point $x \in X$. Further, L_0 is normal in L, since $\mathfrak{l}_0$ is an ideal in $\mathfrak{l}$. Hence for any $g \in L$ and $l \in L_0$, $(glg^{-1})g.x = gl.x = g.x$ and so L_0 fixes $g.x$. Hence L_0 fixes all $x \in X$. Therefore L_0 is trivial, since $L = Isom(X)$. $\square$

As an immediate Corollary of Proposition 1.1, we have the following:

Corollary 1.2. *Suppose Γ is a Zariski-dense subgroup of a semi-simple Lie group $L = Isom(X)$ for X a rank one symmetric space of non-compact type. Further suppose that the limit set $\Lambda_\Gamma \neq \partial X$. Then the commensurator $Comm(\Gamma)$ of Γ is discrete in L.*

Proof: The limit set Λ_Γ is invariant under $Comm(\Gamma)$. Hence if $Comm(\Gamma)$ is dense in L, $\Lambda_\Gamma = \partial X$. Therefore by Proposition 1.1, $Comm(\Gamma)$ is discrete in L. $\square$

Next, we generalize Proposition 1.1 to arbitrary semi-simple $L = Isom(X)$, by introducing an appropriate generalization of the notion of irreducibility. Motivated by the definition of irreducibility of lattices we propose the following.

Definition 1.3. *A Zariski dense subgroup Γ of a semi-simple Lie group L will be called* **strongly irreducible** *if for any proper non-trivial normal subgroup L_0 of L, the projection of Γ to the quotient L/L_0 is indiscrete.*

Now, let Γ be a strongly irreducible Zariski-dense subgroup of a semi-simple Lie group $L = Isom(X)$. As in Proposition 1.1 let L_0 be the connected component of the identity in $\overline{Comm(\Gamma)}$. Suppose $L_0 \neq \{1\}$. The quotient $\overline{Comm(\Gamma)}/L_0$ is discrete. Since $\Gamma \subset \overline{Comm(\Gamma)}$, it follows that $\overline{Comm(\Gamma)} = L_0$. Since Γ is Zariski-dense, $L_0 = L$ and we have shown the following.

Proposition 1.4. *Suppose Γ is a strongly irreducible Zariski-dense subgroup of a semi-simple Lie group $L = Isom(X)$ for X a symmetric space of non-compact type. Then the commensurator $\mathrm{Comm}(\Gamma)$ of Γ is either discrete or dense in L.*

The following is another simple generalization of Proposition 1.1.

Proposition 1.5. *Suppose Γ is a Zariski-dense subgroup of a semi-simple Lie group $L = Isom(X)$ for X a symmetric space of non-compact type. If $\mathrm{Comm}(\Gamma)$ denotes the commensurator of Γ, then there exists a short exact sequence*

$$1 \to L_0 \to \overline{Comm(\Gamma)} \to Q \to 1$$

where L_0 is a semi-simple Lie subgroup of L and Q is discrete in L/L_0.

Proof: As in the proof of Proposition 1.1, let $\overline{Comm(\Gamma)}$ be the closure of $\mathrm{Comm}(\Gamma)$ in the Lie group L and L_0 be the connected component of the identity in $\overline{Comm(\Gamma)}$. If L_0 is trivial, then $\overline{Comm(\Gamma)}$ is totally disconnected and hence $Comm(\Gamma)$ is discrete.

Else, as before, let $\mathfrak{l}_0$ denote the Lie algebra of L_0 and $\mathfrak{l}$ denote the Lie algebra of L. Then $L_0 \subset \overline{Comm(\Gamma)}$ is normal and $\overline{Comm(\Gamma)}/L_0 = Q$ is discrete. $\square$

We now proceed to refine some of these results to higher rank semi-simple Lie groups in the context of Zariski dense subgroups of $\mathrm{Isom}(X)$ acting on the flag variety or the Furstenberg boundary. This gives a geometric perspective on Proposition 1.5. Let G be a semi-simple Lie group, K a maximal compact subgroup and and P a Borel subgroup. Then G/P is called the Furstenberg boundary of the associated symmetric space $X = G/K$. See [Ben97] for details. Limit sets of Zariski-dense subgroups Γ of G have been defined by Benoist [Ben97] as in the rank one case. The limit set of the action of Γ on G/P will be denoted as $\Lambda(G/P, \Gamma)$. The following basic Proposition is due to Benoist, generalizing an analogous statement for $\mathrm{SL}_n(\mathbb{R})$ by Guivarc'h.

Proposition 1.6. *(Benoist [Ben97]) Let $\Gamma \subset G$ be a Zariski dense subgroup. Then $\Lambda(G/P, \Gamma)$ is the unique minimal closed Γ-invariant subset G/P.*

A semi-simple Lie algebra is a direct sum of simple Lie algebras. A closed semi-simple Lie subgroup L_0 of a Lie group L will be called a **(virtual) factor** if its Lie algebra $\mathfrak{l}_0$ is a direct summand of the Lie algebra $\mathfrak{l}$ of L.

Proposition 1.6 will be essential in generalizing Corollary 1.2 to the following.

Proposition 1.7. *Suppose Γ is a Zariski-dense subgroup of a semi-simple Lie group $L = Isom(X)$ for X a symmetric space of non-compact type. Further suppose that the limit set $\Lambda(G/P, \Gamma)$ is not invariant under any non-trivial semisimple (virtual) factor L_0. Then the commensurator $\mathrm{Comm}(\Gamma)$ of Γ is discrete in L.*

Proof: As in the proof of Proposition 1.1, let $\overline{Comm(\Gamma)}$ be the closure of $\mathrm{Comm}(\Gamma)$ in the Lie group L and L_0 be the connected component of the identity in $\overline{Comm(\Gamma)}$. If L_0 is trivial, then $\overline{Comm(\Gamma)}$ is totally disconnected and hence $Comm(\Gamma)$ is discrete.

Else, as before, let $\mathfrak{l}_0$ denote the Lie algebra of L_0 and $\mathfrak{l}$ denote the Lie algebra of L.

2 cases arise:

Case 1: L_0 is compact. Then exactly the same proof as Case 2 of Proposition 1.1 shows that L_0 is trivial.

Case 2: L_0 is non-compact. $\mathrm{Comm}(\Gamma) \cap L_0 = \Gamma_0$ is an infinite normal subgroup of $\mathrm{Comm}(\Gamma)$. Also, Γ_0 is Zariski-dense in L_0. Let Λ_Γ and Λ_{Γ_0} denote the limit sets of Γ and Γ_0 respectively. Both are non-empty.

For every $g \in L$, $g.\Lambda_\Gamma$ is the limit set of $g.\Gamma$ and hence $g.\Gamma g^{-1}$. Since the limit set of a Zariski dense group is the unique minimal invariant set (Proposition 1.6), and since a finite index subgroup of a Zariski dense group is Zariski dense, $g.\Lambda_\Gamma = \Lambda_\Gamma$ for every $g \in \mathrm{Comm}(\Gamma)$, and hence for every $g \in \overline{\mathrm{Comm}(\Gamma)}$. In particular, Λ_Γ is L_0-invariant. But by hypothesis Λ_Γ is not invariant under any such L_0, a contradiction.

Therefore $\mathrm{Comm}(\Gamma)$ is discrete in L. $\square$

Remark 1.8. *In terms of limit sets, invariance of $\Lambda_\Gamma \subset G/P$ under the action of a semi-simple Lie subgroup $L_0 \subset L$ means that $\Lambda_\Gamma = L_0/P_0 \times \Lambda_1$, where*
1) $L = L_0 \times L_1$ (at least virtually)
2) L_0/P_0 is the Furstenberg boundary of L_0
3) Λ_1 is the limit set of the induced action of Γ on the Furstenberg boundary L_1/P_1 of the semisimple L_1.

2. Relations on Boundaries

2.1. **Hyperbolicity and Relative Hyperbolicity.**

Definition 2.1. *For any geodesic metric space (H, d), the* hyperbolic cone *H^h is the metric space $H \times [0, \infty) = H^h$ equipped with a path metric d_h obtained from two pieces of data*
1) $d_{h,t}((x,t),(y,t)) = 2^{-t} d_H(x, y)$, where $d_{h,t}$ is the induced path metric on $H \times \{t\}$. Paths joining $(x, t), (y, t)$ and lying on $H \times \{t\}$ are called horizontal paths.
2) $d_h((x,t),(x,s)) = |t - s|$ for all $x \in H$ and for all $t, s \in [0, \infty)$, and the corresponding paths are called vertical paths.
3) for all $x, y \in H^h$, $d_h(x, y)$ is the path metric induced by the collection of horizontal and vertical paths.

Definition 2.2. *Let X be a proper (i.e. complete and locally compact) geodesic metric space and $\mathcal{H}$ be a collection of uniformly separated subsets of X. X is said to be* hyperbolic relative to $\mathcal{H}$ in the sense of Gromov, *if the space $\mathcal{G}(X, \mathcal{H})$, obtained by attaching the hyperbolic cones H^h to $H \in \mathcal{H}$ by identifying $(z, 0)$ with z for all $H \in \mathcal{H}$ and $z \in H$, is a proper hyperbolic metric space. The collection $\{H^h : H \in \mathcal{H}\}$ is denoted as $\mathcal{H}^h$. The induced path metric on $\mathcal{G}(X, \mathcal{H})$ is also denoted by d_h.*

A group G is hyperbolic relative to a finite collection $H_1, \cdots, H_k$, if the Cayley graph Γ of G with respect to some finite generating set is hyperbolic relative to the collection $\mathcal{H}$ of translates of Cayley subgraphs of $H_1, \cdots, H_k$ (see [Gro85], [Far98] and [Bow97] for details on relative hyperbolicity). Then $\partial G = \partial \mathcal{G}(G, \mathcal{H})$ is called the (relative) hyperbolic (or Bowditch [Bow97])-boundary of G, and $\widehat{G} = \mathcal{G}(G, \mathcal{H}) \cup \partial \mathcal{G}(G, \mathcal{H})$ is the compactification of $\partial \mathcal{G}(G, \mathcal{H})$. The set of distinct pairs of points on ∂G will be denoted as $\partial^2 G$.

Let G be a (relatively) hyperbolic group. A fixed point on ∂G of a hyperbolic element g of G is called a **pole** (See [Gro85] Sec 5.1), and the pair of fixed points $(g_{-\infty}, g_\infty)$ is called a **pole-pair**.

Proposition 2.3. Pole-pairs Dense *(Gromov* [Gro85] *Sec 5.1, p.136) The collection of pairs* $(g_{-\infty}, g_{\infty})$ *as* g *ranges over hyperbolic elements of* G *is dense in* $\partial^2 G$.

2.2. **Cannon-Thurston Maps.** Let (X, d_X) be a proper hyperbolic metric space and G be a (Gromov) hyperbolic group acting freely, properly discontinuously by isometries on X. Let Γ be a Cayley graph of G with respect to some finite generating set. By adjoining the Gromov boundaries ∂X and ∂G to X and Γ, one obtains their compactifications $\widehat{X}$ and $\widehat{G}$ respectively. Choose a basepoint $o \in X$. Let $i : \Gamma \to X$ denote an 'inclusion map' mapping $g \in \Gamma$ to $g.o$ and an edge $[a, b]$ of Γ to a geodesic in X joining $a.o, b.o$.

A **Cannon-Thurston map** $\hat{i}$ from $\widehat{G}$ to $\widehat{X}$ is a continuous extension of i. The restriction of $\hat{i}$ to ∂G will be denoted by ∂i. The map ∂i induces a relation $\mathcal{R}_{CT}$ on ∂G where $x \sim y$ if $\partial i(x) = \partial i(y)$ for $x, y \in \partial G$.

Definition 2.4. *A* **CT leaf** λ_{CT} *is a bi-infinite geodesic in* Γ *whose end-points are identified by* ∂i.

Next, let G be a relatively hyperbolic group, hyperbolic relative to a collection $H_1, \cdots, H_k$ of subgroups. Let, as before, (X, d_X) be a hyperbolic metric space and G act freely, properly discontinuously by isometries on X, such that each conjugate of H_i fixes a unique point of ∂X. We thus think of each conjugate of H_i acting on X as a group of parabolic isometries. Let, as in Section 2.2, $i : \Gamma \to X$ denote the 'inclusion map' mapping $g \in \Gamma$ to $g.o$ for some basepoint $o \in X$. Then i induces a map $i^h : \mathcal{G}(G, \mathcal{H}) \to X$.

A **Cannon-Thurston map** $\hat{i}$ from $\widehat{G}$ to $\widehat{X}$ is a continuous extension of i^h. As before, the restriction of $\hat{i}$ to ∂G will be denoted by ∂i and the induced relation on ∂G by $\mathcal{R}_{CT}$.

Also, the set of *distinct* pairs of points identified by ∂i will be denoted as $\mathcal{R}_{CT}^2$, which is a subset of $\partial^2(G)$.

Lemma 2.5. $i : \Gamma \to X$ *(or* $i^h : \mathcal{G}(G, \mathcal{H}) \to X$*) is a quasi-isometric embedding if and only if* $\mathcal{R}_{CT}^2 = \emptyset$.

Proof: If i (or i^h) is a quasi-isometric embedding then $\partial i : \partial G \to \partial X$ is a homeomorphic embedding (Theorem 7.2H p. 189 of [Gro85]). Hence $\mathcal{R}_{CT}^2 = \emptyset$.

Conversely, if i (or i^h) is *not* a quasi-isometric embedding then there exist $a_n, b_n \in i(\Gamma)$ (or $i^h(\mathcal{G}(G, \mathcal{H}))$) such that the geodesic $[a_n, b_n]_\Gamma \subset \Gamma$ passes through $1 \in \Gamma$, but the geodesic $[a_n, b_n]_X \subset X$ joining $i(a_n), i(b_n)$ in X lies outside $B_n(i(1))$, the n-ball about $i(1)$ in X. Assume (after subsequencing) that $a_n \to a_\infty \in \partial G$ and $b_n \to b_\infty \in \partial G$. It follows that $\partial i(a_\infty) = \partial i(b_\infty)$. Hence $a_\infty \sim b_\infty$ and $\mathcal{R}_{CT}^2 \neq \emptyset$. $\square$

Definition 2.6. $\mathcal{R}_{CT}$ *will be called* **trivial** *if* $x \sim y$ *for all* $x, y \in \partial G$, *and* **quasiconvex** *if* $\mathcal{R}_{CT}^2 = \emptyset$.

2.3. **Properties of the** CT **relation.** We assume in this subsection that G is a (relatively) hyperbolic group acting freely, properly discontinuously by isometries on (X, d_X), a hyperbolic metric space. Further we assume that a Cannon-Thurston map $\hat{i}$ from $\widehat{G}$ to $\widehat{X}$ exists. The restriction of $\hat{i}$ to ∂G is denoted by ∂i. Recall that ∂i induces a relation $\mathcal{R}_{CT}$ on ∂G where $x \sim y$ if $\partial i(x) = \partial i(y)$ for $x, y \in \partial G$. Note

that $\mathcal{R}_{CT}$ is G-invariant. We investigate the basic properties of $\mathcal{R}_{CT}$ now. Assume that $\mathcal{R}_{CT}$ is non-trivial.

The following Lemma is a direct consequence of the continuity of $\hat{i}$.

Lemma 2.7. $\mathcal{R}_{CT}$ *is a closed relation on* ∂G, *i.e. if* $x_n \sim y_n$ *for sequences* $\{x_n\}, \{y_n\} \in \partial G$ *and* $x_n \to x \in \partial G$, $y_n \to y \in \partial G$, *then* $x \sim y \in \partial G$. *Further,* $\mathcal{R}_{CT}^2$ *is a closed subset of* $\partial^2(G)$.

Definition 2.8. G *is said to act on* X *without accidental parabolics if for every hyperbolic element* g *of* G, *its fixed points on* ∂X *are distinct.*

The next Lemma says that poles cannot lie in $\mathcal{R}_{CT}$.

Lemma 2.9. *Suppose* G *acts on* X *without accidental parabolics If* $(x, y) \in \mathcal{R}_{CT}$ *and* $x \neq y$, *then* x *cannot be a pole of* G.

Proof: By G-invariance of $\mathcal{R}_{CT}$ we can assume that the CT-leaf $(x, y) \subset \Gamma$ passes through $1 \in \Gamma$. We argue by contradiction. If x is a pole, there exists a hyperbolic element $g \in G$ such that $g_\infty = x$ and so g acts on X as a hyperbolic isometry. Then the infinite geodesic ray $[1, x) \subset \Gamma$ is a C-quasigeodesic in X for some $C \geq 1$ since G acts without accidental parabolics. Choosing a sequence of points $x_n \in [1, x) \subset (x, y)$ such that $x_n \to \infty$, note that $x_n^{-1}.(x, y)$ and $x_n^{-1}.[1, x)$ converge (up to subsequencing) to the same bi-infinite geodesic $(p, q) \subset \Gamma$. Hence $(p, q) \subset \Gamma$ is a C-quasigeodesic in X. In particular $\partial i(p) \neq \partial i(q)$. But, by Lemma 2.7 $(p, q) \in \mathcal{R}_{CT}$, i.e. $\partial i(p) = \partial i(q)$. This is a contradiction. $\square$

Density

The next Lemma proves density of orbits of cosets of $\mathcal{R}_{CT}$ in the Hausdorff metric.

Lemma 2.10. *Let* G *be a (relatively) hyperbolic group acting on a hyperbolic metric space without accidental parabolics and admitting a Cannon-Thurston map. Assume that the relation* $\mathcal{R}_{CT}$ *is non-trivial, i.e. not all points are in the same equivalence class. Let* $K \subset \mathcal{R}_{CT}$ *be a coset (equivalence class) of the relation. Let* $C_c(\partial G)$ *denote the space of closed subsets of* ∂G *with the Hausdorff metric. Then for all* $x \in \partial G$, *the singleton set* $\{x\}$ *is an accumulation point of* $\{g.K : g \in G\}$.

Proof: Clearly, $K \subset \partial G$ is closed. By Proposition 2.3 and Lemma 2.9, K is nowhere dense in ∂G and, for all $x \in \partial G$, there exist a sequence $\{h_n\} \subset G$ such that
1) the pole pair $(h_{n,-\infty}, h_{n,\infty})$ of h_n is a subset of $\partial G \setminus K$
2) $h_{n,\infty} \to x \in \partial G$ as $n \to \infty$.

By taking sufficiently large powers $h_n^{i_n}$ of h_n, we can ensure that $h_n^{i_n}(K)$ lies in an ϵ_n Hausdorff neighborhood of $h_{n,\infty}$, where $\epsilon_n \to 0$. Hence $h_n^{i_n}(K) \to \{x\} \in C_c(\partial G)$. $\square$

Remark 2.11. *The assumption on absence of accidental parabolics is not too restrictive. In all cases of interest, subgroups* H *of* G *which become parabolic in* X *will be regarded as elements of the the collection* $H_1, \cdots, H_n$ *relative to which* G *is hyperbolic.*

2.4. **Continuity in Uniform Topology.** Let $\Lambda_G \subset \partial X$ denote the limit set of G, i.e. the collection of accumulation points in ∂X of an(y) orbit of G acting on X. The existence of a Cannon-Thurston map ensures that $\partial i : \partial G \to \Lambda_G$

is a quotient map, where pre-images of ∂i are given by elements of $\mathcal{R}_{CT}$. Let $\partial i_c : C_c(\partial G) \to C_c(\Lambda_G)$ be the map induced by i_c from $\mathcal{R}_{CT}$-saturated compact subsets of ∂G to compact subsets of Λ_G both equipped with the Hausdorff metric. Suppose that f is a homeomorphism of ∂G that preserves the cosets of $\mathcal{R}_{CT}$, i.e. for any coset $K \in \mathcal{R}_{CT}$, $f(K)$ is also a coset of $\mathcal{R}_{CT}$. Then the quotient map from Λ_G to itself induced by f will be denoted by $\overline{f}$.

Proposition 2.12. *Suppose that $\mathcal{R}_{CT}$ is non-trivial. Let f_n be a sequence of homeomorphisms of $(\partial G, d)$ that preserves the cosets of $\mathcal{R}_{CT}$, where d denotes some visual metric. Let $\overline{f_n}$ denote the induced homeomorphisms of Λ_G. If $f_n \to id$ in the uniform topology on $Homeo(\partial G)$ then $\overline{f_n} \to id$ in the uniform topology on $Homeo(\Lambda_G)$. Conversely, if $\overline{f_n} \to id$ in the uniform topology on $Homeo(\Lambda_G)$ then for every pole $p \in \partial G$, $d(p, f_n(p)) \to 0$. More generally, if $\{p\}$ is a coset of $\mathcal{R}_{CT}$, $d(p, f_n(p)) \to 0$.*

Proof: The forward direction follows from the definition of quotient maps. Conversely, suppose $\overline{f_n} \to id$ in the uniform topology on $Homeo(\Lambda_G)$ but f_n does not converge to the identity map in the uniform topology on $Homeo(\partial G)$. Suppose that there exists (up to subsequencing) an $\epsilon > 0$ and a pole $x \in \partial G$ (or a point x such that $\{x\}$ is a coset of $\mathcal{R}_{CT}$) such that for all $n \in \mathbb{N}$ $d(f_n(x), x) > \epsilon$.

Then there exists $\eta < \frac{\epsilon}{10}$, say, such that $d_H(B_\eta(x), f_n(B_\eta(x))) > \eta$ where d_H denotes the Hausdorff metric on $C_c(\partial G)$ and $B_\eta(w)$ denotes a closed ball of radius η centered at w.

Assume after subsequencing that $f_n(x) \to y \in \partial G$. Then $x \sim y$, since $\overline{f_n} \to id$. But $x \neq y$.

This contradicts Lemma 2.9, if x is a pole (or more generally, if $\{x\}$ is a coset of $\mathcal{R}_{CT}$). $\square$

3. Commensurators of Kleinian Groups

In this section we prove discreteness of commensurators for finitely generated, discrete, Zariski dense, infinite covolume Kleinian groups.

3.1. **Surface Groups.** It follows from the Scott core Theorem [Sco73] that any finitely generated Kleinian group H is the fundamental group of a compact 3 manifold with boundary. Further, a geometrically finite group H_{gf} can be chosen abstractly isomorphic to H such that the isomorphism preserves parabolics (see [MMS85] [McC86] [MF87] for relative versions of the Scott core Theorem). Hence abstractly G is hyperbolic relative to its parabolic subgroups (see [Far98] for instance). The convex core of $\mathbb{H}^3/H$ will be denoted by M and the convex core of $\mathbb{H}^3/H_{gf}$ will be denoted by K, so that the inclusion $K \subset M$ is a homotopy equivalence.

Theorem 3.1. [Mj06b] [Mj10] *Let G be a finitely generated Kleinian group. Then G admits a Cannon-Thurston map from the (relative) hyperbolic boundary of G to its limit set. Further, the Cannon-Thurston map from the (relative) hyperbolic boundary of G to its limit set identifies precisely the end-points of leaves of the ending laminations.*

By Corollary 1.2, it suffices to consider the case where all ends of M are degenerate as these are precisely the manifolds that come from Kleinian groups with empty domain of discontinuity. Our study of $Comm(H)$ splits into two cases:

Case I: Some component of the boundary ∂K of K is an incompressible geometrically finite surface

Case II: K is topologically a compression body whose lower boundary is a (possibly empty) collection of tori.

This dichotomy stems from the fact that if some boundary component Q of K is incompressible relative to rank one cusps, then some other component of ∂K is an incompressible geometrically finite surface unless K is topologically a compression body whose lower boundary is a (possibly empty) collection of tori. This follows from standard 3-manifold topology [Hem] by choosing a maximal collection of disjoint homotopically distinct compressing disks with boundary curves on Q. In particular, if some maximal cusp of K is of rank one, we are in Case I.

We deal with Case I in this subsection and indicate the modifications for Case II in the next subsection.

Lemma 3.2. *Let M, K be as above. Let $\Sigma \subset Bdy(K)$ be a finite area surface component of the boundary of K that is incompressible away from cusps. Further suppose that the ideal (relative) hyperbolic boundary $\partial(\widetilde{\Sigma}) = S^1$ is a proper subset of the (relative) hyperbolic boundary $\partial(\widetilde{K})$.*

Let $\overline{f_n} \in Comm(H)$ be a sequence of commensurators converging to the identity in $\mathrm{Isom}(\mathbb{H}^3)$ and let f_n be the induced homeomorphisms on the (relative) hyperbolic boundary $\partial(\widetilde{K})$. Then $f_n \to Id \in Homeo(S^1)$.

Proof: Suppose f_n does not converge to the Identity map $Id \in Homeo(S^1)$. Then by Proposition 2.12 it suffices to show that there exists (up to subsequencing) a pole $p \in \partial(\widetilde{K})$ and an $\epsilon > 0$ such that $d(f_n(p), p) \geq \epsilon$ for all n, where d is a visual metric on $\partial(\widetilde{K})$.

Since f_n does not converge to the identity, we may assume (after passing to a further subsequence if necessary) that there exists a translate $T(= g.\partial(\widetilde{\Sigma}))$ of $\partial(\widetilde{\Sigma})$ such that $f_n(T) \neq T$ for all n. Since T has a definite visual diameter and the poles of $g\rho(\pi_1(\Sigma))g^{-1}$ are dense in $g.\partial(\widetilde{\Sigma})$, there exists a pole $p \in g.\partial(\widetilde{\Sigma})$ and an $\epsilon > 0$ such that $d(f_n(p), p) \geq \epsilon$ for all n $\square$

It remains to deal with doubly degenerate surface groups. By passing to a subgroup of index at most two in the commensurator group if necessary, we can assume without loss of generality that ends of doubly degenerate groups are preserved. Thus, in any case, for a sequence of commensurators g_n converging to the identity an end corresponding to a strictly type-preserving representation $\rho : \pi_1(S) \to Isom(\mathbb{H}^3)$ is "preserved". $\rho(\pi_1(S))$ is simply the subgroup of H corresponding to an incompressible end of M. By Theorem 3.1 we may assume that $\{g_n\}$ preserves the relation $\mathcal{R}_{CT}$ corresponding to the Cannon-Thurston map for the end. Let $\mathcal{L}$ be the ending lamination whose end-points are identified by the Cannon-Thurston map.

Claim 3.3. *Let $\overline{f_n} \in Comm(H)$ be a sequence of commensurators converging to the identity in $\mathrm{Isom}(\mathbb{H}^3)$ and let f_n be the induced homeomorphisms on the (relative) hyperbolic boundary $\partial \pi_1(S)(= S^1)$ of the group $\pi_1(S)$. Then $f_n \to Id \in Homeo(S^1)$.*

Proof of Claim: Suppose f_n does not converge to the Identity map $Id \in Homeo(S^1)$. Then by Proposition 2.12 it suffices to show that there exists (up to subsequencing)

a pole $p \in S^1$ and an $\epsilon > 0$ such that $d(f_n(p), p) \geq \epsilon$ for all n, where d is the usual visual metric on S^1.

Since f_n does not converge to the identity, we may assume (after passing to a further subsequence if necessary) that there exists an ideal polygon Δ such that
a) The sides of Δ are leaves of the ending lamination $\mathcal{L}$
b) $f_n|_\Delta \neq Id$ for all n.

Then there exist ideal adjacent vertices A, B of Δ such that $f_n(A) \neq A$ and $f_n(B) \neq B$ for all n. Let Δ_1 be another polygon whose sides lie in $\mathcal{L}$ and whose end-points lie in the arc (AB) of the circle S^1 that contains no other ideal points of Δ. Also let A_1, B_1 be adjacent vertices of Δ_1 such that $(A_1 B_1) \subset (AB)$. See figure below.
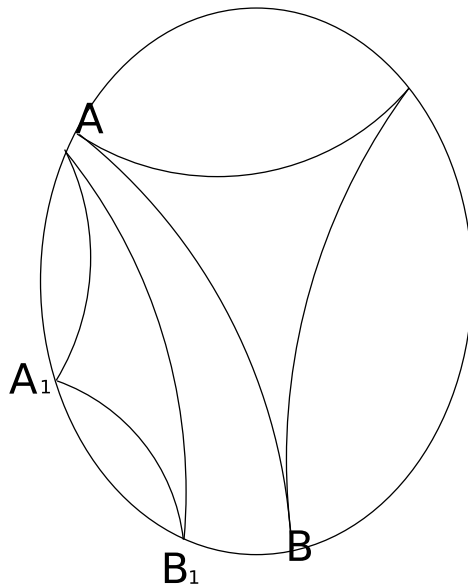
Since $f_n(A) \neq A$ and $f_n(B) \neq B$ for all n it follows that any pole p in the arc $(A_1 B_1)$ is moved at least $\epsilon = min(d(A, A_1), d(A, B_1), d(B, A_1), d(B, B_1))$ by f_n for all n. This proves the Claim. $\square$

We are finally in a position to prove the following.

Theorem 3.4. *Let G be a finitely generated Kleinian group such that G is Zariski dense in $\mathrm{PSL}_2(\mathbb{C})$ and $\mathbb{H}^3/G$ has infinite volume. Further, suppose that some end of $M = \mathbb{H}^3/G$ is incompressible. Then the commensurator $\mathrm{Comm}(G)$ of G is discrete in $\mathrm{PSL}_2(\mathbb{C})$.*

Proof: As above, suppose $\overline{f_n} \in Comm(H)$ be a sequence of commensurators converging to the identity in $\mathrm{Isom}(\mathbb{H}^3)$ and let f_n be the induced homeomorphisms on the (relative) hyperbolic boundary $\partial \pi_1(S)(= S^1)$ of the group $\pi_1(S)$. Then by Claim 3.3 it follows that for any ideal polygon Δ with boundary in the ending lamination there exists $N = N(\Delta)$ such that f_n fixes all the vertices of Δ for all $n \geq N$. Let $z_\Delta \in S^2_\infty$ denote the common image of the ideal vertices of Δ under the Cannon-Thurston map of Theorem 3.1. The collection of translates of z_Δ is dense in S^2. Choose ideal polygons $\Delta_1, \cdots, \Delta_4$ such that the collection $\{z_{\Delta_1}, \cdots, z_{\Delta_4}\}$

is in general position on S^2_∞, i.e. not all lying on the same circle. Then for all $n \geq max_{i=1\cdots4}\{N(\Delta_i)\}$, $\overline{f_n} = Id$. The Theorem follows. $\square$

Theorem 3.4 was proven by Leininger, Long and Reid [LLR09] under the additional assumption that G is non-free and without parabolics.

3.2. **Compressible Core.** It remains to prove the analogue of Theorem 3.4 in Case II, i.e. when the (relative) core K is topologically a compression body whose lower boundary is a (possibly empty) collection of tori. As in the previous subsection, H will denote the Kleinian group. The associated manifold $M = K \cup E$, where E is homeomorphic to $S \times [0, \infty)$ and $E \cap K = S \times \{0\}$. Further E is a simply degenerate end. Let σ be a homotopically essential simple closed curve on the boundary $Bdy(K)$ of K.

Background: In [Min02], Minsky constructs a model for the simply degenerate end E. We consider the family $\mathcal{F}$ of pleated surfaces exiting the end E and let $\widetilde{\mathcal{F}}$ be the collection of lifts of $\mathcal{F}$ to the universal cover $\widetilde{M}$ of M. The elements of $\widetilde{\mathcal{F}}$ are lifts $\widetilde{\Sigma}$ of pleated surfaces Σ in E.

For any $g \in \text{Comm}(H)$, $H_g = g^{-1}Hg \cap H$ is of finite index in H. Hence $M_g = \mathbb{H}^3/H_g$ is a finite-sheeted cover of M. Let $\mathcal{F}_g$ denote the lifts of elements of $\mathcal{F}$ to M_g. Since $\sigma \subset Bdy(K) = Bdy(E)$ is homotopically trivial in K, any loop in E freely homotopic to σ **in** E lifts isometrically to loops in $\widetilde{M}$ as well as in M_g. $\sigma_\alpha \subset \Sigma_\alpha$ will denote the unique geodesic representative in the homotopy class of σ in E carried by the pleated surface $\Sigma_\alpha \in \mathcal{F}$.

In [Mj10] we constructed the following.

(1) A subcollection $\{\Sigma_\beta, \beta \in \mathcal{B}\}$ of the collection $\mathcal{F}$.
(2) An auxiliary Gromov hyperbolic metric d_G on $\widetilde{M}$ called the graph metric.
(3) Let D_σ denote a disk bounded by (a lift of) σ in $\widetilde{K}$. We also constructed a quasidisk $A_\sigma = (D_\sigma \bigcup_{\beta \in \mathcal{B}} \sigma_\beta)$ such that A_σ is quasiconvex in $(\widetilde{M}, d_G)$. This construction is natural with respect to taking covers (finite or infinite) as σ bounds in K. Hence any cover of K contains isometric copies of A_σ. Since A_σ is quasiconvex in $(\widetilde{M}, d_G)$, the limit set ∂A_σ is a proper subset of $\mathbb{S}^2_\infty$ with diameter greater than zero.
(4) It is also shown in [Mj10] that if $h \to \infty$ in Γ_H, then the visual diameter of $h.\partial A_\sigma \subset \mathbb{S}^2_\infty$ tends to zero. In particular, the collection $\{h.\partial A_\sigma\}$ is discrete in $\mathbb{S}^2_\infty$.

We therefore conclude

Lemma 3.5. *Let $g_n \in \text{Comm}(H)$ such that $g_n \to Id$. Then for any $h_1, \cdots h_k \in H$, there exists $N > 0$ such that $g_n.(h_i.\partial A_\sigma) = h_i.\partial A_\sigma$ for all $i = 1 \cdots k$ and $n \geq N$.*

Remark 3.6. *We may think of the collection $\{h.\partial A_\sigma\}$ as an H-invariant pattern on $\mathbb{S}^2_\infty$ along the lines of* [Mj09]

The last step of the argument is similar to Theorem 3.4. Let Δ be an ideal polygon contained in $Bdy(\widetilde{K})$, whose sides are leaves of the ending lamination (thought of as an element of the Masur domain).

Choose a collection of lifts $\sigma_1, \cdots \sigma_m$ of σ to $\widetilde{K}$ such that

(1) The associated compressing disks $D_1, \cdots D_m$ are the corresponding lifts of the disk D bounded by σ in K.

(2) The boundary of $(\widetilde{K} \setminus \bigcup_{i=1\cdots m} D_i)$ consists of one bounded piece Σ_b homeomorphic to a sphere with finitely many holes and a finite collection of unbounded components.

(3) Consider the singular foliation associated to the ending lamination. Then the polygon Δ naturally corresponds to an l-prong $\mathcal{P}$ – a union of l geodesic rays starting at a common point $p \in Bdy(\widetilde{K})$. p will be called the base of the l-prong $\mathcal{P}$. We assume that p lies in the interior of Σ_b.

Let $A_1, \cdots, A_m$ be the quasidisks associated to $D_1, \cdots D_m$. Let $g_n \in \mathrm{Comm}(H)$ such that $g_n \to Id$. Then, by Lemma 3.5, there exists $N > 0$ such that $g_n.(\partial A_i) = \partial A_i$ for all $n \geq N$.

Since any commensurator of G induces a homeomorphism of one finite sheeted cover of K to another, it moves $\widetilde{K}$ to within a bounded Hausdorff distance of itself in $\widetilde{M}$. Hence we can isotope the image to $\widetilde{K}$ by a bounded isotopy.

After a bounded isotopy, we may assume that $g_n(D_i) = D_i$ for $i = 1\cdots m$ and for all $n \geq N$. Hence the base p of the l-prong $\mathcal{P}$ is moved to some point in Σ_b by g_n for all $n \geq N$.

However, there exist only finitely many l-prongs ($l > 2$) whose base lies in Σ_b. Also $g_n(D_i)$ can be isotoped to D_i by a bounded isotopy of $\widetilde{M}$. Hence there exists $M \geq N$ such that $g_n(\mathcal{P})$ is homotopic to $\mathcal{P}$ by a bounded homotopy for all $n \geq M$.

As in Claim 3.3 let f_n be the homeomorphisms of the (relative) hyperbolic boundary $\partial \widetilde{K}$ induced by g_n. Then for all $n \geq M$, $f_n(x) = x$ for all ideal end-points x of $\mathcal{P}$.

Let $q : \partial H \to \Lambda_H$ be the Cannon-Thurston map of Theorem 3.1 from the relative hyperbolic boundary of H to its limit set Λ_H. Then $g_n(q(x)) = q(x)$ for all $n \geq M$.

Choose four prongs $\mathcal{P}_1, \cdots, \mathcal{P}_4$ and ideal end-points $x_1, \cdots x_4$ such that the images $q(x_i)$ are in general position on $\mathbb{S}^2_\infty$. Then there exists $M_1 \geq M$ such that $g_n(q(x_i)) = q(x_i)$ for all $n \geq M_1$. Therefore $g_n = Id$ for all $n \geq M$.

We have finally proved the following.

Theorem 3.7. *Let G be a finitely generated Kleinian group such that G is Zariski dense in $\mathrm{PSL}_2(\mathbb{C})$ and $\mathbb{H}^3/G$ has infinite volume. Then the commensurator $\mathrm{Comm}(G)$ of G is discrete in $\mathrm{PSL}_2(\mathbb{C})$.*

Using the Thurston-Canary covering theorem [Can96], Theorem 3.7 can be strengthened a la Leininger, Long and Reid [LLR09] to show that unless G corresponds to a fiber subgroup of a 3-manifold fibering over the circle, then, in fact, $[\mathrm{Comm}(G) : G] < \infty$. We discuss this below.

Let $g \in Comm(G)$ be a commensurator of G. Then the group $G_1 =< G, g >$ is a finitely generated discrete Kleinian group. Let $M = \mathbb{H}^3/G$ and $M_1 = \mathbb{H}^3/G_1$. If M is an infinite cover of M, i.e. if $[G_1 : G] = \infty$, then the Thurston-Canary covering theorem [Can96] implies that G is virtually a fiber subgroup of the finite-volume 3-manifold group G_1, i.e. M_1 has a finite sheeted cover that fibers over the circle with fiber subgroup of finite index in G. Else, inductively, for all $g_1, \cdots, g_k \in Comm(G)$, if we denote $G_k =< G, g_1, \cdots, g_k >$, then $[G_k : G] < \infty$. Let $M_k = \mathbb{H}^3/G_1$. Again, by the Thurston-Canary covering theorem [Can96], every end of M_k is covered by an end of M. Since M has a finite number of ends, we may assume after passing to a subsequence, that all the M_k have the same number of ends. Since each end of M_k is homeomorphic to $S \times [0, \infty)$ for some finite volume hyperbolic surface S

and since each S can cover only finitely many hyperbolic surfaces, it follows that the sequence M_k must stabilize. We have thus shown the following.

Theorem 3.8. *Let G be a finitely generated Kleinian group such that G is Zariski dense in $\mathrm{PSL}_2(\mathbb{C})$ and $\mathbb{H}^3/G$ has infinite volume. Then $[\mathrm{Comm}(G) : G] < \infty$ unless G is virtually a fiber subgroup, in which case $\mathrm{Comm}(G)$ is the fundamental group of a virtually fibered finite volume hyperbolic 3-manifold.*

Finally, a note of caution: Just the existence of Cannon-Thurston maps is not sufficient to ensure discreteness of commensurators. A non-uniform arithmetic lattice in $\mathbb{H}^2$ has dense commensurator. However, work of Floyd [Flo80] ensures the existence of Cannon-Thurston maps in this situation. Thus the proof of Theorem 3.7 implicitly uses the following fact in an essential way:

Any polygon whose sides are leaves of an ending lamination has more than two sides.

This is exactly what goes wrong for non-uniform lattices in $\mathbb{H}^2$ and prevents us from using the separation arguments that go into the proof of Theorem 3.7.

3.3. **Examples, Applications and Questions.** We give an example to show that certain naive analogues of Theorem 3.8 cannot be expected in higher dimensions. The examples of fiber groups in Theorem 3.8 have empty domain of discontinuity. We first furnish an example that illustrates: *Domain of discontinuity non-empty does not imply $[\mathrm{Comm}(\Gamma) : \Gamma] < \infty$*
Example of a group Γ whose limit set is a proper subset of ∂X, but $[\mathrm{Comm}(\Gamma) : \Gamma] = \infty$ Let M be a finite volume hyperbolic 3-manifold fibering over the circle and admitting a totally geodesic embedded incompressible surface Σ. Let $\rho : \pi_1(M) \to Isom(\mathbb{H}^3)$ be the associated representation. Let $\rho' : \pi_1(M) \to Isom(\mathbb{H}^4)$ be ρ composed with the embedding of $\mathrm{Isom}(\mathbb{H}^3)$ in $\mathrm{Isom}(\mathbb{H}^4)$. Then a bending deformation of ρ', keeping $\rho'(\pi_1(\Sigma))$ fixed gives a new discrete faithful representation $\rho'' : \pi_1(M) \to Isom(\mathbb{H}^4)$ with Zariski dense image and such that $\rho''(\pi_1(M)) = G$ is quasiconvex. Hence $\Lambda_G \neq \partial\mathbb{H}^4$. Now, let F be the fiber of M. Then $\rho''(\pi_1(F)) = H$ is normal in G and hence $\Lambda_H = \Lambda_G \neq \partial\mathbb{H}^4$. By Corollary 1.2, $\mathrm{Comm}(H)$ is discrete. But G normalizes H. Therefore $G \subset Comm(H)$. Hence $[\mathrm{Comm}(H) : H] = \infty$.
Absence of examples in dimension 4 We indicate that a naive generalization of Theorem 3.8 cannot exist in dimension 4. In fact, in [Gro00] Gromov comments:

> Non-arithmetic Γ are especially plentiful for $n = 3$ by Thurston's theory and often have unexpected features, e.g. some $V = \mathbb{H}^3/\Gamma$ fiber over S^1 (which is hard to imagine ever happening for large n).

The next Proposition provides some justification.

Proposition 3.9. *Let M be a closed aspherical four manifold fibering over the circle with fiber F. Then $\pi_1(M)$ cannot be Gromov-hyperbolic. In particular M cannot be a rank one locally symmetric space. More generally if M is a closed aspherical n- manifold fibering over an $(n-3)$-manifold with infinite fundamental group, then $\pi_1(M)$ cannot be Gromov-hyperbolic.*

Proof: Observe that $F \times \mathbb{R}$ covers M and $\widetilde{F} \times \mathbb{R} = \widetilde{M}$ is aspherical. Hence $\pi_i(F) = 0$ for all $i > 1$. Hence F is a closed aspherical 3-manifold. Suppose $\pi_1(M)$ is Gromov-hyperbolic. Then $\pi_1(M)$ cannot contain a copy of $\mathbb{Z} \oplus \mathbb{Z}$. Hence F is a closed aspherical atoroidal 3-manifold with infinite fundamental group. By Perelman's

Geometrization Theorem, it follows that F admits a hyperbolic structure. Hence $Out(\pi_1(F))$ is finite.

But $M = (F \times I)/\phi$ for a diffeomorphism ϕ inducing an infinite order element of $Out(\pi_1(F))$ as otherwise $\pi_1(M)$ would contain a copy of $\mathbb{Z} \oplus \mathbb{Z}$. This is a contradiction.

The same argument gives the more general conclusion in the statement of the Proposition. $\square$

Remark 3.10. *The Chern-Gauss-Bonnet formula proves that an even dimensional hyperbolic manifold cannot fiber over the circle as such a manifold must have Euler characteristic zero.*

An alternate way of increasing Fibered Faces

An application of the ideas developed in this paper is an alternate argument (cf. Long-Reid [LR08], Agol [Ago08]) for increasing the number of fibered faces of an arithmetic hyperbolic 3-manifold fibering over the circle.

Let N be a finite volume *arithmetic* hyperbolic 3-manifold fibering over the circle. Equip N with the suspension flow of the associated pseudo-Anosov monodromy. A theorem of Fried [FLP79] says the following:

Let F_1 and F_2 be two embedded incompressible connected surfaces in a fibered 3 manifold M. Then F_1 and F_2 lie in the same fiber face of the unit Thurston norm ball if and only the flow lines of the associated suspension pseudo-Anosov flows are isotopic.

Take a stable polygon $\mathcal{P}$ (of more than two, possibly infinitely many sides, as in the punctured case) on a lift of the fiber $\widetilde{F} \subset \widetilde{F}$. Flow it forward and backward under the suspension flow to obtain a set $S(\mathcal{P})$. If F is a punctured surface and $\mathcal{P}$ has infinitely many sides, adjoin a copy of the horoball $\mathcal{B}$ which has the same common horospherical base-point at infinity as $\mathcal{P}$. By [Mj06a], $\mathcal{B} \cup S(\mathcal{P})$ is a quasiconvex subset of $\mathbb{H}^3$. Let $\partial \mathcal{P}$ denote the limit set of $\mathcal{B} \cup S(\mathcal{P})$. Then the collection of translates $\{g.\partial\mathcal{P} : g \in \rho(\pi_1(S))\}$ is a discrete collection of sets in the Hausdorff topology on $\mathbb{S}^2_\infty$.

This collection is a discrete pattern [Mj09] which is an invariant of the fiber face of the unit Thurston norm ball, i.e. F_1 and F_2 lie in the same fiber face of the unit Thurston norm ball if and only if they give rise to the same invariant collection of translates $\{g.\partial\mathcal{P} : g \in \rho(\pi_1(F_i))\}$.

Since the 3-manifold N is arithmetic, $\mathrm{Comm}(\pi_1(N))$ is dense. Therefore the $\mathrm{Comm}(\pi_1(N))$-orbit of $\partial\mathcal{P}$ is not discrete. Hence, as in [Ago08] Theorem 7.1, or [LR08], intersecting $\pi_1(N)$ with finitely many $g\pi_1(N)g^{-1}$ for appropriate $g \in \mathrm{Comm}(\pi_1(N))$ one can increase the number of fiber faces arbitrarily.

Questions

Together, Proposition 1.7 and Theorem 3.7 prove discreteness of commensurators for all known examples of finitely presented, Zariski dense, infinite covolume discrete subgroups of $\mathrm{Isom}(X)$ for X a symmetric space of non-compact type. We therefore venture the following

Question 3.11. *If Γ is a Zariski dense infinite covolume subgroup of $\mathrm{Isom}(X)$ for an irreducible symmetric space X of non-compact type, is $\mathrm{Comm}(\Gamma)$ discrete?*

For finitely presented groups, Question 3.11 subsumes the following (by Proposition 1.7 and Theorem 3.7).

Question 3.12. *If Γ is a finitely presented, infinite covolume, Zariski dense infinite covolume subgroup of $\mathrm{Isom}(X)$ for an irreducible symmetric space $X \neq \mathbb{H}^3$ of non-compact type, can the limit set be all of the Furstenberg boundary?*

Misha Kapovich [Kap98] has an example of a finitely generated but not finitely presented normal subgroup of a lattice in $\mathbb{CH}^2$. Normality ensures that the limit set is all of the visual boundary.

If we leave the domain of finitely generated groups, then there do exist examples to which Question 3.11 applies.

Question 3.13. *Let Γ_0 be an infinite index normal subgroup of a rank one (possibly arithmetic) lattice Γ. Is $\mathrm{Comm}(\Gamma_0)$ discrete? For instance Γ_0 could be the commutator subgroup $[\Gamma, \Gamma]$ when $H_1(\Gamma, \mathbb{Q}) \neq 0$.*

A particularly simple instance when even the special case of Question 3.13 seems unknown is when $\Gamma \subset SL_2(\mathbb{R})$ is a congruence subgroup of $SL_2(\mathbb{Z})$.

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