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Jacobian of meromorphic curves

SHREERAM S ABHYANKAR* and ABDALLAH ASSI†

* Mathematics Department, Purdue University, West Lafayette, IN 47907, USA
E-mail: ram@cs.purdue.edu
† Université d'Angers, Mathématiques, 49045 Angers cedex 01, France
E-mail: assi@univ-angers.fr

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Abstract. The contact structure of two meromorphic curves gives a factorization of their jacobian.

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1. Introduction

Let $J(F,G) = J_{(X,Y)}(F,G)$ be the jacobian of $F = F(X,Y)$ and $G = G(X,Y)$ with respect to X and Y, i.e., let $J(F,G) = F_X G_Y - F_Y G_X$ where subscripts denote partial derivatives. Here, to begin with, F and G are plane curves, i.e., polynomials in X and Y over an algebraically closed ground field k of characteristic zero. More generally, we let F and G be meromorphic curves, i.e., polynomials in Y over the (formal) meromorphic series field $k((X))$.

In terms of the the contact structure of F and G, we shall produce a factorization of $J(F,G)$. Note that if $G = -X$ then $J(F,G) = F_Y$; in this special case, our results generalize some results of Merle [Me], Delgado [De], and Kuo–Lu [KL] who studied the situation when F has one (Merle) or two (Delgado) or more (Kuo–Lu) branches. These authors restricted their attention to the analytic case, i.e., when F is a polynomial in Y over the (formal) power series ring $k[[X]]$. With an eye on the Jacobian conjecture, we are particularly interested in the meromorphic case.

The main technique we use is the method of Newton polygon, i.e., the method of deformations, characteristic sequences, truncations, and contact sets given in Abhyankar's 1977 Kyoto paper [Ab]. In § 2–5 we shall review the relevant material from [Ab]. In § 6 we shall introduce the tree of contacts and in § 7–9 we shall show how this gives rise to the factorizations.

The said Jacobian conjecture predicts that if the jacobian of two bivariate polynomials $F(X,Y)$ and $G(X,Y)$ is a nonzero constant then the variables X and Y can be expressed as polynomials in F and G, i.e., if $0 \neq J(F,G) \in k$ for F and G in $k[X,Y]$ then $k[F,G] = k[X,Y]$. We hope that the results of this paper may contribute towards a better understanding of this bivariate conjecture, and hence also of its obvious multivariate incarnation.

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2. Deformations

We are interested in studying polynomials in indeterminates X and Y over an algebraically closed ground field k of characteristic zero. To have more elbow room to manoeuver, we consider the larger ring $R = k((X))[Y]$ of polynomials in Y over $k((X))$, i.e., with coefficients in $k((X))$, where $k((X))$ is the meromorphic series field in X over k.

Given any

$$g = g(X,Y) = \sum_{i\in\mathbb{Z}} g^{[i]}X^i = \sum_{j\in\mathbb{Z}} g^{((j))}Y^j \in R,$$

with

$$g^{[i]} = g^{[i]}(Y) \in k[Y] \quad \text{and} \quad g^{((j))} = g^{((j))}(X) \in k((X)),$$

we put

$$\text{Supp}_X g = \{i \in \mathbb{Z} : g^{[i]} \neq 0\} \quad \text{and} \quad \text{Supp}_Y g = \{j \in \mathbb{Z} : g^{((j))} \neq 0\},$$

and we call this the X-support and the Y-support of g respectively. Note that these supports are bounded from below and above respectively, and upon letting

$$\gamma^\sharp = \text{ord}_X g = \text{the } X\text{-order of } g \quad \text{and} \quad \gamma = \deg_Y g = \text{the } Y\text{-degree of } g$$

we have

$$\gamma^\sharp = \begin{cases} \min(\text{Supp}_X g) & \text{if } g \neq 0 \\ \infty & \text{if } g = 0 \end{cases} \quad \text{and} \quad \gamma = \begin{cases} \max(\text{Supp}_Y g) & \text{if } g \neq 0 \\ -\infty & \text{if } g = 0. \end{cases}$$

Now

$$g^{[i]}(Y) = \sum_{j\in\mathbb{Z}} g^{(i,j)}Y^j \quad \text{and} \quad g^{((j))}(X) = \sum_{i\in\mathbb{Z}} g^{(i,j)}X^i \quad \text{with} \quad g^{(i,j)} \in k$$

and we put

$$\text{Supp}(g) = \text{Supp}_{(X,Y)} g = \{(i,j) \in \mathbb{Z}\times\mathbb{Z} : g^{(i,j)} \neq 0\}$$

and we call this the support, or the (X,Y)-support, of g. We put

$$\text{inco}_X g = \begin{cases} g^{[\gamma^\sharp]} & \text{if } g \neq 0 \\ 0 & \text{if } g = 0 \end{cases} \quad \text{and} \quad \text{deco}_Y g = \begin{cases} g^{((\gamma))} & \text{if } g \neq 0 \\ 0 & \text{if } g = 0 \end{cases}$$

and we call this the X-initial-coefficient and the Y-degree-coefficient of g respectively. Upon letting

$$\begin{aligned} \hat{\gamma}^\sharp &= \text{ord}(g) = \text{the (total) order of } g \\ &= \text{ord}_{(X,Y)} g = \text{the } (X,Y)\text{-order of } g \end{aligned}$$

we have

$$\hat{\gamma}^\sharp = \begin{cases} \min\{i+j : (i,j) \in \text{Supp}(g)\} & \text{if } g \neq 0 \\ \infty & \text{if } g = 0 \end{cases}$$

and we put

$$\mathrm{info}(g) = \mathrm{info}_{(X,Y)}g = \begin{cases} \sum_{i+j=\hat{\gamma}^\sharp} g^{(i,j)}X^iY^j & \text{if } g \neq 0 \\ 0 & \text{if } g = 0 \end{cases}$$

and we call this the initial-form, or the (X, Y)-initial-form, of g.[1] If $g \in k[X, Y]$ then upon letting

$$\begin{aligned} \hat{\gamma} &= \deg(g) = \text{the (total) degree of } g \\ &= \deg_{(X,Y)}g = \text{the } (X, Y)\text{-degree of } g \end{aligned}$$

we have

$$\hat{\gamma} = \begin{cases} \max\{i + j : (i, j) \in \mathrm{Supp}(g)\} & \text{if } g \neq 0 \\ -\infty & \text{if } g = 0 \end{cases}$$

and we put

$$\mathrm{defo}(g) = \mathrm{defo}_{(X,Y)}g = \begin{cases} \sum_{i+j=\hat{\gamma}} g^{(i,j)}X^iY^j & \text{if } g \neq 0 \\ 0 & \text{if } g = 0 \end{cases}$$

and we call this the degree-form, or the (X, Y)-degree-form, of g.[2]

Given any $z = z(X) \in k((X))$, we write

$$z = z(X) = \sum_{i\in\mathbb{Z}} z[i]X^i \quad \text{with} \quad z[i] \in k,$$

and we put

$$z[i] = 0 \text{ for all } i \in \mathbb{Q} \setminus \mathbb{Z},$$

and we let

$$\epsilon(z) = \begin{cases} \text{the set of all } (U, V, W) \in \mathbb{Z}^3 \text{ such that } U > 0 < V \\ \text{and } iV/U \in \mathbb{Z} \text{ for all } i \in \mathrm{Supp}_X z \text{ with } i < WU/V \end{cases}$$

and we call this the edge of z, and for any $(U, V, W) \in \epsilon(z)$ we let

$$z^\dagger(X, U, V, W) = \sum_{i<WU/V} z[i]X^{iV/U} \in k((X))$$

and

$$z^\ddagger(X, U, V, W, Y) = z^\dagger(X, U, V, W) + X^W Y \in R$$

[1] In an obvious manner, the definitions of $\mathrm{Supp}_X g$, $\mathrm{ord}_X g$, $\mathrm{inco}_X g$, $\mathrm{Supp}_Y g$, $\deg_Y g$, $\mathrm{deco}_Y g$, $\mathrm{Supp}_{(X,Y)} g$, $\mathrm{ord}_{(X,Y)} g$, and $\mathrm{info}_{(X,Y)} g$, can be extended to any g in $k((X))[Y, Y^{-1}]$, and for any such g we can also define $\mathrm{ord}_Y g$ and $\mathrm{inco}_Y g$, and then we have: $g = 0 \Leftrightarrow \mathrm{Supp}_X g = \emptyset \Leftrightarrow \mathrm{ord}_X g = \infty \Leftrightarrow \mathrm{inco}_X g = 0 \Leftrightarrow \mathrm{Supp}_Y g = \emptyset \Leftrightarrow \mathrm{ord}_Y g = \infty \Leftrightarrow \deg_Y g = -\infty \Leftrightarrow \mathrm{inco}_Y g = 0 \Leftrightarrow \mathrm{deco}_Y g = 0 \Leftrightarrow \mathrm{Supp}_{(X,Y)} g = \emptyset \Leftrightarrow \mathrm{ord}_{(X,Y)} g = \infty \Leftrightarrow \mathrm{info}_{(X,Y)} g = 0$.

[2] Again, in an obvious manner, the definitions of $\deg_{(X,Y)} g$ and $\mathrm{defo}_{(X,Y)} g$ can be extended to any g in $k[X, X^{-1}, Y, Y^{-1}]$, and for any such g we can also define $\deg_X g$ and $\mathrm{deco}_X g$, and then we have: $g = 0 \Leftrightarrow \deg_X g = -\infty \Leftrightarrow \mathrm{deco}_X g = 0 \Leftrightarrow \deg_{(X,Y)} g = -\infty \Leftrightarrow \mathrm{defo}_{(X,Y)} g = 0$.

and

$$z^{\dagger *}(X,U,V,W) = \sum_{i\le WU/V} z[i]X^{iV/U} \in k((X))$$

and we call these the (U,V,W)-truncation, the (U,V,W)-deformation, and the strict (U,V,W)-truncation of $z(X)$ respectively. Given any $H = H(X,Y) \in R$, we are interested in calculating $\text{ord}_X H(X^V, z^\ddagger(X,U,V,W,Y))$ and $\text{inco}_X H(X^V, z^\ddagger(X,U,V,W,Y))$.[3] For this purpose we proceed to give a review on characteristic sequences.

3. Characteristic sequences

Let $\hat{R}^\natural$ be the set of all monic polynomials in Y over $k((X))$, i.e., those nonzero members of R in whom the coefficient of the highest Y-degree term is 1. Let $R^\natural$ be the set of all irreducible monic polynomials in Y over $k((X))$, i.e., those members of $\hat{R}^\natural$ which generate prime ideals in R; note that their Y-degrees are positive integers.

Given any $f = f(X,Y) \in R^\natural$ of Y-degree n, by Newton's theorem

$$f(X^n, Y) = \prod_{1\le j\le n} [Y - z_j(X)] \quad \text{with} \quad z_j(X) \in k((X)),$$

where we note that $\text{Supp}_X z_j$ is independent of j. Let $m(f) = m_i(f)_{0\le i\le h(m(f))+1}$ be the newtonian sequence of characteristic exponents of f relative to n as defined on page 300 of [Ab], let $d(m(f)) = d_i(m(f))_{0\le i\le h(d(m(f)))+2}$ be the GCD-sequence of $m(f)$ as defined on page 297 of [Ab], let $q(m(f)) = q_i(m(f))_{0\le i\le h(q(m(f)))+1}$ be the difference sequence of $m(f)$ as defined on page 301 of [Ab], let $s(q(m(f))) = s_i(q(m(f)))_{0\le i\le h(s(q(m(f))))+1}$ be the inner product sequence of $q(m(f))$ as defined on page 302 of [Ab], and let $r(q(m(f))) = r_i(q(m(f)))_{0\le i\le h(r(q(m(f))))+1}$ be the normalized inner product sequence of $q(m(f))$ as defined on p. 302 of [Ab].[4] Note that then

$$\begin{aligned} h(d(m(f))) &= h(m(f)) = h(q(m(f))) \\ &= h(s(q(m(f)))) = h(r(q(m(f)))) = \text{a nonnegative integer} \end{aligned}$$

and

$$d_0(m(f)) = 0 \quad \text{and} \quad d_{h(d(m(f)))+1}(m(f)) = 1$$

[3] To motivate the definitions of $\epsilon(z)$ and $z^\ddagger$, given any $H = H(X,Y) = \sum_{i,j} H^{(i,j)} X^i Y^j \in R$ with $H^{(i,j)} \in k$, let $\Gamma^\sharp$ and $\Theta^\sharp(Y)$ be the values of $\text{ord}_X H(X^V, z^\ddagger(X,U,V,W,Y))$ and $\text{inco}_X H(X^V, z^\ddagger(X,U,V,W,Y))$ when $z=0$, i.e., let $\Gamma^\sharp = \text{ord}_X H(X^V, X^W Y)$ and $\Theta^\sharp(Y) = \text{inco}_X H(X^V, X^W Y)$. Also let Γ and $\Theta(X,Y)$ be the weighted order and the weighted initial form of $H(X,Y)$, when we give weights (V,W) to (X,Y), i.e., let $\Gamma = \min\{iV + jW : (i,j) \in \text{Supp}_{(X,Y)} H(X,Y)\}$ and $\Theta(X,Y) = \sum_{iV+jW=\Gamma} H^{(i,j)} X^i Y^j$. Then $\Gamma^\sharp = \Gamma = \text{ord}_{(X,Y)} H(X^V, Y^W)$ and $\Theta(X,Y) = \text{info}_{(X,Y)} H(X^V, Y^W)$. Moreover, $\Theta^\sharp(Y)$ and $\Theta(X,Y)$ determine each other by the formulas $\Theta^\sharp(Y) = \Theta(1,Y)$ and $\Theta(X,Y) = X^{\Gamma/V}\Theta^\sharp(X^{-W/V}Y)$. The parameter U is a normalizing parameter which essentially says that we want to intersect the "meromorphic curve" $H(X,Y) = 0$ with a deformation of the "irreducible meromorphic curve" $f(X,Y) = 0$ where $f(X,Y)$ is a monic irreducible polynomial of degree $U = n$ in Y over $k((X))$; to do this we take a "fractional meromorphic" root $y(X)$ of $f(X,Y) = 0$ with $y(X^n) = z(X) \in k((X))$, and then after "deforming" $y(X)$ at $X^{W/U}$ we substitute the deformation in $f(X,Y)$ for Y. For further motivation see the definitions of $\epsilon(f,\lambda)$ and $t(f,\lambda)$ displayed in the middle of the next section.

[4] It is really not necessary to look up [Ab] for the definitions of the sequences m, d, q, s, r, since they are completely redefined in the next three sentences ending with the displayed item (•).

and

$$d_1(m(f)) = m_0(f) = q_0(m(f)) = s_0(q(m(f))) = r_0(q(m(f))) = n,$$

and

$$\begin{aligned} d_{h(m(f))+2}(m(f)) &= m_{h(m(f))+1}(f) = q_{h(m(f))+1}(m(f)) \\ &= s_{h(m(f))+1}(q(m(f))) = r_{h(m(f))+1}(q(m(f))) = \infty, \end{aligned}$$

and

$$m_1(f) = q_1(m(f)) = s_1(q(m(f)))/n = r_1(q(m(f))) = \min(\mathrm{Supp}_X z_1),$$

with the understanding that the min of the empty set is ∞. Also note that: $h(m(f)) = 0 \Leftrightarrow f(X,Y) = Y$. Finally note that if $f(X,Y) \neq Y$ then for $2 \le i \le h(m(f))$ we have that $d_i(m(f))$, $m_i(f)$, $q_i(m(f))$, $s_i(q(m(f)))$, $r_i(q(m(f)))$ are integers with $d_i(m(f)) > 0$ such that:

$$(\bullet) \qquad \begin{cases} d_i(m(f)) = \mathrm{GCD}(m_0(f), m_1(f), \ldots, m_{i-1}(f)), \\ m_i(f) = \min(\mathrm{Supp}_X z_1 \setminus d_i(m(f))\mathbb{Z}), \\ q_i(m(f)) = m_i(f) - m_{i-1}(f), \\ s_i(q(m(f))) = q_1(m(f))d_1(m(f)) + \cdots + q_i(m(f))d_i(m(f)), \\ \text{and } r_i(q(m(f))) = s_i(q(m(f)))/d_i(m(f)). \end{cases}$$

In the rest of this section we shall use the abbreviations

$$d_i = d_i(m(f)) \quad \text{and} \quad s_i = s_i(q(m(f)))$$

for all relevant values of i. Let the sequence $c(f) = c_i(f)_{1 \le i \le h(c(f))}$ be defined by putting

$$h(c(f)) = h(m(f)) \quad \text{and} \quad c_i(f) = m_i(f)/n \text{ for } 1 \le i \le h(c(f))$$

and let us call this the normalized characteristic sequence of f. Note that then $c_1(f) < c_2(f) < \cdots < c_{h(c(f))}(f)$ are rational numbers, out of which only $c_1(f)$ could be an integer. To obtain an alternative characterization of the noninteger members of this sequence, for any rational number λ, we let

$$p(f,\lambda) = \begin{cases} \text{the unique nonnegative integer } \le h(c(f)) \text{ such that} \\ c_i(f) < \lambda \le c_j(f) \text{ for } 1 \le i \le p(f,\lambda) < j \le h(c(f)) \end{cases}$$

and

$$p^*(f,\lambda) = \begin{cases} \text{the unique nonnegative integer } \le h(c(f)) \text{ such that} \\ c_i(f) \le \lambda < c_j(f) \text{ for } 1 \le i \le p^*(f,\lambda) < j \le h(c(f)) \end{cases}$$

and

$$D(f,\lambda) = n/d_{p+1} \quad \text{with} \quad p = p(f,\lambda)$$

and

$$D^*(f,\lambda) = n/d_{p^*+1} \quad \text{with} \quad p^* = p^*(f,\lambda)$$

and

$$S(f,\lambda) = \begin{cases} (s_p + (n\lambda - m_p(f))d_{p+1})/n^2 & \text{if } p = p(f,\lambda) \neq 0 \\ \lambda & \text{if } p = p(f,\lambda) = 0 \end{cases}$$

and, for any $z \in k((X))$, we let

$$A(f,\lambda,z) = \prod_{i=1}^{p(f,\lambda)} ((d_i/d_{i+1})z[m_i(f)]^{(d_i/d_{i+1})-1})^{d_{i+1}D(f,\lambda)/n}$$

and

$$\hat{A}(f,\lambda,z) = A(f,\lambda,z)^{n/D(f,\lambda)}$$

and

$$E(f,\lambda,z,Y) = Y^{D^*(f,\lambda)/D(f,\lambda)} - z[n\lambda]^{D^*(f,\lambda)/D(f,\lambda)}$$

and

$$\hat{E}(f,\lambda,z,Y) = E(f,\lambda,z,Y)^{n/D^*(f,\lambda)}$$

and we call these the λ-position, the strict λ-position, the λ-degree, the strict λ-degree, the λ-strength, the (λ,z)-reduced-constant, the (λ,z)-constant, the (λ,z)-reduced-polynomial, and the(λ,z)-polynomial of f respectively; note that the above objects p, p^*, D, D^*, S, A, $\hat{A}$, E, and $\hat{E}$ respectively correspond to the objects $p(<)$, $p(\leq)$, D, E, s, B, $\hat{B}$, P, and $\hat{P}$ introduced on pp. 326–328 of [Ab]. We also define the sequence $m(f,\lambda) = m_i(f,\lambda)_{0\leq i\leq h(m(f,\lambda))+1}$ by putting

$$h(m(f,\lambda)) = p(f,\lambda) \text{ and } m_i(f,\lambda) = m_i(f)D(f,\lambda)/n \text{ for } 0\leq i\leq p(f,\lambda)+1$$

with the understanding that $m_i(f,\lambda) = \infty$ for $i = p(f,\lambda)+1$, and we define the sequence $m^*(f,\lambda) = m_i^*(f,\lambda)_{0\leq i\leq h(m^*(f,\lambda))+1}$ by putting

$$h(m^*(f,\lambda)) = p^*(f,\lambda) \text{ and}$$
$$m_i^*(f,\lambda) = m_i(f)D^*(f,\lambda)/n \text{ for } 0\leq i\leq p^*(f,\lambda)+1$$

with the understanding that $m_i^*(f,\lambda) = \infty$ for $i = p^*(f,\lambda)+1$, and we define the sequence $c(f,\lambda) = c_i(f,\lambda)_{1\leq i\leq h(c(f,\lambda))}$ by putting

$$h(c(f,\lambda)) = p(f,\lambda) \quad \text{and} \quad c_i(f,\lambda) = c_i(f) \text{ for } 1\leq i\leq p(f,\lambda)$$

and we define the sequence $c^*(f,\lambda) = c_i^*(f,\lambda)_{1\leq i\leq h(c^*(f,\lambda))}$ by putting

$$h(c^*(f,\lambda)) = p^*(f,\lambda) \quad \text{and} \quad c_i^*(f,\lambda) = c_i(f) \text{ for } 1\leq i\leq p^*(f,\lambda),$$

and we call these sequences the λ-characteristic-sequence, the strict λ-characteristic-sequence, the λ-normalized-characteristic-sequence, and the strict λ-normalized-characteristic-sequence of f respectively. We also let

$$\epsilon(f,\lambda) = \begin{cases} \text{the set of all } (z,U,V,W)\in k((X))\times\mathbb{Z}^3 \text{ such that } U=n, W/V=\lambda, \\ \text{and } (U,V,W)\in\epsilon(z) \text{ where } z=z(X)\in k((X)) \text{ with } f(X^n,z(X))=0 \end{cases}$$

and we call this the λ-edge of$f$. Finally we define $t(f,\lambda) = t(f,\lambda)(X,Y)$ to be the unique member of $R^\natural$ such that

$$t(f,\lambda)(X^V, z^\dagger(X,U,V,W)) = 0 \text{ for some}$$
$$\text{(and hence for all)}(z,U,V,W)\in\epsilon(f,\lambda)$$

and we call this the λ-normalized-truncation of$f$, and we define $t^*(f,\lambda) = t^*(f,\lambda)(X,Y)$ to be the unique member of $R^\natural$ such that

$$t^*(f,\lambda)(X^V, z^{\dagger *}(X,U,V,W)) = 0 \text{ for some (and hence for all) } (z,U,V,W) \in \epsilon(f,\lambda)$$

and we call this the strict λ-normalized-truncation of f; note that on p. 294 of [Ab] we have called these the open and closed $(n\lambda)$-truncations of f respectively.

From the above definitions of the various objects, we see that

$$\begin{cases} p(f,\lambda) \text{ and } p^*(f,\lambda) \text{ are integers with} \\ 0 \le p(f,\lambda) \le p^*(f,\lambda) \le h(m(f)), \text{ and} \\ D(f,\lambda) \text{ and } D^*(f,\lambda) \text{ are positive integers with} \\ n/D^*(f,\lambda) \in \mathbb{Z} \text{ and } D^*(f,\lambda)/D(f,\lambda) \in \mathbb{Z} \end{cases} \tag{NP1}$$

and

$$\begin{cases} t(f,\lambda) \text{ and } t^*(f,\lambda) \text{ are elements of } R^\natural \text{ such that:} \\ m(t(f,\lambda)) = m(f,\lambda) \text{ and } m(t^*(f,\lambda)) = m^*(f,\lambda), \\ c(t(f,\lambda)) = c(f,\lambda) \text{ and } c(t^*(f,\lambda)) = c^*(f,\lambda), \\ \deg_Y t(f,\lambda) = D(f,\lambda) \text{ and } \deg_Y t^*(f,\lambda) = D^*(f,\lambda), \text{ and} \\ h(m(t(f,\lambda))) = h(c(t(f,\lambda))) = p(f,\lambda) \text{ and } h(m(t^*(f,\lambda))) \\ \qquad = h(c(t^*(f,\lambda))) = p^*(f,\lambda) \end{cases} \tag{NP2}$$

and

$$\begin{cases} A(f,\lambda,z) \in k \text{ and } \hat{A}(f,\lambda,z) = A(f,\lambda,z)^{n/D(f,\lambda)} \in k \text{ are such that:} \\ \text{if } f(X^n, z(X)) = 0 \text{ then } A(f,\lambda,z) \neq 0 \neq \hat{A}(f,\lambda,z) \end{cases} \tag{NP3}$$

and

$$\begin{cases} E(f,\lambda,z,Y) = Y^{D^*(f,\lambda)/D(f,\lambda)} - z[n\lambda]^{D^*(f,\lambda)/D(f,\lambda)} \in k[Y] \\ \text{and } \hat{E}(f,\lambda,z,Y) = E(f,\lambda,z,Y)^{n/D^*(f,\lambda)} \in k[Y] \\ \text{are monic polynomials of degrees } D^*(f,\lambda)/D(f,\lambda) \text{ and } n/D(f,\lambda) \\ \text{respectively, where } z[n\lambda] \in k \text{ is such that: } z[n\lambda] \neq 0 \Leftrightarrow n\lambda \in \mathrm{Supp}_X z \end{cases} \tag{NP4}$$

and

$$\begin{cases} D^*(f,\lambda)/D(f,\lambda) > 1 \\ \Leftrightarrow \lambda = c_i(f) \notin \mathbb{Z} \text{ for some } i \in \{1,\dots,h(c(f))\} \\ \Leftrightarrow \hat{E}(f,\lambda,z,Y) \text{ has more than one root in } k \\ \qquad \text{for any } z = z(X) \in k((X)) \text{ with } f(X^n, z(X)) = 0 \end{cases} \tag{NP5}$$

and

$$\begin{cases} S(f,\lambda) \in \mathbb{Q} \text{ is such that:} \\ \text{if } (z,U,V,W) \in \epsilon(f,\lambda) \text{ then } S(f,\lambda)nV \in \mathbb{Z}. \end{cases} \tag{NP6}$$

With this preparation, what we have called Newton polygon (3) on p. 334 of [Ab] can be restated by saying that:

$$\begin{cases}\text{if } (z,U,V,W)\in\epsilon(f,\lambda)\\ \text{then } \mathrm{ord}_X f(X^V,z^\ddagger(X,U,V,W,Y))=S(f,\lambda)nV\\ \text{and } \mathrm{inco}_X f(X^V,z^\ddagger(X,U,V,W,Y))=\hat{A}(f,\lambda,z)\hat{E}(f,\lambda,z,Y).\end{cases} \tag{NP7}$$

In view of (NP3) and (NP5), the last line of (NP7) tells us that the noninteger members of the sequence $c_i(f)_{1\le i\le h(c(f))}$ are exactly those values of λ for which $\mathrm{inco}_X(f(X^V,z(X,U,V,W,Y)))$ has more than one root in k; this then is the alternative characterization we spoke of.

Given any other $f'=f'(X,Y)\in R^\natural$ of Y-degree n', by Newton's theorem

$$f'(X^{n'},Y)=\prod_{1\le j\le n'}[Y-z'_j(X)] \quad \text{with} \quad z'_j(X)\in k((X)).$$

Recall that on p. 287 of [Ab] the contact $\mathrm{cont}(f,f')$ of f with f' is defined by putting

$$\mathrm{cont}(f,f')=\max\{(1/n')\mathrm{ord}_X[z_j(X^{n'})-z'_{j'}(X^n)]:1\le j\le n \text{ and } 1\le j'\le n'\}.$$

We define the normalized contact $\mathrm{noc}(f,f')$ of f with f' by putting $\mathrm{noc}(f,f')=(1/n)\,\mathrm{cont}(f,f')$, i.e., equivalently, by putting

$$\mathrm{noc}(f,f')=\max\{(1/(nn'))\mathrm{ord}_X[z_j(X^{n'})-z'_{j'}(X^n)]:1\le j\le n \text{ and } 1\le j'\le n'\}.$$

We note that if $f\neq f'$ then $\mathrm{noc}(f,f')$ is a rational number, and if $f=f'$ then $\mathrm{noc}(f,f')=\infty$. We also note the isosceles triangle property which we shall tacitly use and which says that

$$\begin{cases} f''\in R^\natural \Rightarrow \mathrm{noc}(f,f'')\ge\min(\mathrm{noc}(f,f'),\mathrm{noc}(f',f''))\\ \text{and}\\ f''\in R^\natural \text{ with } \mathrm{noc}(f,f')\neq\mathrm{noc}(f',f'')\Rightarrow \mathrm{noc}(f,f'')\\ \qquad =\min(\mathrm{noc}(f,f'),\mathrm{noc}(f',f'')).\end{cases} \tag{ITP}$$

In view of the confluence lemmas given on pp. 338–344 of [Ab] we see that

$$\begin{cases}\text{if } \lambda\le\lambda'=\mathrm{noc}(f,f')\\ \text{then } p(f',\lambda)=p(f,\lambda), D(f',\lambda)=D(f,\lambda), S(f',\lambda)=S(f,\lambda),\\ m(f',\lambda)=m(f,\lambda), c(f',\lambda)=c(f,\lambda), t(f',\lambda)=t(f,\lambda),\\ \text{and } A(f',\lambda,z')=A(f,\lambda,z)\\ \text{where we have chosen } z=z(X) \text{ and } z'=z'(X) \text{ in } k((X)) \text{ such that}\\ f(X^n,z(X))=0=f'(X^{n'},z'(X)) \text{ and } (1/(nn'))\mathrm{ord}_X[z(X^{n'})-z'(X^n)]=\lambda'\end{cases} \tag{GNP1}$$

and

$$\begin{cases}\text{if } \lambda<\lambda'=\mathrm{noc}(f,f')\\ \text{then } p^*(f',\lambda)=p^*(f,\lambda), D^*(f',\lambda)=D^*(f,\lambda),\\ m^*(f',\lambda)=m^*(f,\lambda), c^*(f',\lambda)=c^*(f,\lambda), t^*(f',\lambda)=t^*(f,\lambda),\\ \text{and } E(f',\lambda,z',Y)=E(f,\lambda,z,Y)\\ \text{where we have chosen } z=z(X) \text{ and } z'=z'(X) \text{ in } k((X)) \text{ such that}\\ f(X^n,z(X))=0=f'(X^{n'},z'(X)) \text{ and } (1/(nn'))\,\mathrm{ord}_X[z(X^{n'})-z'(X^n)]=\lambda'\end{cases} \tag{GNP2}$$

and

$$\begin{cases} \text{if } \lambda = \lambda' = \text{noc}(f, f') \\ \text{then } E(f', \lambda, z', Y) \text{ and } E(f, \lambda, z, Y) \text{ do not have a common root} \\ \text{where we have chosen } z = z(X) \text{ and } z' = z'(X) \text{ in } k((X)) \text{ such that} \\ f(X^n, z(X)) = 0 = f'(X^{n'}, z'(X)) \text{ and } (1/(nn'))\text{ord}_X[z(X^{n'}) - z'(X^n)] = \lambda' \end{cases} \quad \text{(GNP3)}$$

and

$$\begin{cases} \text{if } \lambda \leq \lambda' = \text{noc}(f, f') \text{ and } (z, U, V, W) \in \epsilon(f, \lambda) \\ \text{then } S(f, \lambda)n'V \in \mathbb{Z}. \end{cases} \quad \text{(GNP4)}$$

What we have called generalized Newton polygon (6) on pp. 346–347 of [Ab] can now be restated by saying that:

$$\begin{cases} \text{if } \lambda \leq \lambda' = \text{noc}(f, f') \text{ and } (z, U, V, W) \in \epsilon(f, \lambda) \\ \text{then } \text{ord}_X f'(X^V, z^\ddagger(X, U, V, W)) = S(f, \lambda)n'V \\ \text{and } \text{inco}_X f'(X^V, z^\ddagger(U, V, W, Y)) = \hat{A}(f', \lambda, z')\hat{E}(f', \lambda, z', Y) \\ \text{where we have chosen } z' = z'(X) \in k((X)) \text{ such that} \\ f'(X^{n'}, z'(X)) = 0 \text{ and } (1/(nn'))\text{ord}_X[z(X^{n'}) - z'(X^n)] = \lambda' \end{cases} \quad \text{(GNP5)}$$

and

$$\begin{cases} \text{if } \lambda > \lambda' = \text{noc}(f, f') \text{ and } (z, U, V, W) \in \epsilon(f, \lambda) \\ \text{then } \text{ord}_X f'(X^V, z^\ddagger(X, U, V, W)) = S(f, \lambda')n'V \\ \text{and } 0 \neq \text{inco}_X f'(X^V, z^\ddagger(U, V, W, Y)) \\ \quad = \hat{A}(f', \lambda', z')\hat{E}(f', \lambda', z', z[n\lambda]) \in k \\ \text{where we have chosen } z' = z'(X) \in k((X)) \text{ such that} \\ f'(X^{n'}, z'(X)) = 0 \text{ and } (1/(nn'))\text{ord}_X[z(X^{n'}) - z'(X^n)] = \lambda' \end{cases} \quad \text{(GNP6)}$$

and

$$\begin{cases} \text{if } \lambda = \lambda' = \text{noc}(f, f') \text{ and } f(X^n, z(X)) = 0 \text{ with } z(X) \in k((X)) \\ \text{then } \text{ord}_X f'(X^n, z(X)) = S(f, \lambda)n'n. \end{cases} \quad \text{(GNP7)}$$

Finally we note that, for the truncations $t(f, \lambda)$ and $t^*(f, \lambda)$, we obviously have

$$\begin{cases} \text{noc}(f, t(f, \lambda)) \geq \lambda \text{ and} \\ \text{noc}(f, t^*(f, \lambda)) > \lambda. \end{cases} \quad \text{(GNP8)}$$

4. Truncations and buds

To continue discussing truncations, we let $R^\flat$ be the set of all buds in R, where by a bud we mean a pair $B = (\sigma(B), \lambda(B))$ with $\emptyset \neq \sigma(B) \subset R^\natural$ and $\lambda(B) \in \mathbb{Q}$ such that $\text{noc}(f, f') \geq \lambda(B)$ for all f and f' in $\sigma(B)$; we call $\sigma(B)$ the stem of B, and $\lambda(B)$ the level of B; we also let $\tau(B) = \{f \in R^\natural : \text{noc}(f, f') \geq \lambda(B) \text{ for all } f' \in \sigma(B)\}$, and we call $\tau(B)$ the flower of B.[5] For any $f \in R^\natural$ and $B \in R^\flat$, we let $\mathrm{noc}(f,B)$ be the rational number defined by saying that if $f \in \tau(B)$ then $\mathrm{noc}(f,B) = \lambda(B)$, whereas if $f \notin \tau(B)$ then $\mathrm{noc}(f,B)$ equals the common value (see (ITP)) of $\mathrm{noc}(f,f')$ as f' varies in $\tau(B)$; we call $\mathrm{noc}(f,B)$ the normalized contact of f with B, and we note that: $\mathrm{noc}(f,B) \neq \lambda(B) \Leftrightarrow \mathrm{noc}(f,B) < \lambda(B) \Leftrightarrow f \notin \tau(B)$.[6] For any $f \in R^\natural$ and $\lambda \in \mathbb{Q}$, we let $\overline{R}(f,\lambda) = \{B' \in R^\flat : f \in \tau(B')$ and $\lambda(B') = \lambda\}$; members of $\overline{R}(f,\lambda)$ may be called λ-buddies of f. For any $f \in R^\natural$ and $B \in R^\flat$, we let $\overline{R}(f,B) = \overline{R}(f,\mathrm{noc}(f,B))$; members of $\overline{R}(f,B)$ may be called B-buddies of f. For any $B \in R^\flat$, we let $\overline{R}(B) = \{B' \in R^\flat : \tau(B') = \tau(B)$ and $\lambda(B') = \lambda(B)\}$; members of $\overline{R}(B)$ may be called buddies of B.

Given any bud B, by (GNP1) we see that there is a unique nonnegative integer $p(B)$, a unique positive integer $D(B)$, a unique rational number $S(B)$, a unique sequence of integers $m(B) = m_i(B)_{0 \le i \le p(B)+1}$ with the exception that $m_{p(B)+1} = \infty$, a unique sequence of rational numbers $c(B) = c_i(B)_{1 \le i \le p(B)}$, a unique member $t(B)$ of $R^\natural$, a unique nonzero element $A(B)$ of k, and a unique nonempty set $\epsilon(B)$ of triples (z, V, W) with $z = z(X) \in k((X))$ and $0 < V \in \mathbb{Z}$ and $W \in \mathbb{Z}$, having the bud properties which say that

$$\begin{cases} \text{for every } f \in \tau(B), \text{ upon letting } \deg_Y f = n, \text{ we have:} \\ p(f,\lambda(B)) = p(B), D(f,\lambda(B)) = D(B), S(f,\lambda(B)) = S(B), \\ m(f,\lambda(B)) = m(B), c(f,\lambda(B)) = c(B), t(f,\lambda(B)) = t(B), \\ A(f,\lambda(B),\tilde{z}) = A(B) \text{ for all } \tilde{z} = \tilde{z}(X) \in k((X)) \text{ with } f(X^n,\tilde{z}(X)) = 0, \\ \text{and } (z,n,V,W) \mapsto (\hat{z},V,W) \text{ gives a surjection of } \epsilon(f,\lambda(B)) \text{ onto } \epsilon(B) \\ \text{where } \hat{z}(X) = z^\dagger(X,n,V,W). \end{cases} \tag{BP1}$$

We call $p(B)$, $D(B)$, $S(B)$, $m(B)$, $c(B)$, $t(B)$, $A(B)$, and $\epsilon(B)$, the position, the degree, the strength, the characteristic sequence, the normalized characteristic sequence, the normalized truncation, the reduced constant, and the edge of B respectively, and we note that then for $t(B)$ we have

$$\begin{cases} t(B) \in \tau(B), \deg_Y t(B) = D(B), m(t(B)) = m(B), \\ c(t(B)) = c(B), h(m(t(B))) = h(c(t(B))) = p(B), \text{ and} \\ \epsilon(B) = \{(z,V,W) : 0 < V \in D(B)\mathbb{Z} \text{ and } W = \lambda(B)V \in \mathbb{Z} \text{ and} \\ \qquad z = z(X) \in k((X)) \text{ with } t(B)(X^V, z(X)) = 0\}. \end{cases} \tag{BP2}$$

Given any bud B, by (BP1) and (BP2) we see that

$$\begin{cases} \text{for any } B' \in R^\flat \text{ we have:} \\ B' \in \overline{R}(B) \Leftrightarrow \overline{R}(B') = \overline{R}(B) \\ \qquad \Leftrightarrow \tau(B') \cap \tau(B) \neq \emptyset \text{ and } \lambda(B') = \lambda(B) \\ \qquad \Rightarrow p(B') = p(B), D(B') = D(B), S(B') = S(B), m(B') = m(B), \\ \qquad\quad c(B') = c(B), t(B') = t(B), A(B') = A(B), \text{ and } \epsilon(B') = \epsilon(B) \end{cases} \tag{BP3}$$

[5] Basically, the stem $\sigma(B)$ of a bud $B = (\sigma(B), \lambda(B))$ is a nonempty set of irreducible meromorphic curves $f(X,Y) = 0$ whose fractional meromorphic roots mutually coincide up to $X^{\lambda(B)}$, and its flower $\tau(B)$ is the set of all irreducible meromorphic curves whose fractional meromorphic roots coincide with the fractional meromorphic roots of members of $\sigma(B)$ up to $X^{\lambda(B)}$.

[6] Equivalently, $\mathrm{noc}(f,B)$ can be defined by saying that, for any $f \in R^\natural$ and $B \in R^\flat$, we have $\mathrm{noc}(f,B) = \min\{\mathrm{noc}(f,f') : f' \in \tau(B)\}$.

and by (NP1), (GNP4), (GNP5) and (GNP6) we see that

$$\begin{cases} \text{for any } f \in \tau(B), \text{ with } \deg_Y f = n, \text{ we have } n/D(B) \in \mathbb{Z}, \\ \text{and for any } (z, V, W) \in \epsilon(B) \text{ we have} \\ \operatorname{ord}_X f(X^V, z(X) + X^W Y) = S(B)nV \in \mathbb{Z} \\ \text{and } \deg_Y \operatorname{inco}_X f(X^V, z(X) + X^W Y) = n/D(B) \end{cases} \qquad \text{(BP4)}$$

and

$$\begin{cases} \text{for any } f' \in R^\natural \setminus \tau(B) \text{ and } (z, V, W) \in \epsilon(B) \text{ we have} \\ 0 \neq \operatorname{inco}_X f'(X^V, z(X) + X^W Y) \in k, \\ \text{and for any } B' \in \overline{R}(f', B), \text{ upon letting } \deg_Y f' = n', \text{ we have} \\ \operatorname{ord}_X f'(X^V, z(X) + X^W Y) = S(B')n'V \in \mathbb{Z}. \end{cases} \qquad \text{(BP5)}$$

Next we let $R^{\flat *}$ be the set of all strict buds in R, where by a strict bud we mean a bud B such that $\operatorname{noc}(f, f') > \lambda(B)$ for all f and f' in $\sigma(B)$; we also let $\tau^*(B) = \{f \in R^\natural$: $\operatorname{noc}(f, f') > \lambda(B)$ for all $f' \in \sigma(B)\}$, and we call $\tau^*(B)$ the strict flower of B.[7] For any $f \in R^\natural$ and $\lambda \in \mathbb{Q}$ we let $\overline{R}^*(f, \lambda) = \{B' \in R^{\flat *} : f \in \tau^*(B')$ and $\lambda(B') = \lambda\}$; members of $\overline{R}^*(f, \lambda)$ may be called strict λ-buddies of f. For any $f \in R^\natural$ and $B \in R^\flat$, we let $\overline{R}^*(f, B) = \overline{R}^*(f, \operatorname{noc}(f, B))$; members of $\overline{R}^*(f, B)$ may be called strict B-buddies of f. For any $B \in R^\flat$, we let $\overline{R}^*(B) = R^{\flat *} \cap \overline{R}(B)$; members of $\overline{R}^*(B)$ may be called strict buddies of B. Finally, for any $B \in R^{\flat *}$, we let $\overline{R}^{**}(B) = \{B' \in \overline{R}^*(B) : \tau^*(B') = \tau^*(B)\}$; members of $\overline{R}^{**}(B)$ may be called doubly strict buddies of B.

Given any strict bud B, by (GNP2) we see that there is a unique nonnegative integer $p^*(B)$, a unique positive integer $D^*(B)$, a unique sequence of integers $m^*(B) = m_i^*(B)_{0 \le i \le p^*(B)+1}$ with the exception that $m^*_{p^*(B)+1} = \infty$, and a unique sequence of rational numbers $c^*(B) = c_i^*(B)_{1 \le i \le p^*(B)}$, a unique member $t^*(B)$ of $R^\natural$, a unique monic polynomial $E(B, Y)$ in $k[Y]$, a unique element $E_0(B)$ in k, and a unique nonempty set $\epsilon^*(B)$ of triples (z, V, W) with $z = z(X) \in k((X))$ and $0 < V \in \mathbb{Z}$ and $W \in \mathbb{Z}$, having the strict bud properties which say that

$$\begin{cases} \text{for every } f \in \tau^*(B), \text{ upon letting } \deg_Y f = n, \text{ we have:} \\ p^*(f, \lambda(B)) = p^*(B) \ge p(B), D^*(f, \lambda(B)) = D^*(B) \in D(B)\mathbb{Z}, \\ m^*(f, \lambda(B)) = m^*(B), c^*(f, \lambda(B)) = c^*(B), t^*(f, \lambda(B)) = t^*(B), \\ E(f, \lambda(B), \tilde{z}, Y) = E(B, Y) = Y^{D^*(B)/D(B)} - E_0(B) \\ \text{for all } \tilde{z} = \tilde{z}(X) \in k((X)) \text{ with } f(X^n, \tilde{z}(X)) = 0, \\ \text{and } (z, n, V, W) \mapsto (\hat{z}, V, W) \text{ gives a surjection of } \epsilon(f, \lambda(B)) \text{ onto } \epsilon^*(B) \\ \text{where } \hat{z}(X) = z^{\dagger *}(X, n, V, W). \end{cases} \qquad \text{(SBP1)}$$

We call $p^*(B)$, $D^*(B)$, $m^*(B)$, $c^*(B)$, $t^*(B)$, $E(B, Y)$, $E_0(B)$, and $\epsilon^*(B)$, the strict position, the strict degree, the strict characteristic sequence, the strict normalized characteristic sequence, the strict normalized truncation, the reduced polynomial, the polynomial

[7] Again, basically, the stem $\sigma(B)$ of a strict bud $B = (\sigma(B), \lambda(B))$ is a nonempty set of irreducible meromorphic curves $f(X, Y) = 0$ whose fractional meromorphic roots mutually coincide through $X^{\lambda(B)}$, and its strict flower $\tau^*(B)$ is the set of all irreducible meromorphic curves whose fractional meromorphic roots coincide with the fractional meromorphic roots of members of $\sigma(B)$ through $X^{\lambda(B)}$.

constant, and the strict edge of B respectively, and we note that then for $t^*(B)$ we have

$$\begin{cases} t^*(B)\in\tau^*(B), \deg_Y t^*(B)=D^*(B), m(t^*(B))=m^*(B), \\ c(t^*(B))=c^*(B), h(m(t^*(B)))=h(c(t^*(B)))=p^*(B), \text{ and} \\ \epsilon^*(B)=\{(z,V,W):0<V\in D^*(B)\mathbb{Z} \text{ and } W=\lambda(B)V\in\mathbb{Z} \text{ and} \\ \quad z=z(X)\in k((X)) \text{ with } t^*(B)(X^V,z(X))=0\}. \end{cases} \tag{SBP2}$$

Given any strict bud B, by (SBP1) and (SBP2) we see that

$$\begin{cases} \text{for any } B'\in R^{\flat *} \text{ we have:} \\ B'\in\overline{R}^{**}(B)\Leftrightarrow\overline{R}^{**}(B')=\overline{R}^{**}(B) \\ \qquad\Leftrightarrow\tau^*(B')\cap\tau^*(B)\neq\emptyset \text{ and } \lambda(B')=\lambda(B) \\ \qquad\Rightarrow p^*(B')=p^*(B), D^*(B')=D^*(B), m^*(B')=m^*(B), \\ \qquad\quad c^*(B')=c^*(B), t^*(B')=t^*(B), E(B',Y)=E(B,Y), \\ \qquad\quad \text{and } \epsilon^*(B')=\epsilon^*(B) \end{cases} \tag{SBP3}$$

and by (NP1) and (GNP5) we see that

$$\begin{cases} \text{for any } f\in\tau^*(B), \text{ with } \deg_Y f=n, \text{ we have } n/D^*(B)\in\mathbb{Z}, \\ \text{and for any } (z,V,W)\in\epsilon(B) \text{ we have} \\ \mathrm{inco}_X f(X^V,z(X)+X^W Y)=A(B)^{n/D(B)}E(B,Y)^{n/D^*(B)}. \end{cases} \tag{SBP4}$$

Given any bud B, by (NP2), (NP4) and (GNP3) we get the mixed bud properties which say that

$$\begin{cases} \text{for any } B'\in\overline{R}^{*}(B) \text{ and } B''\in\overline{R}^{*}(B) \text{ we have :} \\ E(B',Y)\neq E(B'',Y)\Leftrightarrow\tau^*(B')\neq\tau^*(B'') \\ \quad\Leftrightarrow\tau^*(B')\cap\tau^*(B'')=\emptyset \\ \quad\Leftrightarrow E(B',Y) \text{ and } E(B'',Y) \text{ have no common root in } k \end{cases} \tag{MBP1}$$

and

$$\begin{cases} \text{for any } B'\in\overline{R}^{*}(B) \text{ we have:} \\ E_0(B')=0\Leftrightarrow B'\in\overline{R}^{*}(t(B),B) \\ \qquad\Rightarrow p^*(B')=p(B'), D^*(B')=D(B), m^*(B')=m(B'), \\ \qquad\quad c^*(B')=c(B'), t^*(B')=t(B'), \text{ and } \epsilon^*(B')=\epsilon(B') \end{cases} \tag{MBP2}$$

and

$$\begin{cases} \text{for any } B'\in\overline{R}^{*}(B)\setminus\overline{R}^{*}(t(B),B) \text{ and } B''\in\overline{R}^{*}(B)\setminus\overline{R}^{*}(t(B),B) \\ \text{we have : } p^*(B')=p^*(B''), D^*(B')=D^*(B''), m^*(B')=m^*(B''), \\ \text{and } c^*(B')=c^*(B''). \end{cases} \tag{MBP3}$$

5. Contact sets

Given any $F=F(X,Y)\in R$ of Y-degree N, we can write

$$F=\prod_{0\le j\le\chi(F)} F_j \quad \text{where} \quad F_0=F_0(X)\in K((X))$$

and

$$F_j = F_j(X,Y) \in R^\natural \quad \text{with} \quad \deg_Y F_j = N_j \text{ for } 1 \le j \le \chi(F)$$

and $\chi(F)$ is a nonnegative integer such that: $\chi(F) = 0 \Leftrightarrow F \in k((X))$.[8] We define the contact set $C(F)$ of F by putting

$$C(F) = \{c_i(F_j) : 1 \le j \le \chi(F) \text{ and } 1 \le i \le h(c(F_j)) \text{ and } c_i(F_j) \notin \mathbb{Z}\}$$
$$\cup \{\operatorname{noc}(F_j, F_{j'}) : 1 \le j < j' \le \chi(F) \text{ with } F_j \ne F_{j'}\}.$$

Upon letting

$$N^\natural = \prod_{1 \le j \le \chi(F)} N_j$$

(with the usual convention that the product of an empty family is 1), by Newton's theorem we have

$$F(X^{N^\natural}, Y) = F_0(X^{N^\natural}) \prod_{1 \le j \le N} [Y - z_j^\natural(X)] \quad \text{with} \quad z_j^\natural(X) \in k((X))$$

and by the material on p. 300 of [Ab], as an alternative characterization of $C(F)$, we get

$$C(F) = \{(1/N^\natural) \operatorname{ord}_X[z_j^\natural(X) - z_{j'}^\natural(X)] : 1 \le j < j' \le N \text{ with } z_j^\natural(X) \ne z_{j'}^\natural(X)\}.$$

Note that

$$C(F) = \emptyset \Leftrightarrow N_j = 1 \text{ for } 1 \le j \le \chi(F) \text{ and } F_j = F_{j'} \text{ for } 1 \le j < j' \le \chi(F).$$

Given any $G = G(X,Y) \in R$ of Y-degree M, we can write

$$G = \prod_{0 \le j \le \chi(G)} G_j \quad \text{where} \quad G_0 = G_0(X) \in K((X))$$

and

$$G_j = G_j(X,Y) \in R^\natural \quad \text{with} \quad \deg_Y G_j = M_j \text{ for } 1 \le j \le \chi(G)$$

and $\chi(G)$ is a nonnegative integer such that: $\chi(G) = 0 \Leftrightarrow G \in k((X))$. Note that now

$$C(FG) = C(F) \cup C(G) \cup \{\operatorname{noc}(F_j, G_{j'}) : 1 \le j \le \chi(F) \text{ and } 1 \le j' \le \chi(G) \text{ with } F_j \ne G_{j'}\}.$$

Let

$$J(F,G) = J_{(X,Y)}(F,G)$$

be the jacobian of $F = F(X,Y)$ and $G = G(X,Y)$ with respect to X and Y, i.e., let

$$J(F,G) = F_X G_Y - G_X F_Y$$

where subscripts denote partial derivatives. Our aim is to produce a factorization of $J(F,G)$ in terms of the contact set $C(FG)$.

[8] In other words, if $F \in k((X))$ then $\chi(F) = 0$, whereas if $F \notin k((X))$ then $\chi(F)$ equals the number of irreducible factors of F in R.

In case of $F \neq 0$ we can write

$$F = X^{N^\sharp}P + (\text{terms of } X\text{-degree} > N^\sharp)$$

where

$$N^\sharp = \mathrm{ord}_X(F) \quad \text{and} \quad 0 \neq P = P(Y) = \mathrm{inco}_X(F) \in k[Y] \quad \text{with} \quad \deg_Y(P) = \nu.$$

Likewise, in case of $G \neq 0$ we can write

$$G = X^{M^\sharp}Q + (\text{terms of } X\text{-degree} > M^\sharp)$$

where

$$M^\sharp = \mathrm{ord}_X(G) \quad \text{and} \quad 0 \neq Q = Q(Y) = \mathrm{inco}_X(G) \in k[Y] \quad \text{with} \quad \deg_Y(Q) = \mu.$$

Now in case of $F \neq 0 \neq G$ we get

$$\begin{aligned} J(F,G) = {} & X^{N^\sharp + M^\sharp - 1}(N^\sharp P Q_Y - M^\sharp P_Y Q) \\ & + (\text{terms of } X\text{-degree} > N^\sharp + M^\sharp - 1) \end{aligned} \tag{JE1}$$

and hence

$$\mathrm{ord}_X J(F,G) \geq N^\sharp + M^\sharp - 1 \tag{JE2}$$

and

$$\begin{aligned} \mathrm{ord}_X J(F,G) = N^\sharp + M^\sharp - 1 \Leftrightarrow {} & N^\sharp P Q_Y - M^\sharp P_Y Q \neq 0 \\ & \Rightarrow \mathrm{inco}_X J(F,G) = N^\sharp P Q_Y - M^\sharp P_Y Q. \end{aligned} \tag{JE3}$$

These Jacobian estimates are basic in getting a factorization of $J(F,G)$ out of $C(FG)$ or, more precisely, out of the "tree" $T(FG)$ which, in § 6, we shall build from $C(FG)$. Moreover, as we shall explain in § 7, most of this set-up works in getting a factorization of any $H \in R$ out of any tree T. In § 8 we shall apply it to the situation when $H = J(F,G) = F_Y$ with $G = -X$. In § 9 we shall consider the general case of $H = J(F,G)$.

6. Trees

By allowing the level $\lambda(B)$ of a bud B to be $-\infty$ we get the set $R^\flat_\infty$ of all improper buds B; note that any nonempty subset of $R^\natural$ can be the stem $\sigma(B)$ of an improper bud B; moreover, for any improper bud B we have $\lambda(B) = -\infty$ and $\tau(B) = R^\natural$. We put $\hat{R}^\flat = R^\flat \cup R^\flat_\infty$, and we call a member of $\hat{R}^\flat$ a generalized bud. For any $B \in \hat{R}^\flat$ we let $\tau^*(B) = \{f \in \tau(B) : \mathrm{noc}(f,f') > \lambda(B) \text{ for some } f' \in \sigma(B)\}$, and we call $\tau^*(B)$ the strict flower of B; note that for any $B \in R^{\flat *}$ this definition coincides with the definition made earlier; also note that for any $B \in R^\flat_\infty$ we have $\tau^*(B) = R^\natural$. For any $B \in \hat{R}^\flat$ we let $\tau'(B) = \tau(B) \setminus \tau^*(B)$, and we call $\tau'(B)$ the primitive flower of B. Previously we have defined the normalized contact $\mathrm{noc}(f,B)$ for all $f \in R^\natural$ and $B \in R^\flat$; now we extend this by putting $\mathrm{noc}(f,B) = -\infty$ for all $f \in R^\natural$ and $B \in R^\flat_\infty$. For any $f \in R^\natural$ and $B \in \hat{R}^\flat$ we define $R^*(f,B)$ to be the unique member of $\hat{R}^\flat$ whose stem is $\{f\}$ and whose level is $\mathrm{noc}(f,B)$, and we call $R^*(f,B)$ the strict B-friend of f; note that then $R^*(f,B)$ belongs to $R^{\flat *}$ or $R^\flat_\infty$ according as $B \in R^\flat$ or $B \in R^\flat_\infty$. For any $B \in \hat{R}^\flat$ we define $R^*(B)$ to be the set of all $B' \in R^{\flat *} \cup R^\flat_\infty$ such that $\lambda(B') = \lambda(B)$ and $\sigma(B') = \tau^*(B') \cap \sigma(B)$, and we call members $R^*(B)$ strict friends of B; note that $\sigma(B) = \coprod_{B' \in R^*(B)} \sigma(B')$ gives a partition of $\sigma(B)$ into pairwise disjoint nonempty subsets.

Now the set $\hat{R}^\flat$ is prepartially ordered by defining $B' \geq B$ to mean $\lambda(B') \geq \lambda(B)$ and $\tau(B') \subset \tau(B)$.[9] For any B' and B in $\hat{R}^\flat$ we write $B' \gg B$ or $B \ll B'$ to mean $B' > B$ and $\lambda(B') > \lambda(B)$, i.e., to mean $\lambda(B') > \lambda(B)$ and $\tau(B') \subset \tau(B)$. For any $B' \gg B$ in $\hat{R}^\flat$ we define $R^*(B', B)$ to be the unique member of $\hat{R}^\flat$ whose stem is $\sigma(B')$ and whose level is $\lambda(B)$, and we call $R^*(B', B)$ the strict B-friend of B'; note that then $R^*(B', B)$ belongs to $R^{\flat *}$ or $R^\flat_\infty$ according as $\lambda(B) \neq -\infty$ or $\lambda(B) = -\infty$. For any $B' \gg B$ in $\hat{R}^\flat$ we also put $\tau^*(B', B) = \tau^*(R^*(B', B)) \setminus \tau(B')$, and we call $\tau^*(B', B)$ the strict B-flower of B'.

Let $\hat{R}^\sharp$ be the set of all trees in R, where by a tree we mean a subset T of $\hat{R}^\flat$ such that T contains an improper bud, and for any $B' \neq B$ in T with $\lambda(B') = \lambda(B)$ we have $\tau(B') \cap \tau(B) = \emptyset$; note that then the prepartial order $\geq$ induces a partial order on T, and hence in particular T has a unique improper bud; we call this improper bud the root of T and denote it by $R_\infty(T)$; also note that for any B' and B in T we have: $B' > B \Leftrightarrow B' \gg B$. For any tree T, we put $\Lambda(T) = \{\lambda(B) : B \in T\}$ and we call $\Lambda(T)$ the level set of T; we define the height $h(T)$ of T by putting $h(T) = \infty$ if $\Lambda(T)$ is infinite, and $h(T)$ = the cardinality of $\Lambda(T)$ minus 1 if $\Lambda(T)$ is finite; moreover, in case $h(T)$ is a nonnegative integer, i.e., in case $\Lambda(T)$ is a finite set, we let $l(T) = l_i(T)_{0 \leq i \leq h(T)}$ be the strictly increasing sequence $l_0(T) < \cdots < l_{h(T)}(T)$ such that $\{l_0(T), \ldots, l_{h(T)}(T)\} = \Lambda(T)$, and we call $l(T)$ the level sequence of T. Note that a tree T is finite iff its level set $\Lambda(T)$ is finite and T has at most a finite number of generalized buds of any given level. We put

$$R^\sharp = \text{the set of all finite trees in } R.$$

For any generalized bud B in any tree T, we put $\pi(T, B) = \{B' \in T : B' > B\}$ and we call $\pi(T, B)$ the B-preroof of T, and we put $\rho(T, B) = \{B' \in \pi(T, B)$: there is no $B'' \in \pi(T, B)$ with $B' > B''\}$ and we call $\rho(T, B)$ the B-roof of T. For any generalized bud B in any tree T, we also put $\tau(T, B) = \tau(B) \setminus \cup\{\tau(B') : B' \in \rho(T, B)\}$ and we call $\tau(T, B)$ the B-flower of T, and we put $\tau^*(T, B) = \tau^*(B) \setminus \cup\{\tau(B') : B' \in \rho(T, B)\}$ and we call $\tau^*(T, B)$ the strict T-flower of B; note that then $\tau(T, B) = \tau(B) \setminus \cup\{\tau(B') : B' \in \pi(T, B)\}$ and $\tau^*(T, B) = \tau^*(B) \setminus \cup\{\tau(B') : B' \in \pi(T, B)\} = \tau(T, B) \setminus \tau'(B)$.[10]

A tree T is said to be strict if for every $\lambda \in \Lambda(T)$ we have $\sigma(R_\infty(T)) = \cup_{B \in T^{(\lambda)}} \sigma(B)$ where $T^{(\lambda)}$ is the set of all $B \in T$ with $\lambda(B) = \lambda$. Given any $\lambda \in \mathbb{Q} \cup \{-\infty\}$, by (ITP) we see that $f \sim_\lambda f'$ gives an equivalence relation on $R^\natural$ where $f \sim_\lambda f'$ means $\mathrm{noc}(f, f') \geq \lambda$. It follows that, given any $\hat{\sigma} \subset R^\natural$ and $\hat{\Lambda} \subset \mathbb{Q}$, there is a unique strict tree $\hat{T}(\hat{\sigma}, \hat{\Lambda})$ with $\Lambda(\hat{T}(\hat{\sigma}, \hat{\Lambda})) = \{-\infty\} \cup \hat{\Lambda}$ such that $\sigma(R_\infty(\hat{T}(\hat{\sigma}, \hat{\Lambda}))) = \hat{\sigma}$ or $\{Y\}$ according as $\hat{\sigma}$ is nonempty or empty; we call $\hat{T}(\hat{\sigma}, \hat{\Lambda})$ the $\hat{\Lambda}$-tree of $\hat{\sigma}$; note that, if $\hat{\sigma}$ is nonempty then, for every $\lambda \in \hat{\Lambda}$, the stems of the buds of $\hat{T}(\hat{\sigma}, \hat{\Lambda})$ of level λ are the equivalence classes of $\hat{\sigma}$ under $\sim_\lambda$; likewise, if $\hat{\sigma}$ is empty then, for every $\lambda \in \hat{\Lambda}$, the stem of the unique bud of $\hat{T}(\hat{\sigma}, \hat{\Lambda})$ of level λ is $\{Y\}$. We put

$$R^{\sharp *} = \text{the set of all finite strict trees in } R$$

and we note that for any $B \in T \in R^{\sharp *}$ with $\lambda(B) = l_i$ for some $i < h(T)$ we have $\rho(T, B) = \{B' \in T : \lambda(B') = l_{i+1}\}$ and $\sigma(B) = \coprod_{B' \in \rho(T,B)} \sigma(B')$ which is a partition of $\sigma(B)$ into pairwise disjoint nonempty subsets. For any $F \in R$, with its monic irreducible factors

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[9] A set is prepartially ordered by $\geq$ means: $a \geq b$ and $b \geq c$ implies $a \geq c$. It is partially ordered if also: $a \geq b$ and $b \geq a$ implies $a = b$.

[10] For printing convenience we may write $\cup\{\tau(B') : B' \in \rho(T, B)\}$ instead of $\cup_{B' \in \rho(T,B)} \tau(B')$, with similar notation for $\cap$, $\sum$ and $\prod$.

$F_1, \ldots, F_{\chi(F)}$ as in the previous section, we put

$$T(F) = \hat{T}(\{F_1, \ldots, F_{\chi(F)}\}, C(F))$$

and we call $T(F)$ the tree of F, and we note that then $T(F) \in R^{\sharp *}$.

A tree T' is a subtree of a tree T if for every $B' \in T'$ there exists some (and hence a unique) $B \in T$ such that $\sigma(B') \subset \sigma(B)$ and $\lambda(B') = \lambda(B)$. Every tree is clearly a subtree of the universal tree $\hat{T}(R^\natural, \mathbb{Q})$, which is a strict tree of infinite height.[11]

Remark (TR1). Basically we are interested in comparing the tree $T(FG)$ of the product of two members F and G of R with the tree $T(J(F, G))$ of their jacobian. In case of $G = -X$, this reduces to comparing $T(F)$ with $T(F_Y)$.

Remark (TR2). For the benefit of the readers (and ourselves) we shall now describe three examples of the tree $T(F)$ of various types of $F \in \hat{R}^\natural$.

Example (TR3). First, here is an example of $F \in \hat{R}^\natural$ which is irreducible and has only one characteristic exponent, i.e., with $\chi(F) = 1$ and $h(m(F)) = 1$. Namely, let

$$1 < n \in \mathbb{Z} \text{ and } 0 \neq e \in \mathbb{Z} \text{ with } \mathrm{GCD}(n, e) = 1$$

and

$$F = f = f(X, Y) = Y^n + \sum_{1 \le i \le n} w_i(X) Y^{n-i}$$

where $w_i(X) \in k((X))$ is such that

$$\mathrm{ord}_X w_i(X) > ie/n \text{ for } 1 \le i \le n-1 \text{ and } \mathrm{ord}_X w_n(X) = e$$

and let κ be the coefficient of X^e in $w_n(X)$, i.e., let $0 \neq \kappa \in k$ be such that $\mathrm{ord}_X(w_n(X) - \kappa X^e) > e$. Then f is irreducible in $\hat{R}^\natural$, and we have the Newtonian factorization

$$f(X^n, Y) = \prod_{1 \le j \le n} [Y - z_j(X)]$$

where $z_j(X) \in k((X))$ is such that

$$z_j(X) = \omega^j \kappa^* X^e + (\text{terms of degree} > e \text{ in } X)$$

where ω is a primitive n-th root of 1 in k, and κ^* is an n-th root of $-\kappa$ in k.

To see this, first note that $f(X^n, X^e Y) = X^{ne} g(X, Y)$ where

$$g(X, Y) = Y^n + \sum_{1 \le i \le n} v_i(X) Y^{n-i}$$

and $v_i(X) = X^{-ie} w_i(X^n) \in k[[X]]$ is such that

$$v_i(0) = 0 \text{ for } 1 \le i \le n-1 \text{ and } v_n(0) = \kappa.$$

[11] This universal tree is like the Ashwattha Tree of the Bhagwad-Gita. The stem of its root contains the embryos of all the past, present and future creatures in nascent form. Its trunks travel upwards first comprising of large tribes and then of smaller and smaller clans. Its "ultimate" shoots reaching heaven are the individual souls eager to embrace their maker.

Now $g(0,Y)=Y^n-\kappa^{*n}$, and hence we get the desired factorization by applying Hensel's lemma. Since $\mathrm{GCD}(n,e)=1$, we see that f is irreducible in $\hat{R}^\natural$.

The above factorization of f yields $h(m(f))=1$ with

$$m_0(f)=q_0(m(f))=s_0(q(m(f)))=r_0(q(m(f)))=n \text{ and } d_1(m(f))=n$$

and

$$m_1(f)=q_1(m(f))=s_1(q(m(f)))/n=r_1(q(m(f)))=e \text{ and } d_2(m(f))=1.$$

Therefore

$$C(F)=C(f)=\{c_1(f)\} \text{ with } c_1(f)=e/n.$$

Hence $h(T(f))=1$ with

$$l_0(T(f))=-\infty \text{ and } l_1(T(f))=c_1(f)=e/n$$

and upon letting

$$B_i\in\hat{R}^\flat \text{ with } \sigma(B_i)=\{f\} \text{ and } \lambda(B_i)=l_i(T(f)) \text{ for } 0\le i\le 1$$

we have

$$T(F)=T(f)=\{B_0,B_1\}$$

with

$$D'(B_0)=0 \text{ and } D'(B_1)=n-1.$$

Note that for F to be analytic, i.e., for it to belong to the ring $k[[X]][Y]$, the condition $e>0$ is necessary and sufficient. However, for F to be pure meromorphic, i.e., for it to belong to the ring $k[X^{-1}][Y]$, i.e., for the existence of $\Phi(X,Y)\in k[X,Y]$ with $F(X,Y)=\Phi(X^{-1},Y)$, the condition $e<0$ is necessary but not sufficient. As a specific illustration of the analytic case we may take $(n,e)=(4,5)$ and $(w_1(X),\dots,w_{n-1}(X),w_n(X))=(0,\dots,0,X^5)$, giving us $F(X,Y)=Y^4+X^5$. Similarly, as a specific illustration of the pure meromorphic case we may take $(n,e)=(4,-3)$ and $(w_1(X),\dots,w_{n-1}(X),w_n(X))=(0,\dots,0,X^{-3})$, giving us $F(X,Y)=Y^4+X^{-3}$, i.e., $F(X,Y)=\Phi(X^{-1},Y)$ with $\Phi(X,Y)=Y^4+X^3$.

Example (TR4). Next, here is an example of $F\in\hat{R}^\natural$ which is irreducible and has two characteristic exponents, i.e., with $\chi(F)=1$ and $h(m(F))=2$. Namely, let

$$F=f=f(X,Y)=(Y^2-X^{2a+1})^2-X^{3a+b+2}Y \text{ with } a\in\mathbb{Z} \text{ and } 0\le b\in\mathbb{Z}.$$

Then f is irreducible in $\hat{R}^\natural$, and we have the Newtonian factorization

$$f(X^4,Y)=\prod_{1\le j\le 4}[Y-z_j(X)]$$

where $z_j(X)\in k((X))$ is such that

$$z_j(X)=(\iota^jX)^{4a+2}+\tfrac{1}{2}(\iota^jX)^{4a+2b+3}+(\text{terms of degree} > 4a+2b+3 \text{ in } X)$$

where ι is a primitive 4-th root of 1 in k (e.g., $\iota=$ the usual i).

To see this, first note that $f(X^4,X^{4a+2}Y)=X^{16a+8}g(X,Y)$ where

$$g(X,Y)=(Y^2-1)^2-X^{4b+2}Y.$$

Now for $\jmath = 1$ or -1, upon letting $g_\jmath(X,Y) = g(X, Y+\jmath)$ we have

$$g_\jmath(X,Y) = (2\jmath + Y)^2 Y^2 - \jmath(1+\jmath Y)X^{4b+2}$$

and hence (say by the binomial theorem) we get

$$g_\jmath(X,Y) = [(2\jmath + Y)Y - \jmath^*\theta(Y)X^{2b+1}][(2\jmath + Y)Y + \jmath^*\theta(Y)X^{2b+1}]$$

where

$$\theta(Y) = 1 + (\jmath Y/2) - \sum_{i=2}^{\infty} 1 \times 3 \times \cdots \times (2i-3) \times (-\jmath Y/2)^i / i!$$

and $\jmath^* = \iota^2$ or ι according as $\jmath = 1$ or -1, and therefore (say by the Weierstrass preparation theorem) we have

$$g(X,Y) = \prod_{1 \le j \le 4} [Y - y_j(X)]$$

where $y_j(X) \in k[[X]]$ is such that

$$y_j(X) = (-1)^j [1 + \tfrac{1}{2}(\iota^j X)^{2b+1} + (\text{terms of degree} > 2b+1 \text{ in } X)].$$

Since $f(X^4, X^{4a+2}Y) = X^{16a+8} g(X,Y)$, we get the above factorization of $f(X^4, Y)$. Since the GCD of 4 with the support of $z_1(X)$ is 1, we conclude that f is irreducible in $\hat{R}^\natural$, i.e., $\chi(F) = 1$.

The above factorization of f yields $h(m(f)) = 2$ with

$$m_0(f) = q_0(m(f)) = s_0(q(m(f))) = r_0(q(m(f))) = 4 \text{ and } d_1(m(f)) = 4$$

and

$$m_1(f) = q_1(m(f)) = s_1(q(m(f)))/4 = r_1(q(m(f))) = 4a+2 \text{ and } d_2(m(f)) = 2$$

and

$$m_2(f) = 4a + 2b + 3 \text{ and } q_2(m(f)) = 2b+1$$

and

$$s_2(q(m(f))) = 16a+4b+10 \text{ and } r_2(q(m(f))) = 8a+2b+5 \text{ and } d_3(m(f)) = 1.$$

Therefore

$$C(F) = C(f) = \{c_1(f), c_2(f)\}$$

with

$$c_1(f) = (2a+1)/2 \text{ and } c_2(f) = (4a+2b+3)/4.$$

Hence $h(T(f)) = 2$ with $l_0(T(f)) = -\infty$ and

$$l_1(T(f)) = c_1(f) = (2a+1)/2 \text{ and } l_2(T(f)) = c_2(f) = (4a+2b+3)/4$$

and upon letting

$$B_i \in \hat{R}^\flat \text{ with } \sigma(B_i) = \{f\} \text{ and } \lambda(B_i) = l_i(T(f)) \text{ for } 0 \le i \le 2$$

'e have

$$T(F) = T(f) = \{B_0, B_1, B_2\}.$$

with $D'(B_0)=0$ and

$$D'(B_1)=1 \text{ and } D'(B_2)=2.$$

As a specific illustration of the analytic case we may take $(a,b)=(1,0)$, giving us $F(X,Y)=(Y^2-X^3)^2-X^5Y$. Similarly, as a specific illustration of the pure meromorphic case we may take $(a,b)=(-1,1)$, giving us $F(X,Y)=(Y^2-X^{-1})^2-Y$, i.e., $F(X,Y)=\Phi(X^{-1},Y)$ with $\Phi(X,Y)=(Y^2-X)^2-Y$. Note that this Φ is a variable in the sense that $k[X,Y]=k[\Phi,\Psi]$ for some Ψ in $k[X,Y]$; in our situation we can take $\Psi(X,Y)=Y^2-X$.

Example (TR5). Finally, here is an example of $F\in\hat{R}^\natural$ which has two factors, i.e., with $\chi(F)=2$. Namely, let

$$0<n\in\mathbb{Z} \text{ and } a\in\mathbb{Z} \text{ and } 0\leq b\in\mathbb{Z}$$

and

$$F=F(X,Y)=Y^{n+2}+\sum_{2\leq i\leq n+2}u_i(X)Y^{n+2-i}$$

where $u_i(X)\in k((X))$ is such that

$$\text{ord}_X u_i(X)\geq i(a+1) \text{ for } 3\leq i\leq n+1$$

and

$$\text{ord}_X u_2(X)=2a+1 \text{ and } \text{ord}_X u_{n+2}(X)=(n+2)(a+1)+b$$

and let $0\neq\kappa'\in k$ and $0\neq\kappa\in k$ be the coefficients of X^{2a+1} and $X^{(n+2)(a+1)+b}$ in $u_2(X)$ and $u_{n+2}(X)$ respectively. Then

$$F(X,Y)=f(X,Y)f'(X,Y) \text{ with } f(X,Y)\neq f'(X,Y)$$

where

$$f(X,Y)=Y^n+\sum_{1\leq i\leq n}w_i(X)Y^{n-i}\in\hat{R}^\natural \text{ and } w_i(X)\in k((X))$$

with

$$\begin{cases}\text{ord}_X w_i(X)>ie/n \text{ for } 1\leq i\leq n-1 \text{ and } \text{ord}_X w_n(X)=e+b\\ \text{for the integer } e=na+n+1 \text{ for which } \text{GCD}(n,e)=1\end{cases}$$

and

$$f'(X,Y)=Y^2+\sum_{1\leq i\leq 2}w'_i(X)Y^{n-i}\in R^\natural \text{ and } w'_i(X)\in k((X))$$

with

$$\begin{cases}\text{ord}_X w'_1(X)>e'/2 \text{ and } \text{ord}_X w'_2(X)=e'\\ \text{for the integer } e'=2a+1 \text{ for which } \text{GCD}(2,e')=1\end{cases}$$

and $0\neq\kappa'\in k$ and $0\neq\kappa/\kappa'\in k$ are the coefficients of $X^{e'}$ and X^{e+b} in $w'_2(X)$ and $w_n(X)$ respectively. Moreover, if $b=0$ then we also have $f(X,Y)\in R^\natural$.

To see this, first note that $F(X,X^aY)=X^{na+2a}g(X,Y)$ where

$$g(X,Y)=Y^{n+2}+\sum_{2\leq i\leq n+2}v_i(X)Y^{n+2-i}$$

and $v_i(X) = X^{-ia}u_i(X) \in k[[X]]$ is such that

$$\mathrm{ord}_X\, v_i(X) \geq i \text{ for } 3 \leq i \leq n+1$$

and

$$\mathrm{ord}_X\, v_2(X) = 1 \text{ and } \mathrm{ord}_X\, v_{n+2}(X) = n+2+b$$

and $0 \neq \kappa' \in k$ and $0 \neq \kappa \in k$ are the coefficients of X and X^{n+2+b} in $v_2(X)$ and $v_{n+2}(X)$ respectively. Now the initial form $g(X,Y)$ is $\kappa' XY^n$ which factors into the coprime factors $\kappa' X$ and Y^n, and hence by the tangent lemma incarnation of Hensel's lemma (cf. pp. 140–141 of Abhyankar's 1990 AMS book "*Algebraic geometry for scientists and engineers*") we can find $\phi(X,Y)$ and $\phi'(X,Y)$ in $k[[X,Y]]$ such that $g(X,Y) = \phi(X,Y)\ \phi'(X,Y)$ and

$$\phi(X,Y) = Y^n + \phi_{n+1}(X,Y) + (\text{terms of degree} > n+1 \text{ in } X \text{ and } Y)$$

and

$$\phi'(X,Y) = \kappa' X + \phi'_2(X,Y) + (\text{terms of degree} > 2 \text{ in } X \text{ and } Y)$$

where $\phi_{n+1}(X,Y) \in k[X,Y]$ is homogeneous of degree $n+1$ and $\phi'_2(X,Y) \in k[X,Y]$ is homogeneous of degree 2 (with the understanding that the zero polynomial is homogeneous of any degree). Comparing terms of degree $n+2$ in the equation $g(X,Y) = \phi(X,Y)\ \phi'(X,Y)$ we get

$$\kappa' X\phi_{n+1}(X,Y) + Y^n\phi'_2(X,Y) = Y^{n+2} + \sum_{2 \leq i \leq n+2} \kappa_i X^i Y^{n+2-i}$$

where $\kappa_2 \in k$ is the coefficient of X^2 in $v_2(X) - \kappa' X$, and $\kappa_i \in k$ is the coefficient of X^i in $v_i(X)$ for $3 \leq i \leq n+2$. Successively putting $X = 0$ and $Y = 0$ in the above equation we see that $\phi'_2(0,Y) = Y^2$ and $\phi_{n+1}(X,0) = \kappa_{n+2}X^{n+1}$. Therefore, in view of the Weierstrass preparation theorem, we can find $\theta(X,Y)$ and $\theta'(X,Y)$ in $k[[X,Y]]$ with $\theta(0,0) \neq 0 \neq \theta'(0,0)$ such that upon letting $\tilde{f}(X,Y) = \theta(X,Y)\phi(X,Y)$ and $\tilde{f}'(X,Y) = \theta'(X,Y)\phi'(X,Y)$ we have $g(X,Y) = \tilde{f}(X,Y)\,\tilde{f}'(X,Y)$ and

$$\tilde{f}(X,Y) = Y^n + \sum_{1 \leq i \leq n} \tilde{w}_i(X)Y^{n-i} \text{ and } \tilde{w}_i(X) \in k[[X]]$$

with

$$\mathrm{ord}_X\, \tilde{w}_i(X) > i(n+1)/n \text{ for } 1 \leq i \leq n-1 \text{ and } \mathrm{ord}_X\, \tilde{w}_n(X) = n+1+b$$

and

$$\tilde{f}'(X,Y) = Y^2 + \sum_{1 \leq i \leq 2} \tilde{w}'_i(X)Y^{2-i} \text{ and } \tilde{w}'_i(X) \in k[[X]]$$

with

$$\mathrm{ord}_X\, \tilde{w}'_1(X) > 1/2 \text{ and } \mathrm{ord}_X\, \tilde{w}'_2(X) = 1$$

and $0 \neq \kappa' \in k$ and $0 \neq \kappa/\kappa' \in k$ are the coefficients of X and X^{n+1+b} in $\tilde{w}'_2(X)$ and $\tilde{w}_n(X)$ respectively. Now upon letting $f(X,Y) = X^{na}\tilde{f}(X, X^{-a}Y)$ and $f'(X,Y) = X^{2a}\tilde{f}'(X, X^{-a}Y)$, we get the desired factorization of $F(X,Y)$. Since $(n, e+b) \neq (2, e')$, we also get $f \neq f'$. By (TR3) it follows that f' is irreducible in $\hat{R}^{\natural}$, and if $b = 0$ then so is f.

Now assuming $b = 0$ and $n > 1$, in view of (TR3), the factorization of F tells us that $h(T(F)) = 2$ with $l_0(T(F)) = -\infty$ and

$$l_1(T(F)) = a + (1/2) \text{ and } l_2(T(F)) = a + 1 + (1/n)$$

and upon letting

$$\begin{cases} B_0 \in \hat{R}^{\flat} \text{ with } \sigma(B_0) = \{f, f'\} \text{ and } \lambda(B_0) = l_0(T(F)), \\ \text{and } B_1 \in \hat{R}^{\flat} \text{ with } \sigma(B_1) = \{f, f'\} \text{ and } \lambda(B_1) = l_1(T(F)), \\ \text{and } B_2 \in \hat{R}^{\flat} \text{ with } \sigma(B_2) = \{f\} \text{ and } \lambda(B_2) = l_2(T(F)), \\ \text{and } B_2' \in \hat{R}^{\flat} \text{ with } \sigma(B_2') = \{f'\} \text{ and } \lambda(B_2') = l_2(T(F)), \end{cases}$$

we have

$$T(F) = \{B_0, B_1, B_2, B_2'\}$$

with $D'(B_0) = 0$ and

$$D'(B_1) = 2 \text{ and } D'(B_2) = n-1 \text{ and } D'(B_2') = 0.$$

As a specific illustration of the pure meromorphic case, taking $a=-1$ and $(u_2(X), u_{n+2}(X)) = (\kappa' X^{-1}, \kappa)$ with $\kappa' \neq 0 \neq \kappa$ in k and $u_i(X) = \kappa_i \in k$ for $3 \leq i \leq n+1$, we get $F(X,Y) = \Phi(X^{-1}, Y)$ where

$$\Phi(X,Y) = Y^{n+2} + \kappa' XY^n + \kappa + \sum_{3 \leq i \leq n+1} \kappa_i Y^{n+2-i} \in k[X,Y]$$

with

$$0 \neq \kappa' \in k \text{ and } 0 \neq \kappa \in k \text{ and } \kappa_i \in k \text{ for } 3 \leq i \leq n+1.$$

7. Slices

Given any $H = H(X,Y) \in R$ of Y-degree O, we can write

$$H = \prod_{0 \leq j \leq \chi(H)} H_j \quad \text{where} \quad H_0 = H_0(X) \in k((X)) \tag{SP1}$$

and

$$H_j = H_j(X,Y) \in R^{\natural} \quad \text{with} \quad \deg_Y H_j = O_j \text{ for } 1 \leq j \leq \chi(H) \tag{SP2}$$

and $\chi(H)$ is a nonnegative integer such that: $\chi(H) = 0 \Leftrightarrow H \in k((X))$. Now

$$H = H_0 H_\infty \quad \text{with} \quad H_\infty = \prod_{1 \leq j \leq \chi(H)} H_j \tag{SP3}$$

where we note that $H_\infty \in \hat{R}^{\natural}$, and we call H_∞ the monic part of H.

We put

$$\Omega_B(H) = \prod_{1 \leq j \leq \chi(H) \text{ with } H_j \in \tau(B)} H_j \quad \text{for all } B \in \hat{R}^{\flat} \tag{SP4}$$

and we call $\Omega_B(H)$ the B-slice of H, and we note that then $\Omega_B(H) \in \hat{R}^{\natural}$, and we recall that

$$\begin{cases} \text{for all } B \in \hat{R}^{\flat} \text{ we have} \\ \tau(B) = \{f \in R^{\natural} : \operatorname{noc}(f, f') \geq \lambda(B) \text{ for all } f' \in \sigma(B)\}. \end{cases} \tag{SP5}$$

We also put

$$\Omega_{(T,B)}(H) = \prod_{1\le j\le\chi(H) \text{ with } H_j\in\tau(T,B)} H_j \quad \text{for all } B\in T\in R^\sharp \tag{SP6}$$

and we call $\Omega_{(T,B)}(H)$ the (T,B)-slice of H, and we note that then $\Omega_{(T,B)}(H)\in\hat{R}^\natural$, and we recall that

$$\begin{cases} \text{for all } B\in T\in\hat{R}^\sharp \text{ we have} \\ \tau(T,B)=\tau(B)\setminus\cup_{B'\in\rho(T,B)}\tau(B')=\tau(B)\setminus\cup_{B'\in\pi(T,B)}\tau(B') \\ \text{where } \pi(T,B)=\{B'\in T: B'>B\} \\ \text{and } \rho(T,B)=\{B'\in\pi(T,B)\text{: there is no } B''\in\pi(T,B) \text{ with } B'>B''\}. \end{cases} \tag{SP7}$$

Clearly we have the slice properties which say that

$$H_\infty=\Omega_B(H) \quad \text{for all } B\in R^\flat_\infty \tag{SP8}$$

and

$$\Omega_B(H)=\Omega_{(T,B)}(H)\prod_{B'\in\rho(T,B)}\Omega_{B'}(H) \quad \text{for all } B\in T\in R^\sharp \tag{SP9}$$

and hence[12]

$$H_\infty=\prod_{B\in T}\Omega_{(T,B)}(H) \quad \text{for all } T\in R^\sharp \tag{SP10}$$

where

$$\begin{cases} \text{for all } B\in T\in R^\sharp \text{ we have} \\ \deg_Y\Omega_{(T,B)}(H)=\deg_Y\Omega_B(H)-\sum_{B'\in\rho(T,B)}\deg_Y\Omega_{B'}(H). \end{cases} \tag{SP11}$$

By (BP4) we also see that

$$\begin{cases} \text{for all } B\in R^\flat \text{ and } (z,V,W)\in\epsilon(B) \text{ and } f\in\tau(B) \text{ we have} \\ \deg_Y f=D(B)\deg_Y \operatorname{inco}_X f(X^V,z(X)+X^W Y)\in D(B)\mathbb{Z} \end{cases} \tag{SP12}$$

[12] In the innocent looking formula (SP10), there is more than meets the eye. Indeed it is the central theme of the paper. It says that any finite tree T gives rise to a factorization of the monic part H_∞ of any meromorphic curve H into the pairwise coprime monic factors $\Omega_{(T,B)}(H)$ with B varying in T. Formula (SP20) gives a further factorization of $\Omega_{(T,B)}$ into the two coprime monic factors $\Omega'_B(H)$ and $\Omega^*_{(T,B)}(H)$. When the finite tree T is strict, formula (SP30) gives a still further factorization of $\Omega^*_{(T,B)}(H)$ into the pairwise coprime monic factors $\Omega^*_{(B',B)}(H)$. Item (SP50) gives a condition for the factorization of H_∞ to consist only of the factors $\Omega'_B(H)$, and item (SP80) gives a companion to this condition. The remaining items (SP1)–(SP9), (SP11)–(SP19), (SP21)–(SP29), (SP31)–(SP49), and (SP51)–(SP79), give us details about these factors, such as their Y-degrees, and hence in particular the information as to which of these factors are trivial (i.e., are reduced to 1) and which are not. Out of these items, the most noteworthy are labelled as (SP40), (SP60), and (SP70). Now roughly speaking, $\Omega_B(H)$ collects together those irreducible monic factors of H whose normalized contact with members of $\sigma(B)$ is at least $\lambda(B)$, and out of these only those are kept in $\Omega'_B(H)$ whose normalized contact with members of $\sigma(B)$ is exactly $\lambda(B')$, while the remaining are put in $\Omega^*_B(H)$. A similar description prevails for $\Omega_{(T,B)}(H)$, $\Omega^*_{(T,B)}(H)$, and $\Omega^*_{(B',B)}(H)$. As we shall see in the next two sections, more details about these factorizations can be given when T and H are somehow related.

and hence by (BP5) we see that

$$\begin{cases} \text{for all } B \in R^{\flat} \text{ and } (z,V,W) \in \epsilon(B), \text{ upon letting } H_B = \Omega_B(H), \\ \text{we have that } 0 \neq \operatorname{inco}_X H_B(X^V, z(X) + X^W Y) \in k[Y] \\ \text{with } \deg_Y \Omega_B(H) = D(B) \deg_Y \operatorname{inco}_X H_B(X^V, z(X) + X^W Y) \in D(B)\mathbb{Z} \\ \text{and } \operatorname{inco}_X H(X^V, z(X) + X^W Y) = \mu \operatorname{inco}_X H_B(X^V, z(X) + X^W Y) \\ \text{where } \mu \in k \text{ is such that: } \mu = 0 \Leftrightarrow H = 0. \end{cases} \tag{SP13}$$

The factorization (SP10) can be refined further. To see this we first put

$$\Omega'_B(H) = \prod_{1 \le j \le \chi(H) \text{ with } H_j \in \tau'(B)} H_j \quad \text{for all } B \in \hat{R}^{\flat} \tag{SP14}$$

and we call $\Omega'_B(H)$ the primitive B-slice of H, and we note that then $\Omega'_B(H) \in \hat{R}^{\natural}$, and we recall that

$$\begin{cases} \text{for all } B \in \hat{R}^{\flat} \text{ we have} \\ \tau'(B) = \{ f \in \tau(B) : \operatorname{noc}(f, f') = \lambda(B) \text{ for all } f' \in \sigma(B) \}. \end{cases} \tag{SP15}$$

Next we put

$$\Omega^*_B(H) = \prod_{1 \le j \le \chi(H) \text{ with } H_j \in \tau^*(B)} H_j \quad \text{for all } B \in \hat{R}^{\flat} \tag{SP16}$$

and we call $\Omega^*_B(H)$ the strict B-slice of H, and we note that then $\Omega^*_B(H) \in \hat{R}^{\natural}$, and we recall that

$$\begin{cases} \text{for all } B \in \hat{R}^{\flat} \text{ we have} \\ \tau^*(B) = \tau(B) \setminus \tau'(B) \\ \qquad = \{ f \in \tau(B) : \operatorname{noc}(f, f') > \lambda(B) \text{ for some } f' \in \sigma(B) \}. \end{cases} \tag{SP17}$$

We also put

$$\Omega^*_{(T,B)}(H) = \prod_{1 \le j \le \chi(H) \text{ with } H_j \in \tau^*(T,B)} H_j \quad \text{for all } B \in T \in R^{\sharp} \tag{SP18}$$

and we call $\Omega^*_{(T,B)}(H)$ the strict (T,B)-slice of H, and we note that then $\Omega^*_{(T,B)}(H) \in \hat{R}^{\natural}$, and we recall that

$$\begin{cases} \text{for all } B \in T \in \hat{R}^{\sharp} \text{ we have} \\ \tau^*(T,B) = \tau(T,B) \setminus \tau'(B) \\ \qquad = \tau^*(B) \setminus \cup_{B' \in \rho(T,B)} \tau(B') = \tau^*(B) \setminus \cup_{B' \in \pi(T,B)} \tau(B'). \end{cases} \tag{SP19}$$

Now clearly

$$\Omega_{(T,B)}(H) = \Omega'_B(H) \Omega^*_{(T,B)}(H) \quad \text{for all } B \in T \in R^{\sharp} \tag{SP20}$$

where

$$\begin{cases} \text{for all } B \in R^{\flat} \text{ we have} \\ \deg_Y \Omega'_B(H) = \deg_Y \Omega_B(H) - \deg_Y \Omega^*_B(H). \end{cases} \tag{SP21}$$

and

$$\begin{cases} \text{for all } B\in T\in R^{\sharp} \text{ we have} \\ \deg_Y \Omega^*_{(T,B)}(H) = \deg_Y \Omega^*_B(H) - \sum_{B'\in\rho(T,B)} \deg_Y \Omega_{B'}(H). \end{cases} \tag{SP22}$$

To describe the above Y-degrees more precisely, given any $z = z(X) \in k((X))$, $0 < V \in \mathbb{Z}$, and $W \in \mathbb{Z}$, we define the modified X-initial-coefficient of H relative to $[z, V, W]$, to be denoted by $\text{minco}_X[z, V, W](H)$, by putting

$$\text{minco}_X[z, V, W](H) = \text{inco}_X H(X^V, z(X) + X^W Y)$$

and, given any $\hat{\sigma} \subset R$, we define the strict X-initial-coefficient of $(H, \hat{\sigma})$ relative to $[z, V, W]$, to be denoted by $\text{sinco}_X[z, V, W](H, \hat{\sigma})$, and the primitive X-initial-coefficient of $(H, \hat{\sigma})$ relative to $[z, V, W]$, to be denoted by $\text{pinco}_X[z, V, W](H, \hat{\sigma})$, by saying that

$$\begin{cases} \text{if } H = 0 \text{ then} \\ \text{we have } \text{sinco}_X[z, V, W](H, \hat{\sigma}) = 1 = \text{pinco}_X[z, V, W](H, \hat{\sigma}) \end{cases}$$

whereas

$$\begin{cases} \text{if } H \neq 0 \text{ then, upon letting} \\ \text{minco}_X[z, V, W](H) = \mu_0 \prod_{1\le i\le \nu} (Y - \mu_i) \text{ with } 0 \neq \mu_0 \in k \text{ and } \mu_i \in k \\ \text{and } \Theta_f(Y) = \text{minco}_X[z, V, W](f) \text{ for all } f \in \hat{\sigma} \\ \text{and } \sigma^* = \{i \in \{1, \ldots, \nu\} : \Theta_f(\mu_i) = 0 \text{ for some } f \in \hat{\sigma}\} \\ \text{and } \sigma' = \{i \in \{1, \ldots, \nu\} : \Theta_f(\mu_i) \neq 0 \text{ for all } f \in \hat{\sigma}\}, \\ \text{we have } \text{sinco}_X[z, V, W](H, \hat{\sigma}) = \prod_{i\in\sigma^*} (Y - \mu_i) \\ \text{and } \text{pinco}_X[z, V, W](H, \hat{\sigma}) = \prod_{i\in\sigma'} (Y - \mu_i). \end{cases}$$

Recall that

$$\begin{cases} \text{for all } B\in R^{\flat} \text{ we have } 0 \neq A(B) \in k, \\ \text{and for all } B \in R^{\flat *} \text{ we have} \\ E(B, Y) = Y^{D^*(B)/D(B)} - E_0(B) \text{ with } E_0(B) \in k. \end{cases} \tag{SP23}$$

Now by (MBP2) we see that

$$\begin{cases} \text{for any } B \in R^{\flat *} \text{ we have:} \\ E_0(B) = 0 \Leftrightarrow B \in R^*(t(B), B) \\ \qquad \Rightarrow D^*(B) = D(B) \text{ and } t^*(B) = t(B) \\ \qquad\quad \text{and } \lambda(B) \neq c_i(B) \text{ for } 1 \le i \le p^*(B) \end{cases} \tag{SP24}$$

and by (MBP1) we see that

$$\begin{cases} \text{for any } B' \text{ and } B'' \text{ in } R^{\flat *} \text{ with } \tau(B') = \tau(B'') \text{ and } \lambda(B') = \lambda(B'') \\ \text{we have: } \tau^*(B') \cap \tau^*(B'') = \emptyset \Leftrightarrow E(B', Y) \text{ and } E(B'', Y) \text{ have no} \\ \text{common root in } k \end{cases} \tag{SP25}$$

and by (SBP4) we see that

$$\begin{cases}\text{for all } B\in R^{\flat *} \text{ with } (z,V,W)\in\epsilon(B) \text{ and } f\in\tau^*(B) \text{ with } \deg_Y f=n\\ \text{we have } \text{minco}_X[z,V,W](f)=A(B)E(B,Y)^{n/D^*(B)}.\end{cases} \tag{SP26}$$

By (SP25) and (SP26) we conclude that

$$\begin{cases}\text{for all } B\in R^{\flat} \text{ and } (z,V,W)\in\epsilon(B) \text{ we have}\\ \text{that } \text{sinco}_X[z,V,W](H,\sigma(B))\in k[Y] \text{ is monic in } Y\\ \text{with } \deg_Y\Omega_B^*(H)=D(B)\ \deg_Y \text{sinco}_X[z,V,W](H,\sigma(B))\in D(B)\mathbb{Z}\\ \text{and } \text{pinco}_X[z,V,W](H,\sigma(B))\in k[Y] \text{ is monic in } Y\\ \text{with } \deg_Y\Omega_B'(H)=D(B)\ \deg_Y \text{pinco}_X[z,V,W](H,\sigma(B))\in D(B)\mathbb{Z}.\end{cases} \tag{SP27}$$

The factorization (SP20) can be refined still further when the finite tree T is strict. To see this we put

$$\Omega^*_{(B',B)}(H)=\prod_{1\le j\le\chi(H) \text{ with } H_j\in\tau^*(B',B)} H_j \quad \text{for all } B'\gg B \text{ in } \hat{R}^{\flat} \tag{SP28}$$

and we call $\Omega^*_{(B',B)}(H)$ the strict (B',B)-slice of H, and we note that then $\Omega^*_{(B',B)}(H)\in\hat{R}^{\natural}$, and we recall that

$$\begin{cases}\text{for all } B'\gg B \text{ in } \hat{R}^{\flat} \text{ we have}\\ \tau^*(B',B)=\tau^*(R^*(B',B))\setminus\tau(B') \text{ where}\\ R^*(B',B)\in\hat{R}^{\flat} \text{ is given by } \sigma(R^*(B',B))=\sigma(B') \text{ and } \lambda(R^*(B',B))=\lambda(B).\end{cases} \tag{SP29}$$

Now clearly

$$\Omega^*_{(T,B)}(H)=\prod_{B'\in\rho(T,B)}\Omega^*_{(B',B)}(H) \quad \text{for all } B\in T\in R^{\sharp *} \tag{SP30}$$

where

$$\begin{cases}\text{for all } B\in T\in\hat{R}^{\sharp} \text{ and } B'\in\rho(T,B) \text{ we have } B'\gg B \text{ in } \hat{R}^{\flat},\\ \text{and in turn for all } B'\gg B \text{ in } \hat{R}^{\flat} \text{ we have}\\ \deg_Y\Omega^*_{(B',B)}(H)=\deg_Y\Omega^*_{R^*(B',B)}(H)-\deg_Y\Omega_{B'}(H)\end{cases} \tag{SP31}$$

and

$$\begin{cases}\text{for all } B\in T\in R^{\sharp *} \text{ with } \lambda(B)=l_{h(T)} \text{ we have } \rho(T,B)=\emptyset,\\ \text{whereas for all } B\in T\in R^{\sharp *} \text{ with } \lambda(B)=l_i \text{ for some } i<h(T) \text{ we have}\\ \rho(T,B)=\{B'\in T:\lambda(B')=\lambda_{i+1}\} \text{ and } \sigma(B)=\coprod_{B'\in\rho(T,B)}\sigma(B')\\ \text{which is a partition of } \sigma(B) \text{ into pairwise disjoint nonempty subsets.}\end{cases} \tag{SP32}$$

To get more information about $\Omega_B(H)$, first we recall that

$$\begin{cases}\text{for any } f\in R^{\natural} \text{ and } B\in\hat{R}^{\flat},\\ R^*(f,B) \text{ is the unique member of } R^{\flat *}\cup R^{\flat}_{\infty}\\ \quad \text{with } \sigma(R^*(f,B))=\{f\}\\ \text{such that } \lambda(R^*(f,B))=\min\{\text{noc}(f,f'):f'\in\tau(B)\}\end{cases} \tag{SP33}$$

and we put

$$S(H,B)=\begin{cases}\mathrm{ord}_X H_0(X)+\sum_{1\le j\le\chi(H)} O_j S(R^*(H_j,B)) & \text{in case } B\in R^\flat\\ \mathrm{ord}_X H_0(X)+\sum_{1\le j\le\chi(H)} \mathrm{ord}_X H_j(X,Y) & \text{in case } B\in R^\flat_\infty\end{cases} \tag{SP34}$$

(with the understanding that if $H=0$ then $S(H,B)=\infty$), and we call $S(H,B)$ the B-strength of H, and by (BP4) and (BP5) we see that

$$\begin{cases}\text{for any } B\in R^\flat \text{ and } (z,V,W)\in\epsilon(B) \text{ we have}\\ \mathrm{ord}_X H(X^V,z(X)+X^WY)=VS(H,B),\\ \text{and for any } B\in R^\flat_\infty \text{ we have}\\ \mathrm{ord}_X H(X,Y)=S(H,B).\end{cases} \tag{SP35}$$

We also put

$$A^{**}(H,B)=\prod_{1\le j\le\chi(H) \text{ with } H_j\in\tau^*(B)} A(R^*(H_j,B)) \quad \text{for all } B\in R^{\flat *} \tag{SP36}$$

and we call $A^{**}(H,B)$ the doubly strict B-constant of H, and we put

$$D^{**}(H,B)=(\deg_Y\Omega^*_B(H))/D^*(B) \quad \text{for all } B\in R^{\flat *} \tag{SP37}$$

and we call $D^{**}(H,B)$ the doubly strict B-degree of H, and we note that, in view of (SBP4),

$$\begin{cases}\text{for any } B\in R^{\flat *} \text{ we have}\\ 0\ne A^{**}(H,B)\in k \text{ and } 0\le D^{**}(H,B)\in\mathbb{Z}\\ \text{with: } D^{**}(H,B)>0\Leftrightarrow \Omega^*_B(H)\ne 1\end{cases} \tag{SP38}$$

and

$$\begin{cases}\text{for any } B\in R^{\flat *} \text{ and } (z,V,W)\in\epsilon(B) \text{ we have}\\ \mathrm{minco}_X[z,V,W](\Omega^*_B(H))=A^{**}(H,B)E(B,Y)^{D^{**}(H,B)}.\end{cases} \tag{SP39}$$

To collect together information about the Y-degrees of $\Omega_B(H)$, $\Omega'_B(H)$, $\Omega^*_B(H)$, in view of (SP13) and (SP27), we see that

$$\begin{cases}\text{for any } B\in R^\flat \text{ we have}\\ \deg_Y\Omega_B(H)=\deg_Y\Omega'_B(H)+\deg_Y\Omega^*_B(H)\\ \text{and for any } (z,V,W)\in\epsilon(B) \text{ we have}\\ \deg_Y\Omega'_B(H)=D(B)\ \deg_Y \mathrm{pinco}_X[z,V,W](H,\sigma(B))\\ \text{and } \deg_Y\Omega^*_B(H)=D(B)\ \deg_Y \mathrm{sinco}_X[z,V,W](H,\sigma(B))\\ \text{and if } H\ne 0 \text{ then we also have}\\ \deg_Y \mathrm{minco}_X[z,V,W](H)=\deg_Y \mathrm{pinco}_X[z,V,W](H,\sigma(B))\\ \quad +\deg_Y \mathrm{sinco}_X[z,V,W](H,\sigma(B))\\ \text{and } \deg_Y\Omega_B(H)=D(B)\deg_Y \mathrm{minco}_X[z,V,W](H).\end{cases} \tag{SP40}$$

Next we recall that

$$\begin{cases} \text{for any } B\in \hat{R}^{\flat} \text{ we have} \\ R^*(B)=\{B'\in R^{\flat *}\cup R^{\flat}_{\infty}: \lambda(B')=\lambda(B) \text{ and } \sigma(B')=\tau^*(B')\cap\sigma(B)\} \\ \text{and } \sigma(B)=\coprod_{B'\in R^*(B)} \sigma(B') \\ \text{which is a partition of } \sigma(B) \text{ into pairwise disjoint nonempty subsets} \end{cases} \quad \text{(SP41)}$$

and we put

$$R^*(H,B)=\{B'\in R^*(B): \Omega^*_{B'}(H)\neq 1\} \quad \text{for all } B\in \hat{R}^{\flat} \quad \text{(SP42)}$$

where we note that $R^*(H,B)$ is a finite set whose members may be called the strict B-friends of H. We also put

$$D'(B)=\begin{cases} -D(B)+\sum_{B'\in R^*(B)} D^*(B') & \text{for all } B\in R^{\flat} \\ -1+\sum_{B'\in R^*(B)} 1 & \text{for all } B\in R^{\flat}_{\infty} \end{cases} \quad \text{(SP43)}$$

(with the understanding that if $R^*(B)$ is an infinite set then $D'(B)=\infty$), and we call $D'(B)$ the primitive degree of B.[13] Moreover we put

$$D''(B)=\begin{cases} -D(B)+\sum_{f\in\sigma(B)} \deg_Y f & \text{for all } B\in R^{\flat} \\ -1+\sum_{f\in\sigma(B)} \deg_Y f & \text{for all } B\in R^{\flat}_{\infty} \end{cases} \quad \text{(SP44)}$$

(with the understanding that if $\sigma(B)$ is an infinite set then $D''(B)=\infty$), and we call $D''(B)$ the doubly primitive degree of B,[14] and we note that[15]

$$\begin{cases} \text{if } B\in \hat{R}^{\flat} \text{ is such that } R^*(H,B)=R^*(B) \\ \text{and } \Omega_B(H) \text{ is devoid of multiple factors in } R, \\ \text{then } \deg_Y\Omega_B(H)=\begin{cases} D(B)+D''(B) & \text{in case } B\in R^{\flat} \\ 1+D''(B) & \text{in case } B\in R^{\flat}_{\infty}. \end{cases} \end{cases} \quad \text{(SP45)}$$

Now clearly

$$\begin{cases} \text{for all } B\in \hat{R}^{\flat} \\ \text{we have } \Omega_B(H)=\Omega'_B(H)\Omega^*_B(H) \\ \text{and } \Omega^*_B(H)=\prod_{B'\in R^*(H,B)} \Omega^*_{B'}(H) \end{cases} \quad \text{(SP46)}$$

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[13] Observe that if $R^*(B)$ is a finite set then $D'(B)$ is a nonnegative integer; in particular, if $B\in R^{\flat}_{\infty}$ then $\text{card}(R^*(B))=1$, where card denotes cardinality, and hence $D'(B)=0$. We shall use $D'(B)$ mainly when $R^*(H,B)=R^*(B)$; in that situation, $R^*(B)$ is obviously a finite set and hence $D'(B)$ is a nonnegative integer. For the use of $D'(B)$ in such a situation, see (SP60) and (SP70).

[14] Observe that if $\sigma(B)$ is a finite set then $D''(B)$ is a nonnegative integer. Also observe that definitions (SP43) and (SP44) would look more natural if for every $B\in R^{\flat}_{\infty}$ we put $D(B)=D^*(B)=1$.

[15] An element Φ in R is devoid of multiple factors in R means that $\Phi\neq 0$ and the ideal ΦR is its own radical in R.

and, in view of (SP23) to (SP26),

$$\begin{cases}\text{for all } B'\in R^*(B) \text{ with } B\in R^\flat \text{ we know that}\\ E(B',Y)\in k[Y] \text{ is monic in } Y \text{ having no multiple root in } k\\ \text{and } \deg_Y E(B',Y)=D^*(B')/D(B')=D^*(B')/D(B)>0\end{cases}\tag{SP47}$$

and

$$\begin{cases}\text{for all } B'\neq B'' \text{ in } R^*(B) \text{ with } B\in R^\flat \text{ we know that}\\ E(B',Y) \text{ and } E(B'',Y) \text{ have no common root in } k\end{cases}\tag{SP48}$$

and

$$\begin{cases}\text{for all } B\in R^\flat \text{ with } (z,V,W)\in\epsilon(B) \text{ we know that}\\ \text{minco}_X[z,V,W](\Omega'_B(H)) \text{ is a nonzero member of } k[Y]\\ \text{which has no common root with } E(B',Y) \text{ in } k \text{ for any } B'\in R^*(B).\end{cases}\tag{SP49}$$

By (SP10) and (SP20) we see that

$$\begin{cases}\text{for any } T\in R^\sharp \text{ we have:}\\ H_\infty=\prod_{R_\infty(T)\neq B\in T}\Omega'_B(H)\Leftrightarrow \deg_Y H_\infty=\sum_{R_\infty(T)\neq B\in T}\deg_Y\Omega'_B(H).\end{cases}\tag{SP50}$$

In the next section we shall show that (SP50) is applicable when $H=F_Y$ where $F\in R$ is devoid of multiple factors. In the section after that we shall apply (SP30) to the case when H is the jacobian $J(F,G)$ of F and G in R. To prepare for all this, until further notice, given any $B\in R^\flat$ with $(z,V,W)\in\epsilon(B)$, for every $\Phi=\Phi(X,Y)\in R$ let us put

$$\begin{cases}\tilde\Phi=\tilde\Phi(X,Y)=\Phi(X^V,z(X)+X^WY) \quad\text{and}\\ I(\Phi)=\text{minco}_X[z,V,W](\Phi)=\text{inco}_X\tilde\Phi.\end{cases}\tag{SP51}$$

Then by (SP13) we see that

$$\begin{cases}\deg_Y\Omega_B(H)=D(B)\deg_Y I(\Omega_B(H))\\ \text{and } I(H)=\mu I(\Omega_B(H))\\ \text{where } \mu\in k \text{ is such that}: \mu=0\Leftrightarrow H=0\end{cases}\tag{SP52}$$

and by the chain rule for partial derivatives we see that

$$\begin{cases}\text{if } I(H)\notin k\\ \text{then } I(H_Y)=(I(H))_Y.\end{cases}\tag{SP53}$$

Also clearly

$$\begin{cases}\text{for any } \Psi(Y)\in k[Y]\setminus k \text{ we have}\\ 0\neq\Psi_Y(Y)\in k[Y] \text{ with } \deg_Y\Psi_Y(Y)=-1+\deg_Y\Psi(Y)\end{cases}\tag{SP54}$$

and

$$\begin{cases}\text{for any } \Psi(Y)\in k[Y] \text{ and } \mu\in k \text{ and } 0<\nu\in\mathbb{Z} \text{ we have:}\\ \Psi(Y)=(Y-\mu)^\nu\Psi'(Y) \text{ where } \Psi'(Y)\in k[Y] \text{ with } \Psi'(\mu)\neq 0\\ \Rightarrow \Psi_Y(Y)=(Y-\mu)^{\nu-1}\Psi''(Y) \text{ where } \Psi''(Y)\in k[Y] \text{ with } \Psi''(\mu)\neq 0.\end{cases}\tag{SP55}$$

By (SP46) we see that

$$I(\Omega_B(H)) = I(\Omega'_B(H)) \prod_{B' \in R^*(H,B)} I(\Omega^*_{B'}(H)). \tag{SP56}$$

Moreover, by (SP38), (SP39) and (SP47),

$$\begin{cases} \text{for all } B' \in R^*(H,B) \text{ we know that} \\ I(\Omega^*_{B'}(H)) = A^{**}(H,B')E(B',Y)^{D^{**}(H,B')} \\ \text{where } 0 \neq A^{**}(H,B') \in k, 0 < D^{**}(H,B') \in \mathbb{Z}, \\ E(B',Y) \in k[Y] \text{ is monic in } Y \text{ having no multiple roots in } k, \\ \text{and } \deg_Y E(B',Y) = D^*(B')/D(B) > 0. \end{cases} \tag{SP57}$$

Likewise, by (SP48) and (SP49),

$$\begin{cases} \text{for all } B' \neq B'' \text{ in } R^*(H,B) \text{ we know that} \\ E(B',Y) \text{ and } E(B'',Y) \text{ have no common root in } k, \\ \text{and we also know that } I(\Omega'_B(H)) \text{ is a nonzero member of } k[Y] \\ \text{which has no common root with } \prod_{B' \in R^*(H,B)} E(B',Y) \text{ in } k. \end{cases} \tag{SP58}$$

By (SP51) to (SP58), we conclude that

$$\begin{cases} \text{if } \Omega_B(H) \neq 1 \\ \text{then } \deg_Y I(\Omega_B(H_Y)) = -1 + \deg_Y I(\Omega_B(H)) \\ \text{and } I(\Omega_B(H_Y)) = L(Y) \prod_{B' \in R^*(H,B)} E(B',Y)^{D^{**}(H,B')-1} \\ \text{where } 0 \neq L(Y) \in k[Y] \text{ has no common root with} \\ \quad \prod_{B' \in R^*(H,B)} E(B',Y) \text{ in } k \\ \text{and } \deg_Y L(Y) = -1 + \deg_Y I(\Omega'_B(H)) + \sum_{B' \in R^*(H,B)} \deg_Y E(B',Y) \\ \text{with } \deg_Y E(B',Y) = D^*(B')/D(B) \text{ for all } B' \in R^*(H,B). \end{cases} \tag{SP59}$$

Now if $R^*(H,B) = R^*(B)$ then clearly $\Omega_B(H) \neq 1 = \Omega'_B(H)$ and hence in particular $\deg_Y I(\Omega'_B(H)) = 0$ and therefore in the situation of (SP59) we get

$$\deg_Y L(Y) = -1 + \sum_{B' \in R^*(B)} [D^*(B')/D(B)] = D'(B)/D(B)$$

and in view of (SP26) we have $\text{pinco}_X[z,V,W](H_Y,\sigma(B)) = \mu L(Y)$ with $0 \neq \mu \in k$, and hence

$$\deg_Y \text{pinco}_X[z,V,W](H_Y,\sigma(B)) = \deg_Y L(Y) = D'(B)/D(B).$$

Thus

$$\begin{cases} \text{if } R^*(H,B) = R^*(B), \\ \text{then } \Omega_B(H) \neq 1 \\ \text{and } \deg_Y \text{pinco}_X[z,V,W](H_Y,\sigma(B)) = D'(B)/D(B). \end{cases} \tag{SP60}$$

In view of (SP40) and (SP51), by (SP59) we see that

$$\begin{cases} \text{if } \Omega_B(H) \neq 1 \\ \text{then } \deg_Y \operatorname{minco}_X[z,V,W](H_Y) = -1 + \deg_Y \operatorname{minco}_X[z,V,W](H) \\ \text{and } \deg_Y \Omega_B(H_Y) = -D(B) + \deg_Y \Omega_B(H). \end{cases} \quad \text{(SP61)}$$

In view of (SP51), by (SP56) to (SP59) we also see that

$$D^{**}(H_Y, B') = -1 + D^{**}(H, B') \quad \text{for all } B' \in R^*(H,B). \quad \text{(SP62)}$$

By (SP62) it follows that

$$\begin{cases} \text{if } B \in R^{\flat *} \text{ and } D^{**}(H,B) > 0 \\ \text{then } D^{**}(H_Y, B) = -1 + D^{**}(H,B). \end{cases} \quad \text{(SP63)}$$

In view of (SP40), by (SP60) and (SP61) we see that

$$\begin{cases} \text{if } R^*(H,B) = R^*(B), \\ \text{then } \deg_Y \operatorname{sinco}_X[z,V,W](H_Y, \sigma(B)) = -1 \\ \quad - [D'(B)/D(B)] + \deg_Y \operatorname{minco}_X[z,V,W](H). \end{cases} \quad \text{(SP64)}$$

In view of (SP40), by (SP60) and (SP64) we see that

$$\begin{cases} \text{if } R^*(H,B) = R^*(B), \\ \text{then } \deg_Y \Omega_B(H_Y) = -D(B) + \deg_Y \Omega_B(H) \\ \text{and } \deg_Y \Omega'_B(H_Y) = D'(B) \\ \text{and } \deg_Y \Omega^*_B(H_Y) = -D(B) - D'(B) + \deg_Y \Omega_B(H). \end{cases} \quad \text{(SP65)}$$

Turning to the jacobian, upon letting

$$\Delta = J(H,G) \quad \text{and} \quad \overline{\Delta} = J(\tilde{H}, \tilde{G}) \quad \text{with } G \in R$$

by the chain rule for jacobians we get

$$\overline{\Delta}(X,Y) = VX^{V+W-1}\tilde{\Delta}(X,Y)$$

and now, assuming that $\Omega_B(H) \neq 1 = \Omega_B(G)$ and $G \neq 0 \neq S(G,B)$, we have

$$\tilde{H}(X,Y) = I(H)X^{VS(H,B)} + (\text{terms of degree} > VS(H,B) \text{ in } X)$$

with $I(H) \in k[Y] \setminus k$, and

$$\tilde{G}(X,Y) = I(G)X^{VS(G,B)} + (\text{terms of degree} > VS(G,B) \text{ in } X)$$

with $0 \neq I(G) \in k$ and $VS(G,B) \neq 0$, and hence we get

$$\begin{aligned} \overline{\Delta}(X,Y) = &- VS(G,B)I(G)(I(H))_Y X^{VS(H,B)+VS(G,B)-1} \\ &+ (\text{terms of degree} \geq [VS(H,B) + VS(G,B)] \text{ in } X) \end{aligned}$$

and therefore by (SP52) we have

$$I(J(H,G)) = \mu I(H_Y) \quad \text{with } 0 \neq \mu \in k$$

and hence by (SP51) we get

$$\operatorname{minco}_X[z,V,W](J(H,G)) = \mu \operatorname{minco}_X[z,V,W](H_Y) \quad \text{with } 0 \neq \mu \in k.$$

Thus

$$\begin{cases}\text{if } \Omega_B(H)\neq 1 \text{ and } 0\neq G\in R \text{ is such that } \Omega_B(G)=1 \text{ and } S(G,B)\neq 0\\ \text{then } \mathrm{minco}_X[z,V,W](J(H,G))=\mu\ \mathrm{minco}_X[z,V,W](H_Y) \text{ with } 0\neq\mu\in k\\ \text{and hence } \mathrm{pinco}_X[z,V,W](J(H,G),\sigma(B))=\mathrm{pinco}_X[z,V,W](H_Y,\sigma(B))\\ \text{and } \mathrm{sinco}_X[z,V,W](J(H,G),\sigma(B))=\mathrm{sinco}_X[z,V,W](H_Y,\sigma(B)).\end{cases}\qquad\text{(SP66)}$$

In view of (SP40), by (SP66) we see that

$$\begin{cases}\text{if } \Omega_B(H)\neq 1 \text{ and } 0\neq G\in R \text{ is such that } \Omega_B(G)=1 \text{ and } S(G,B)\neq 0\\ \text{then } \deg_Y \mathrm{minco}_X[z,V,W](J(H,G))=\deg_Y \mathrm{minco}_X[z,V,W](H_Y)\\ \text{and } \deg_Y \mathrm{pinco}_X[z,V,W](J(H,G),\sigma(B))=\deg_Y \mathrm{pinco}_X[z,V,W](H_Y,\sigma(B))\\ \text{and } \deg_Y \mathrm{sinco}_X[z,V,W](J(H,G),\sigma(B))\\ \quad=\deg_Y \mathrm{sinco}_X[z,V,W](H_Y,\sigma(B))\\ \text{and } \deg_Y\Omega_B(J(H,G))=\deg_Y\Omega_B(H_Y)\\ \text{and } \deg_Y\Omega'_B(J(H,G))=\deg_Y\Omega'_B(H_Y)\\ \text{and } \deg_Y\Omega^*_B(J(H,G))=\deg_Y\Omega^*_B(H_Y).\end{cases}\qquad\text{(SP67)}$$

To consider another similar case, just for a moment let $j(H,G)$ stand for $(HG)_Y$; then upon letting

$$\delta=j(H,G) \quad\text{and}\quad \bar{\delta}=j(\tilde{H},\tilde{G}) \quad\text{with } G\in R$$

by the product rule for derivatives we get

$$\bar{\delta}(X,Y)=X^W\tilde{\delta}(X,Y)$$

and now, assuming that $\Omega_B(H)\neq 1=\Omega_B(G)$ and $G\neq 0$, we have

$$\tilde{H}(X,Y)=I(H)X^{VS(H,B)}+\text{(terms of degree} > VS(H,B) \text{ in } X)$$

with $I(H)\in k[Y]\setminus k$, and

$$\tilde{G}(X,Y)=I(G)X^{VS(G,B)}+\text{(terms of degree} > VS(G,B) \text{ in } X)$$

with $0\neq I(G)\in k$, and hence we get

$$\begin{aligned}\bar{\delta}(X,Y)&=I(G)(I(H))_YX^{VS(H,B)+VS(G,B)}\\ &\qquad+\text{(terms of degree} > [VS(H,B)+VS(G,B)] \text{ in } X)\end{aligned}$$

and therefore by (SP52) we have

$$I((HG)_Y)=\mu I(H_Y) \quad\text{with } 0\neq\mu\in k$$

and hence by (SP51) we get

$$\mathrm{minco}_X[z,V,W]((HG)_Y)=\mu\ \mathrm{minco}_X[z,V,W](H_Y) \quad\text{with } 0\neq\mu\in k.$$

Thus

$$\begin{cases}\text{if } \Omega_B(H)\neq 1 \text{ and } 0\neq G\in R \text{ is such that } \Omega_B(G)=1\\ \text{then } \mathrm{minco}_X[z,V,W]((HG)_Y)=\mu\ \mathrm{minco}_X[z,V,W](H_Y) \text{ with } 0\neq\mu\in k\\ \text{and hence } \mathrm{pinco}_X[z,V,W]((HG)_Y,\sigma(B))\\ \quad=\mathrm{pinco}_X[z,V,W](H_Y,\sigma(B))\\ \text{and } \mathrm{sinco}_X[z,V,W]((HG)_Y,\sigma(B))=\mathrm{sinco}_X[z,V,W](H_Y,\sigma(B)).\end{cases}\qquad\text{(SP68)}$$

In view of (SP40), by (SP68) we see that

$$\begin{cases} \text{if } \Omega_B(H) \neq 1 \text{ and } 0 \neq G \in R \text{ is such that } \Omega_B(G) = 1 \\ \text{then } \deg_Y \text{minco}_X[z,V,W]((HG)_Y) = \deg_Y \text{minco}_X[z,V,W](H_Y) \\ \text{and } \deg_Y \text{pinco}_X[z,V,W]((HG)_Y,\sigma(B)) = \deg_Y \text{pinco}_X[z,V,W](H_Y,\sigma(B)) \\ \text{and } \deg_Y \text{sinco}_X[z,V,W]((HG)_Y,\sigma(B)) \\ \quad = \deg_Y \text{sinco}_X[z,V,W](H_Y,\sigma(B)) \\ \text{and } \deg_Y \Omega_B((HG)_Y) = \deg_Y \Omega_B(H_Y) \\ \text{and } \deg_Y \Omega'_B((HG)_Y) = \deg_Y \Omega'_B(H_Y) \\ \text{and } \deg_Y \Omega^*_B((HG)_Y) = \deg_Y \Omega^*_B(H_Y). \end{cases} \tag{SP69}$$

Abandoning notation (SP51), let us summarize results (SP60) to (SP69) as lemmas (SP70) to (SP75) stated below.

Lemma (SP70). *If* $B \in R^\flat$ *and* $F \in R$ *are such that* $R^*(F,B) = R^*(B)$, *then* $\Omega_B(F) \neq 1$ *and we have*

$$\begin{cases} \deg_Y \Omega_B(F_Y) = -D(B) + \deg_Y \Omega_B(F) \\ \textit{and } \deg_Y \Omega'_B(F_Y) = D'(B) \\ \textit{and } \deg_Y \Omega^*_B(F_Y) = -D(B) - D'(B) + \deg_Y \Omega_B(F) \end{cases}$$

and for every $(z,V,W) \in \epsilon(B)$ *we have*

$$\begin{cases} \deg_Y \text{minco}_X[z,V,W](F_Y) = -1 + \deg_Y \text{minco}_X[z,V,W](F) \\ \textit{and } \deg_Y \text{pinco}_X[z,V,W](F_Y,\sigma(B)) = D'(B)/D(B) \\ \textit{and } \deg_Y \text{sinco}_X[z,V,W](F_Y,\sigma(B)) \\ \quad = -1 - [D'(B)/D(B)] + \deg_Y \text{minco}_X[z,V,W](F). \end{cases}$$

Lemma (SP71). *If* $B \in R^\flat$ *and* $F \in R$ *are such that* $\Omega_B(F) \neq 1$, *then we have*

$$\deg_Y \Omega_B(F_Y) = -D(B) + \deg_Y \Omega_B(F)$$

and for every $(z,V,W) \in \epsilon(B)$ *we have*

$$\deg_Y \text{minco}_X[z,V,W](F_Y) = -1 + \deg_Y \text{minco}_X[z,V,W](F).$$

Lemma (SP72). *Given any* $B \in R^\flat$ *and* $F \in R$, *for every* $B' \in R^*(F,B)$ *we have*

$$D^{**}(F_Y,B) = -1 + D^{**}(F,B).$$

Lemma (SP73). *If* $B \in R^{\flat *}$ *and* $F \in R$ *are such that* $D^{**}(F,B) > 0$, *then we have*

$$D^{**}(F_Y,B) = -1 + D^{**}(F,B).$$

Lemma (SP74). *If* $B \in R^\flat$ *and* $F \in R$ *are such that* $\Omega_B(F) \neq 1$, *and* $0 \neq G \in R$ *is such that* $\Omega_B(G) = 1$ *and* $S(G,B) \neq 0$, *then we have*

$$\begin{cases} \deg_Y \Omega_B(J(F,G)) = \deg_Y \Omega_B(F_Y) \\ \textit{and } \deg_Y \Omega'_B(J(F,G)) = \deg_Y \Omega'_B(F_Y) \\ \textit{and } \deg_Y \Omega^*_B(J(F,G)) = \deg_Y \Omega^*_B(F_Y) \end{cases}$$

and for every $(z, V, W) \in \epsilon(B)$ *we have*

$$\begin{cases} \deg_Y \operatorname{minco}_X[z,V,W](J(F,G),\sigma(B)) = \deg_Y \operatorname{minco}_X[z,V,W](F_Y,\sigma(B)) \\ \textit{and } \deg_Y \operatorname{pinco}_X[z,V,W](J(F,G),\sigma(B)) = \deg_Y \operatorname{pinco}_X[z,V,W](F_Y,\sigma(B)) \\ \textit{and } \deg_Y \operatorname{sinco}_X[z,V,W](J(F,G),\sigma(B)) = \deg_Y \operatorname{sinco}_X[z,V,W](F_Y,\sigma(B)) \end{cases}$$

and actually we have

$$\begin{cases} \operatorname{minco}_X[z,V,W](J(F,G)) = \mu \operatorname{minco}_X[z,V,W](F_Y) \textit{ with } 0 \neq \mu \in k, \\ \textit{and } \operatorname{pinco}_X[z,V,W](J(F,G),\sigma(B)) = \operatorname{pinco}_X[z,V,W](F_Y,\sigma(B)), \\ \textit{and } \operatorname{sinco}_X[z,V,W](J(F,G),\sigma(B)) = \operatorname{sinco}_X[z,V,W](F_Y,\sigma(B)). \end{cases}$$

Lemma (SP75). *If* $B \in R^\flat$ *and* $F \in R$ *are such that* $\Omega_B(F) \neq 1$, *and* $0 \neq G \in R$ *is such that* $\Omega_B(G) = 1$, *then we have*

$$\begin{cases} \deg_Y \Omega_B((FG)_Y) = \deg_Y \Omega_B(F_Y) \\ \textit{and } \deg_Y \Omega'_B((FG)_Y) = \deg_Y \Omega'_B(F_Y) \\ \textit{and } \deg_Y \Omega^*_B((FG)_Y) = \deg_Y \Omega^*_B(F_Y) \end{cases}$$

and for every $(z, V, W) \in \epsilon(B)$ *we have*

$$\begin{cases} \deg_Y \operatorname{minco}_X[z,V,W]((FG)_Y) = \deg_Y \operatorname{minco}_X[z,V,W](F_Y) \\ \textit{and } \deg_Y \operatorname{pinco}_X[z,V,W]((FG)_Y,\sigma(B)) = \deg_Y \operatorname{pinco}_X[z,V,W](F_Y,\sigma(B)) \\ \textit{and } \deg_Y \operatorname{sinco}_X[z,V,W]((FG)_Y,\sigma(B)) = \deg_Y \operatorname{sinco}_X[z,V,W](F_Y,\sigma(B)) \end{cases}$$

and actually we have

$$\begin{cases} \operatorname{minco}_X[z,V,W]((FG)_Y) = \mu \operatorname{minco}_X[z,V,W](F_Y) \textit{ with } 0 \neq \mu \in k, \\ \textit{and } \operatorname{pinco}_X[z,V,W]((FG)_Y,\sigma(B)) = \operatorname{pinco}_X[z,V,W](F_Y,\sigma(B)), \\ \textit{and } \operatorname{sinco}_X[z,V,W]((FG)_Y,\sigma(B)) = \operatorname{sinco}_X[z,V,W](F_Y,\sigma(B)). \end{cases}$$

Now, as a consequence of (SP45) and (SP70) we shall prove the following lemma:

Lemma (SP76). *If* $B \in \hat{R}^\flat$ *and* $F \in R$ *are such that* $R^*(F,B) = R^*(B)$ *and* F *is devoid of multiple factors in R, then* $\Omega_B(F) \neq 1$ *and* $\deg_Y \Omega_B(F_Y) = D''(B)$ *and* $\deg_Y \Omega'_B(F_Y) = D'(B)$.

Namely, if $B \in R^\flat$ and $F \in R$ are such that $R^*(F,B) = R^*(B)$ then by (SP70) we get $\Omega_B(F) \neq 1$ and $\deg_Y \Omega_B(F_Y) = -D(B) + \deg_Y \Omega_B(F)$ and $\deg_Y \Omega'_B(F_Y) = D'(B)$; if F is also devoid of multiple factors in R, then by (SP45) we know that $\deg_Y \Omega_B(F) = D(B) + D''(B)$ and hence we get $\deg_Y \Omega_B(F_Y) = D''(B)$. Likewise, if $B \in R^\flat_\infty$ and $F \in R$ are such that $\Omega_B(F) \neq 1$ then clearly $\deg_Y \Omega_B(F_Y) = -1 + \deg_Y \Omega_B(F)$; if F is also devoid of multiple factors in R, then by (SP45) we know that $\deg_Y \Omega_B(F) = 1 + D''(B)$ and hence we get $\deg_Y \Omega_B(F_Y) = D''(B)$. This completes the proof of (SP76).[16]

Next, as a consequence of (SP74) and (SP75) we shall prove the following lemma:

Lemma (SP77). *Given any* $F \in R \setminus k((X))$ *and* $0 \neq G \in R$, *upon letting* $T = T(FG)$, *we have the following.*

[16] We may tacitly use the obvious facts that: (1) if $B \in \hat{R}^\flat$ and $F \in R$ are such that $R^*(F,B) = R^*(B)$ then $\Omega_B(F) \neq 1$; (2) for every $B \in R^\flat_\infty$ we have $D'(B) = 0$; (3) for every $B \in R^\flat_\infty$ and every $G \in R$ we have $\Omega_B(G) = 1$ and hence $\deg_Y \Omega'_B(G) = 0$.

(SP77.1) *If* $B\in T$ *is such that* $\Omega_B(G)=1$ *and* $S(G,B)\neq 0$ *then*

$$\Omega_B(F)\neq 1$$

and

$$\deg_Y\Omega_B(J(F,G))=\deg_Y\Omega_B(F_Y) \quad \text{and} \quad \deg_Y\Omega'_B(J(F,G))=\deg_Y\Omega'_B(F_Y)$$

and for every $B'\in\pi(T,B)$ *we have*

$$\Omega_{B'}(F)\neq 1=\Omega_{B'}(G) \quad \text{and} \quad S(G,B')\neq 0$$

and

$$\deg_Y\Omega_{B'}(J(F,G))=\deg_Y\Omega_{B'}(F_Y) \quad \text{and} \quad \deg_Y\Omega'_{B'}(J(F,G))=\deg_Y\Omega'_{B'}(F_Y).$$

(SP77.2) *If* $B\in T$ *is such that* $\Omega_B(G)=1$ *then*

$$\Omega_B(F)\neq 1$$

and

$$\deg_Y\Omega_B((FG)_Y)=\deg_Y\Omega_B(F_Y) \quad \text{and} \quad \deg_Y\Omega'_B((FG)_Y)=\deg_Y\Omega'_B(F_Y)$$

and for every $B'\in\pi(T,B)$ *we have*

$$\Omega_{B'}(F)\neq 1=\Omega_{B'}(G)$$

and

$$\deg_Y\Omega_{B'}((FG)_Y)=\deg_Y\Omega_{B'}(F_Y) \quad \text{and} \quad \deg_Y\Omega'_{B'}((FG)_Y)=\deg_Y\Omega'_{B'}(F_Y).$$

Namely, if $B\in T$ is such that $\Omega_B(G)=1$ then obviously $\Omega_B(F)\neq 1$ and for every $B'\in\pi(T,B)$ we have $\Omega_{B'}(F)\neq 1=\Omega_{B'}$, and by (BP5) we also see that if $B\in T$ is such that $\Omega_B(G)=1$ then for every $B'\in\pi(T,B)$ we have $S(G,B')=S(G,B)$. Therefore, in case of $B\in T\setminus\{R_\infty(T)\}$, our assertions follow from (SP74) and (SP75). Moreover, if $B=R_\infty(T)$ and $\Omega_B(G)=1$ and $S(G,B)\neq 0$, then by the equation $J(F,G)=F_XG_Y-F_YG_X$ we see that $J(F,G)=-F_YG_X$ with $0\neq G_X\in k((X))$ and hence $\deg_Y\Omega_B(J(F,G))=\deg_Y\Omega_B(F_Y)$ and $\deg_Y\Omega'_B(J(F,G))=0=\deg_Y\Omega'_B(F_Y)$. Likewise, if $B=R_\infty(T)$ and $\Omega_B(G)=1$, then by the equation $(FG)_Y=FG_Y+F_YG$ we see that $(FG)_Y=F_YG$ with $0\neq G\in k((X))$ and hence $\deg_Y\Omega_B((FG)_Y)=\deg_Y\Omega_B(F_Y)$ and $\deg_Y\Omega'_B((FG)_Y)=0=\deg_Y\Omega'_B(F_Y)$. This completes the proof of (SP77).

Now we shall prove the following lemma:

Lemma (SP78). *For any* $\Phi\in R$, *upon letting* $T=T(\Phi)$, *we have the following.*

(SP78.1) *Given any* $B\in T$ *with* $\pi(T,B)\neq\emptyset$, *for every* $B'\in\rho(T,B)$ *there is a unique* $\alpha(B')\in R^*(B)$ *with* $\sigma(\alpha(B'))=\sigma(B')$, *and* $B'\mapsto\alpha(B')$ *gives a bijection of* $\rho(T,B)$ *onto* $R^*(B)$. *Moreover, for every* $B'\in\rho(T,B)$ *we have*

$$D(B')=\begin{cases} D^*(\alpha(B')) & \text{in case } B\in R^\flat \\ 1 & \text{in case } B\in R^\flat_\infty \end{cases}$$

and hence we have

$$D'(B)=\begin{cases} -D(B)+\sum_{B'\in\rho(T,B)} D(B') & \text{in case } B\in R^\flat \\ -1+\sum_{B'\in\rho(T,B)} D(B') & \text{in case } B\in R^\flat_\infty \end{cases}$$

where we note that if $B\in R^\flat_\infty$ *then* $\operatorname{card}(\rho(T,B))=\operatorname{card}(R^*(B))=1$.

(SP78.2) *Given any* $B \in T$ *with* $\pi(T,B) = \emptyset$, *for every* $f \in \sigma(B)$ *there is a unique* $\beta(f) \in R^*(B)$ *with* $f \in \sigma(\beta(f))$, *and* $f \mapsto \beta(f)$ *gives a bijection of* $\sigma(B)$ *onto* $R^*(B)$. *Moreover, for every* $f \in \sigma(B)$ *we have*

$$\deg_Y f = \begin{cases} D^*(\beta(f)) & \text{in case } B \in R^\flat \\ 1 & \text{in case } B \in R^\flat_\infty \end{cases}$$

and hence we have

$$D''(B) = D'(B).$$

(SP78.3) *For every* $B \in T$ *we have*

$$D''(B) = D'(B) + \sum_{B' \in \pi(T,B)} D'(B').$$

Namely, the proofs of (SP78.1) and (SP78.2) are straightforward. We shall prove (SP78.3) by induction on $\operatorname{card}(\pi(T,B))$. In case of $\operatorname{card}(\pi(T,B)) = 0$ our assertion follows from (SP78.2). So let $\operatorname{card}(\pi(T,B)) > 0$ and assume true for all smaller values of $\operatorname{card}(\pi(T,B))$. Then, by the induction hypothesis, for every $B' \in \rho(T,B)$ we have

$$D''(B') = D'(B') + \sum_{B'' \in \pi(T,B')} D'(B'')$$

and hence by the definition of $D''(B')$ we get

$$D'(B') + \sum_{B'' \in \pi(T,B')} D'(B'') = -D(B') + \sum_{f \in \sigma(B')} \deg_Y f.$$

Summing both sides of the above equation as B' varies over $\rho(T,B)$, we get

$$\sum_{B' \in \pi(T,B)} D'(B') = -\left[\sum_{B' \in \rho(T,B)} D(B')\right] + \left[\sum_{f \in \sigma(B)} \deg_Y f\right]$$

and by (SP78.1) we have

$$D'(B) = \begin{cases} -D(B) + \sum_{B' \in \rho(T,B)} D(B') & \text{in case } B \in R^\flat \\ -1 + \sum_{B' \in \rho(T,B)} D(B') & \text{in case } B \in R^\flat_\infty. \end{cases}$$

By adding the above two equations we get

$$D'(B) + \sum_{B' \in \pi(T,B)} D'(B') = \begin{cases} -D(B) + \sum_{f \in \sigma(B)} \deg_Y f & \text{in case } B \in R^\flat \\ -1 + \sum_{f \in \sigma(B)} \deg_Y f & \text{in case } B \in R^\flat_\infty \end{cases}$$

and hence by the definition of $D''(B)$ we conclude that

$$D''(B) = D'(B) + \sum_{B' \in \pi(T,B)} D'(B').$$

This completes the proof of (SP78.3).

As an immediate consequence of (SP78.3) we get the following lemma:

Lemma (SP79). *Let* $B \in T = T(\Phi)$ *with* $\Phi \in R$ *be such that:* $\deg_Y \Omega_B(H) = D''(B)$, $\deg_Y \Omega'_B(H) = D'(B)$, *and* $\deg_Y \Omega'_{B'}(H) = D'(B')$ *for all* $B' \in \pi(T, B)$. *Then*

$$\deg_Y \Omega_B(H) = \deg_Y \Omega'_B(H) + \sum_{B' \in \pi(T,B)} \deg_Y \Omega'_{B'}(H).$$

For any $B \in T = T(\Phi)$ with $\Phi \in R$, it is clear that $\Omega'_B(H) \prod_{B' \in \pi(T,B)} \Omega'_{B'}(H)$ divides $\Omega_B(H)$ in R and hence, as a companion to (SP50), and as a principle applicable in the situation of (SP79), we get the following lemma:

Thus Lemma (SP80). *For any* $B \in T = T(\Phi)$ *with* $\Phi \in R$ *we have:*

$$\begin{cases} \Omega_B(H) = \Omega'_B(H) \prod_{B' \in \pi(T,B)} \Omega'_{B'}(H) \\ \Leftrightarrow \deg_Y \Omega_B(H) = \deg_Y \Omega'_B(H) + \sum_{B' \in \pi(T,B)} \deg_Y \Omega'_{B'}(H). \end{cases}$$

8. Factorization of the derivative

If $T = T(F)$ where $F \in R \setminus k((X))$, then for every $B \in T$ we clearly have $R^*(F, B) = R^*(B)$. Therefore by (SP76), (SP79) and (SP80) we get the following derivative factorization theorem.

Theorem (DF1). *Let* $T = T(F)$ *where* $F \in R \setminus k((X))$ *is devoid of multiple factors in R. Then we have the following.*

(DF1.1) *For any* $B \in T$ *we have*

$$\deg_Y \Omega_B(F_Y) = D''(B) \quad \textit{and} \quad \deg_Y \Omega'_B(F_Y) = D'(B)$$

and

$$\Omega_B(F_Y) = \Omega'_B(F_Y) \prod_{B' \in \pi(T,B)} \Omega'_{B'}(F_Y).$$

where for every $B' \in \pi(T, B)$ *we have*

$$\deg_Y \Omega_{B'}(F_Y) = D''(B') \quad \textit{and} \quad \deg_Y \Omega'_{B'}(F_Y) = D'(B').$$

(DF1.2) *By taking* $B = R_\infty(T)$ *in* (DF1.1), *for the monic part* $(F_Y)_\infty$ *of* F_Y *we get*

$$(F_Y)_\infty = \prod_{B \in T \setminus \{R_\infty(T)\}} \Omega'_B(F_Y).$$

Remark (DF2). In the factorization (DF1.2), the factor $\Omega'_B(F_Y)$ really occurs, i.e., its Y-degree $D'(B)$ is nonzero, if and only if either: (*) $\mathrm{card}(R^*(B)) = 1$ and for the unique $B' \in R^*(B)$ we have $D^*(B') > D(B)$, or: (**) $\mathrm{card}(R^*(B)) > 1$. Note that in the irreducible case, i.e., when $F = f \in R^\natural$, (*) is always satisfied. Moreover, in the nontrivial irreducible case, i.e., when $F = f \in R^\natural$ with $\deg_Y f = n > 1$, let us put

$$\hat{h} = \begin{cases} h(c(f)) & \text{if } c_1(f) \notin \mathbb{Z} \\ h(c(f)) - 1 & \text{if } c_1(f) \in \mathbb{Z} \end{cases}$$

and for $1 \leq i \leq \hat{h}$ let us put

$$\hat{c}_i = \begin{cases} c_i(f) & \text{if } c_1(f) \notin \mathbb{Z} \\ c_{i+1}(f) & \text{if } c_1(f) \in \mathbb{Z} \end{cases}$$

and

$$\hat{r}_i = \begin{cases} r_i(q(m(f))) & \text{if } r_1(f) \notin \mathbb{Z} \\ r_{i+1}(q(m(f))) & \text{if } r_1(f) \in \mathbb{Z} \end{cases}$$

and for $1 \leq i \leq \hat{h}+1$ let us put

$$\hat{d}_i = \begin{cases} d_i(m(f)) & \text{if } c_1(f) \notin \mathbb{Z} \\ d_{i+1}(m(f)) & \text{if } c_1(f) \in \mathbb{Z}. \end{cases}$$

Then $\hat{h}$ is a positive integer, $\hat{c}_1 < \hat{c}_2 < \cdots < \hat{c}_{\hat{h}}$ are in $\mathbb{Q} \setminus \mathbb{Z}$, and $n = \hat{d}_1 > \hat{d}_2 > \cdots > \hat{d}_{\hat{h}+1} = 1$ are integers with $\hat{d}_i \equiv 0 \,(\operatorname{mod} \hat{d}_{i+1})$ for $1 \leq i \leq \hat{h}$. Let $B_0 = (\sigma(B_0), \lambda(B_0)) \in R^{\flat}_{\infty}$ with $\sigma(B_0) = \{f\}$ and $\lambda(B_0) = -\infty$. For $1 \leq i \leq \hat{h}$ let $B_i = (\sigma(B_i), \lambda(B_i)) \in R^{\flat}$ with $\sigma(B_i) = \{f\}$ and $\lambda(B_i) = \hat{c}_i$. Then $T = T(f) = \{B_0, B_1, \ldots, B_{\hat{h}}\}$ with $R_{\infty}(T) = B_0 < B_1 < \cdots < B_{\hat{h}}$, and for $1 \leq i \leq \hat{h}$ we have

$$S(B_i) = (\hat{d}_i \hat{r}_i)/n^2 \quad \text{and} \quad D(B_i) = n/\hat{d}_i$$

and

$$D^*(B_i') = n/\hat{d}_{i+1} \quad \text{where} \quad \{B_i'\} = R^*(B_i).$$

Now

$$(f_Y)_{\infty} = (1/n) f_Y = \prod_{1 \leq i \leq \hat{h}} \Omega'_{B_i}(f_Y)$$

and for $1 \leq i \leq \hat{h}$ we have

$$\deg_Y \Omega'_{B_i}(f_Y) = D'(B_i) = -D(B_i) + D^*(B_i') = (n/\hat{d}_{i+1}) - (n/\hat{d}_i) > 0.$$

Let us factor f_Y into irreducible factors by writing

$$f_Y = n \prod_{1 \leq j \leq \chi} f^{(j)} \quad \text{with } f^{(j)} \in R^{\natural}$$

and for $1 \leq i \leq \hat{h}$ let us put

$$i^* = \{j \in \{1, \ldots, \chi\} : \operatorname{noc}(f, f^{(j)}) = \hat{c}_i\}.$$

Then

$$\{1, \ldots, \chi\} = \coprod_{1 \leq i \leq \hat{h}} i^*$$

is a partition into pairwise disjoint nonempty sets, and for $1 \leq i \leq \hat{h}$ we have

$$\Omega'_{B_i}(f_Y) = \prod_{j \in i^*} f^{(j)} \quad \text{with} \quad 0 < \deg_Y f^{(j)} \in (n/\hat{d}_i)\mathbb{Z} \quad \text{for all } j \in i^*$$

and

$$\mathrm{int}(f,\Omega'_{B_i}(f_Y)) = nS(B_i)\deg_Y\Omega'_{B_i}(f_Y) = [(\hat{d}_i/\hat{d}_{i+1})-1]\hat{r}_i$$

where int denotes intersection multiplicity.[17]

Example (DF3). Now, if we are in the nontrivial irreducible case of $F=f\in R^\natural$ with $\deg_Y f=n>1$, and if $h(c(f))=1$ with $c_1(f)\notin\mathbb{Z}$, then the conclusions of the above Remark (DF2) say that $\Omega'_{B_1}(f_Y)=(1/n)f_Y$ with $\mathrm{int}(f,f_Y)=(n-1)m_1(f)$, and $\mathrm{noc}(f,f^{(j)})=c_1(f)=m_1(f)/n$ for every irreducible factor $f^{(j)}$ of f_Y. To verify this in a particular situation, by taking $(w_1(X),\dots,w_{n-1}(X),w_n(X))=(0,\dots,0,X^e)$ in Example (TR3) of § 6 we have

$$F(X,Y)=f(X,Y)=Y^n+X^e\in R^\natural \text{ where } 0\neq e\in\mathbb{Z} \text{ with } \mathrm{GCD}(n,e)=1$$

and hence $h(T(f))=h(f)=1$ with $m_1(f)=e$ and

$$l_0(T(f))=-\infty \text{ and } l_1(T(f))=c_1(f)=e/n$$

and upon letting

$$B_i\in\hat{R}^\flat \text{ with } \sigma(B_i)=\{f\} \text{ and } \lambda(B_i)=l_i(T(f)) \text{ for } 0\le i\le 1$$

we have

$$T(F)=T(f)=\{B_0,B_1\}$$

with

$$D'(B_0)=0 \text{ and } D'(B_1)=n-1.$$

Now clearly $f_Y=nY^{n-1}$, and hence $\mathrm{Res}_Y(f,f_Y)=n^nX^{(n-1)e}$ and $f^{(j)}=Y$ for $1\le j\le\chi=n-1$, and therefore $\mathrm{int}(f,f_Y)=\mathrm{ord}_X\mathrm{Res}_Y(f,f_Y)=(n-1)e=(n-1)m_1(f)$ and $\mathrm{noc}(f,f^{(j)})=(1/n)\mathrm{ord}_Xf(X,0)=m_1(f)/n$ for $1\le j\le\chi=n-1$. This completes the verification.

Example (DF4). Next, if we are in the nontrivial irreducible case of $F=f\in R^\natural$ with $\deg_Y f=n>1$, and if $h(c(f))=2$ with $c_1(f)\notin\mathbb{Z}$, then the conclusions of the above Remark (DF2) say that $\Omega'_{B_1}(f_Y)\Omega'_{B_2}(f_Y)=(1/n)f_Y$ and for $1\le i\le 2$ we have that: $\deg_Y\Omega'_{B_i}(f_Y)=D'(B_i)=(n/d_{i+1})-(n/d_i)>0$ and $\mathrm{int}(f,\Omega'_{B_i}(f_Y))=[(d_i/d_{i+1})-1]r_i$ where $\Omega'_{B_i}(f_Y)$ is the product of all those irreducible factors $f^{(j)}$ of f_Y for which $\mathrm{noc}(f,f^{(j)})=c_i(f)$, and moreover for each of these $f^{(j)}$ we have $0<\deg_Y f^{(j)}\in(n/d_i)\mathbb{Z}$. To verify this in a particular situation, in Example (TR4) of § 6 we have

$$F(X,Y)=f(X,Y)=(Y^2-X^{2a+1})^2-X^{3a+b+2}Y\in R^\natural \text{ with } a\in\mathbb{Z} \text{ and } 0\le b\in\mathbb{Z}$$

with

$$\begin{cases} n=d_1=4 \text{ and } d_2=2 \text{ and } d_3=1, \text{ and} \\ c_1(f)=(2a+1)/2 \text{ and } c_2(f)=(4a+2b+3)/4, \text{ and} \\ [(d_1/d_2)-1]r_1=4a+2 \text{ and } [(d_2/d_3)-1]r_2=8a+2b+5 \end{cases}$$

[17] The intersection multiplicity $\mathrm{int}(f,g)$ of $f\in R^\natural$ with $g\in R$ is defined by putting $\mathrm{int}(f,g)=\mathrm{ord}_X\mathrm{Res}_Y(f,g)$, where $\mathrm{Res}_Y(f,g)$ denotes the Y-resultant of f and g; equivalently, for any $z(X)\in k((X))$ with $f(X^n,z(X))=0$ where $\deg_Y f=n$, we have $\mathrm{int}(f,g)=\mathrm{ord}_Xg(X^n,z(X))$; see pp. 286–287 of [Ab]. By (GNP7) we see that, given any $B\in R^\flat$ and $H\in R$, for every $f\in\sigma(B)$ we have $\mathrm{int}(f,\Omega'_B(H))=nS(B)\deg_Y\Omega'_B(H)$ where $\deg_Y f=n$.

and $h(T(f)) = 2$ with $l_0(T(f)) = -\infty$ and

$$l_1(T(f)) = c_1(f) = (2a+1)/2 \text{ and } l_2(T(f)) = c_2(f) = (4a+2b+3)/4$$

and upon letting

$$B_i \in \hat{R}^\flat \text{ with } \sigma(B_i) = \{f\} \text{ and } \lambda(B_i) = l_i(T(f)) \text{ for } 0 \le i \le 2$$

we have

$$T(f) = T(F) = \{B_0, B_1, B_2\}.$$

with $D'(B_0) = 0$ and

$$D'(B_1) = 1 \text{ and } D'(B_2) = 2.$$

Now

$$f_Y = 4Y(Y^2 - X^{2a+1}) - X^{3a+b+2}$$

and hence by (TR5) of § 6 we see that $f_Y = 4Yf^{(1)}f^{(2)}$ where

$$f^{(1)}(X,Y) = Y - v(X) \in R^\natural \text{ and } v(X) \in k((X))$$

with

$$\mathrm{ord}_X v(X) = a+b+1$$

and

$$f^{(2)}(X,Y) = Y^2 + \sum_{1\le i\le 2} v'_i(X)Y^{2-i} \in R^\natural \text{ and } v'_i(X) \in k((X))$$

with

$$\mathrm{ord}_X v'_1(X) > (2a+1)/2 \text{ and } \mathrm{ord}_X v'_2(X) = 2a+1.$$

Comparing coefficients of Y^2 and Y in the equation $f_Y = 4f^{(1)}f^{(2)}$ we see that $v'_1(X) - v(X) = 0$ and $v'_2(X) - v'_1(X)v(X) = -X^{2a+1}$, and hence

$$f^{(2)}(X,Y) = Y^2 + v(X)Y - X^{2a+1} + v(X)^2.$$

Applying the quadratic equation formula to the above equation we get the roots of $f^{(2)}(X^4, Y)$ to be

$$\begin{aligned}
Y &= (-1/2)v(X^4) \pm (1/2)\sqrt{4X^{8a+4} - 3v(X^4)^2} \\
&= (-1/2)v(X^4) \pm X^{4a+2}\sqrt{1 - (3/4)X^{-8a-4}v(X^4)^2} \\
&= (-1/2)[\mu X^{4a+4b+4} + (\text{terms of degree } > 4a+4b+4 \text{ in } X)] \\
&\quad \pm X^{4a+2}\sqrt{1 - (3/4)\mu^2 X^{8b+4} + (\text{terms of degree} > 8b+4 \text{ in } X)} \\
&= (-1/2)[\mu X^{4a+4b+4} + (\text{terms of degree} > 4a+4b+4 \text{ in } X)] \\
&\quad \pm X^{4a+2}[1 - (3/8)\mu^2 X^{8b+4} + (\text{terms of degree} > 8b+4 \text{ in } X)] \\
&= (-1/2)[\mu X^{4a+4b+4} + (\text{terms of degree} > 4a+4b+4 \text{ in } X)] \\
&\quad \pm [X^{4a+2} - (3/8)\mu^2 X^{4a+8b+6} + (\text{terms of degree} > 4a+8b+6 \text{ in } X)]
\end{aligned}$$

where for the third equality we are using the fact that $\text{ord}_X v(X) = a+b+1$ and hence

$$v(X^4) = \mu X^{4a+4b+4} + (\text{terms of degree} > 4a+4b+4 \text{ in } X) \text{ with } 0 \neq \mu \in k$$

and for the fourth equality we are using the binomial theorem for exponent $1/2$. It follows that

$$f^{(2)}(X^4, Y) = [Y - y_1(X)][Y - y_2(X)] \text{ where } y_1 \in k((X)) \text{ and } y_2(X) \in k((X))$$

are such that

$$y_1(X) = X^{4a+2} - (\mu/2)X^{4a+4b+4} + (\text{terms of degree} > 4a+4b+4 \text{ in } X)$$

and

$$y_2(X) = -X^{4a+2} - (\mu/2)X^{4a+4b+4} + (\text{terms of degree} > 4a+4b+4 \text{ in } X).$$

We also have

$$f^{(1)}(X^4, Y) = Y - v(X) \text{ where } v(X) \in k((X)) \text{ with } \text{ord}_X = 4a+4b+4.$$

Finally by (TR4) of § 6 we have

$$f(X^4, Y) = \prod_{1 \leq j \leq 4} [Y - z_j(X)]$$

with

$$z_j(X) = (\iota^j X)^{4a+2} + \tfrac{1}{2}(\iota^j X)^{4a+2b+3} + (\text{terms of degree} > 4a+2b+3 \text{ in } X)$$

where ι is a primitive 4-th root of 1 in k. By the above expressions for the roots of f and $f^{(1)}$ we get

$$\text{int}(f, f^{(1)}) = 4a+2 \text{ and } \text{noc}(f, f^{(1)}) = (2a+1)/2.$$

Likewise, by the above expressions for the roots of f and $f^{(2)}$ we get

$$\text{int}\,(f, f^{(2)}) = 8a+2b+5 \text{ and } \text{noc}(f, f^{(2)}) = (4a+2b+3)/4.$$

It follows that

$$\Omega'_{B_i}(f_Y) = f^{(i)} \text{ with } \deg_Y f^{(i)} = D'(B_i) \text{ for } 1 \leq i \leq 2$$

and this completes the verification.

Example (DF5). Finally, let us turn to the case of $F \in \hat{R}^\natural$ having two factors, i.e., such that $F = ff'$ with $f \in R^\natural$ and $f' \in R^\natural$. At the same time let us arrange matters so that F is pure meromorphic, i.e.,

$$F(X,Y) = \Phi(X^{-1}, Y) \text{ with } \Phi(X,Y) \in k[X,Y].$$

To do this, in Example (TR5) of § 6, let us take $n > 1$ with $b = 0$ and $a = -1$, and

$$\Phi(X,Y) = Y^{n+2} + \kappa' XY^n + \hat{\kappa} Y + \kappa + \sum_{3 \leq i \leq n} \kappa_i Y^{n+2-i} \in k[X,Y]$$

with

$$0 \neq \kappa' \in k \text{ and } 0 \neq \hat{\kappa} \in k \text{ and } 0 \neq \kappa \in k \text{ and } \kappa_i \in k \text{ for } 3 \leq i \leq n.$$

As explained in (TR5) of § 6, we then have

$$F(X,Y)=f(X,Y)f'(X,Y) \text{ with } f(X,Y)\neq f'(X,Y)$$

where

$$f(X,Y)=Y^n+\sum_{1\le i\le n} w_i(X)Y^{n-i}\in R^{\natural} \text{ and } w_i(X)\in k((X))$$

with

$$\mathrm{ord}_X w_i(X) > ie/n \text{ for } 1\le i\le n-1 \text{ and } \mathrm{ord}_X w_n(X)=e=1$$

and

$$f'(X,Y)=Y^2+\sum_{1\le i\le 2} w_i'(X)Y^{2-i}\in R^{\natural} \text{ and } w_i'(X)\in k((X))$$

with

$$\mathrm{ord}_X w_1'(X) > e'/2 \text{ and } \mathrm{ord}_X w_2'(X)=e'=-1$$

and $0\neq\kappa'\in k$ and $0\neq\kappa/\kappa'\in k$ are the coefficients of $X^{e'}$ and X^e in $w_2'(X)$ and $w_n(X)$ respectively. As explained in (TR5) of § 6, we also have $h(T(F))=2$ with $l_0(T(F))=-\infty$ and

$$l_1(T(F))=-1/2 \text{ and } l_2(T(F))=1/n$$

and upon letting

$$\begin{cases} B_0\in\hat{R}^{\flat} \text{ with } \sigma(B_0)=\{f,f'\} \text{ and } \lambda(B_0)=l_0(T(F)),\\ \text{and } B_1\in\hat{R}^{\flat} \text{ with } \sigma(B_1)=\{f,f'\} \text{ and } \lambda(B_1)=l_1(T(F)),\\ \text{and } B_2\in\hat{R}^{\flat} \text{ with } \sigma(B_2)=\{f\} \text{ and } \lambda(B_2)=l_2(T(F)),\\ \text{and } B_2'\in\hat{R}^{\flat} \text{ with } \sigma(B_2')=\{f'\} \text{ and } \lambda(B_2')=l_2(T(F)),\end{cases}$$

we have

$$T(F)=\{B_0,B_1,B_2,B_2'\}$$

with $D'(B_0)=0$ and

$$D'(B_1)=2 \text{ and } D'(B_2)=n-1 \text{ and } D'(B_2')=0.$$

Now

$$F_Y=(n+2)Y^{n+1}+n\kappa'X^{-1}Y^{n-1}+\hat{\kappa}+\sum_{3\le i\le n}(n+2-i)\kappa_iY^{n+1-i}$$

and hence by (TR5) of § 6 we see that $F_Y=(n+2)f^{(1)}f^{(2)}$ with $f^{(1)}\neq f^{(2)}$ where

$$f^{(2)}(X,Y)=Y^{n-1}+\sum_{1\le i\le n-1} v_i(X)Y^{n-1-i}\in R^{\natural} \text{ and } v_i(X)\in k((X))$$

with

$$\mathrm{ord}_X v_i(X) > ie/n \text{ for } 1\le i\le n-2 \text{ and } \mathrm{ord}_X v_{n-1}(X)=e=1$$

and

$$f^{(1)}(X,Y)=Y^2+\sum_{1\le i\le 2} v_i'(X)Y^{2-i}\in R^{\natural} \text{ and } v_i'(X)\in k((X))$$

with

$$\mathrm{ord}_X v_1'(X) > e'/2 \text{ and } \mathrm{ord}_X v_2'(X) = e' = -1$$

and $0 \neq n\kappa'/(n+2) \in k$ and $0 \neq \hat{\kappa}/(n\kappa') \in k$ are the coefficients of $X^{e'}$ and X^e in $v_2'(X)$ and $v_{n-1}(X)$ respectively. In view of (TR3) of § 6 we see that

$$f(X^n, Y) = \prod_{1 \le j \le n} [Y - z_j(X)]$$

where $z_j(X) \in k((X))$ is such that

$$z_j(X) = \omega^j \kappa^* X + (\text{terms of degree} > 1 \text{ in } X)$$

where ω is a primitive n-th root of 1 in k, and κ^* is an n-th root of $-\kappa/\kappa'$ in k, and

$$f'(X^2, Y) = \prod_{1 \le j \le 2} [Y - z_j'(X)]$$

where $z_j'(X) \in k((X))$ is such that

$$z_j'(X) = (-1)^j \kappa'^* X^{-1} + (\text{terms of degree} > -1 \text{ in } X)$$

where κ'^* is a square root of $-\kappa'$ in k. In view of (TR3) of § 6 we also see that

$$f^{(2)}(X^{n-1}, Y) = \prod_{1 \le j \le n-1} [Y - y_j(X)]$$

where $y_j(X) \in k((X))$ is such that

$$y_j(X) = \hat{\omega}^j \hat{\kappa}^* X + (\text{terms of degree} > 1 \text{ in } X)$$

where $\hat{\omega}$ is a primitive $(n-1)$-th root of 1 in k, and $\hat{\kappa}^*$ is an $(n-1)$-th root of $-\hat{\kappa}/(n\kappa')$ in k, and

$$f^{(1)}(X^2, Y) = \prod_{1 \le j \le 2} [Y - y_j'(X)]$$

where $y_j'(X) \in k((X))$ is such that

$$y_j'(X) = (-1)^j \hat{\kappa}'^* X^{-1} + (\text{terms of degree } > -1 \text{ in } X)$$

where $\hat{\kappa}'^*$ is a square root of $-n\kappa'/(n+2)$ in k. By the above expressions of the roots of $f, f', f^{(1)}, f^{(2)}$ we get

$$\begin{cases} \mathrm{int}\,(f, f^{(1)}) = -n \text{ and } \mathrm{noc}(f, f^{(1)}) = -1/2, \text{ and} \\ \mathrm{int}\,(f', f^{(1)}) = -2 \text{ and } \mathrm{noc}(f', f^{(1)}) = -1/2, \text{ and} \\ \mathrm{int}\,(f, f^{(2)}) = (n-1) \text{ and } \mathrm{noc}\,(f, f^{(2)}) = 1/n, \text{ and} \\ \mathrm{int}\,(f', f^{(2)}) = -(n-1) \text{ and } \mathrm{noc}(f', f^{(2)}) = -1/2, \end{cases}$$

and hence

$$\Omega'_{B_i}(F_Y) = f^{(i)} \text{ with } \deg_Y f^{(i)} = D'(B_i) \text{ for } 1 \le i \le 2$$

which verifies Theorem (DF1) in the present situation.

To get an example of $\hat{F}\in\hat{R}^\natural$ having three factors, we take $\hat{F}(X,Y)=\hat{\Phi}(X^{-1},Y)$ with $\hat{\Phi}(X,Y)=\Phi(X,Y)-\Phi(0,0)\in k[X,Y]$. Then by (TR5) of § 6 we get $\hat{F}=\hat{f}\hat{f}'\hat{f}''$ with $\hat{f}''\neq\hat{f}\neq\hat{f}'\neq\hat{f}''$ where $\hat{f}''(X,Y)=Y\in R^\natural$ and

$$\hat{f}(X,Y)=Y^{n-1}+\sum_{1\le i\le n-1}\hat{w}_i(X)Y^{n-1-i}\in R^\natural \text{ and } \hat{w}_i(X)\in k((X))$$

with

$$\mathrm{ord}_X\,\hat{w}_i(X)>ie/n \text{ for } 1\le i\le n-2 \text{ and } \mathrm{ord}_X\,\hat{w}_{n-1}(X)=e=1$$

and

$$\hat{f}'(X,Y)=Y^2+\sum_{1\le i\le 2}\hat{w}'_i(X)Y^{2-i}\in R^\natural \text{ and } \hat{w}'_i(X)\in k((X))$$

with

$$\mathrm{ord}_X\,\hat{w}'_1(X)>e'/2 \text{ and } \mathrm{ord}_X\,\hat{w}'_2(X)=e'=-1$$

and $0\neq\kappa'\in k$ and $0\neq\hat{\kappa}/\kappa'\in k$ are the coefficients of $X^{e'}$ and X^e in $\hat{w}'_2(X)$ and $\hat{w}_{n-1}(X)$ respectively. In view of (TR3) of § 6 we see that

$$\hat{f}(X^{n-1},Y)=\prod_{1\le j\le n-1}[Y-\hat{z}_j(X)]$$

where $\hat{z}_j(X)\in k((X))$ is such that

$$\hat{z}_j(X)=\hat{\omega}^j\hat{\kappa}^*X+(\text{terms of degree} > 1 \text{ in } X)$$

where $\hat{\omega}$ is a primitive $(n-1)$-th root of 1 in k, and $\hat{\kappa}^*$ is an $(n-1)$-th root of $-\hat{\kappa}/\kappa'$ in k, and

$$\hat{f}'(X^2,Y)=\prod_{1\le j\le 2}[Y-\hat{z}'_j(X)]$$

where $\hat{z}'_j(X)\in k((X))$ is such that

$$\hat{z}'_j(X)=(-1)^j\hat{\kappa}'^*X^{-1}+(\text{terms of degree} > -1 \text{ in } X)$$

where $\hat{\kappa}'^*$ is a square root of $-\kappa'$ in k. By the above expressions of the roots of $\hat{f}$ and $\hat{f}'$ it follows that $h(T(\hat{F}))=2$ with $l_0(T(\hat{F}))=-\infty$ and

$$l_1(T(\hat{F}))=-1/2 \text{ and } l_2(T(\hat{F}))=1/(n-1)$$

and upon letting

$$\begin{cases} \hat{B}_0\in\hat{R}^\flat \text{ with } \sigma(\hat{B}_0)=\{\hat{f},\hat{f}',\hat{f}''\} \text{ and } \lambda(\hat{B}_0)=l_0(T(\hat{F})),\\ \text{and } \hat{B}_1\in\hat{R}^\flat \text{ with } \sigma(\hat{B}_1)=\{\hat{f},\hat{f}',\hat{f}''\} \text{ and } \lambda(\hat{B}_1)=l_1(T(\hat{F})),\\ \text{and } \hat{B}_2\in\hat{R}^\flat \text{ with } \sigma(\hat{B}_2)=\{\hat{f},\hat{f}''\} \text{ and } \lambda(\hat{B}_2)=l_2(T(\hat{F})),\\ \text{and } \hat{B}'_2\in\hat{R}^\flat \text{ with } \sigma(\hat{B}'_2)=\{\hat{f}'\} \text{ and } \lambda(\hat{B}'_2)=l_2(T(\hat{F})),\end{cases}$$

we have

$$T(\hat{F})=\{\hat{B}_0,\hat{B}_1,\hat{B}_2,\hat{B}'_2\}$$

with $D'(\hat{B}_0)=0$ and

$$D'(\hat{B}_1)=2 \text{ and } D'(\hat{B}_2)=n-1 \text{ and } D'(\hat{B}'_2)=0.$$

Now $\hat{F}_Y = F_Y = (n+2)f^{(1)}f^{(2)}$, and by the above expressions of the roots of $\hat{f}, \hat{f}'$, $f^{(1)}, f^{(2)}$ we get

$$\begin{cases} \text{int}\,(\hat{f}, f^{(1)}) = -(n-1) \text{ and } \text{noc}(\hat{f}, f^{(1)}) = -1/2, \text{ and} \\ \text{int}\,(\hat{f}', f^{(1)}) = -2 \text{ and } \text{noc}\,(\hat{f}', f^{(1)}) = -1/2, \text{ and} \\ \text{int}\,(\hat{f}'', f^{(1)}) = -1 \text{ and } \text{noc}\,(\hat{f}'', f^{(1)}) = -1/2, \text{ and} \\ \text{int}\,(\hat{f}, f^{(2)}) = (n-1) \text{ and } \text{noc}\,(\hat{f}, f^{(2)}) = 1/(n-1), \text{ and} \\ \text{int}\,(\hat{f}', f^{(2)}) = -(n-1) \text{ and } \text{noc}\,(\hat{f}', f^{(2)}) = -1/2, \text{ and} \\ \text{int}\,(\hat{f}'', f^{(2)}) = 1 \text{ and } \text{noc}\,(\hat{f}'', f^{(2)}) = 1/(n-1), \end{cases}$$

and hence

$$\Omega'_{\hat{B}_i}(\hat{F}_Y) = f^{(i)} \text{ with } \deg_Y f^{(i)} = D'(\hat{B}_i) \text{ for } 1 \le i \le 2$$

which again verifies Theorem (DF1) in the present situation.

9. Factorization of the jacobian

If $T = T(FG)$ where $F \in R \setminus k((X))$ and $0 \neq G \in R$, then for every $B \in T$ with $\Omega_B(G) = 1$ we clearly have $R^*(F, B) = R^*(B)$. Therefore by (SP76), (SP77), (SP79) and (SP80) we get the following jacobian factorization theorem.

Theorem (JF1). *Let $T = T(FG)$ where $F \in R \setminus k((X))$ is devoid of multiple factors in R, and $0 \neq G \in R$. Then we have the following.*

(JF1.1) *If $B \in T$ is such that $\Omega_B(G) = 1$ and $S(G, B) \neq 0$ then we have*

$$\deg_Y \Omega_B(J(F,G)) = \deg_Y \Omega_B((FG)_Y) = \deg_Y \Omega_B(F_Y) = D''(B)$$

and

$$\deg_Y \Omega'_B(J(F,G)) = \deg_Y \Omega'_B((FG)_Y) = \deg_Y \Omega'_B(F_Y) = D'(B)$$

and

$$\Omega_B(J(F,G)) = \Omega'_B(J(F,G)) \prod_{B' \in \pi(T,B)} \Omega'_{B'}(J(F,G))$$

where for every $B' \in \pi(T, B)$ we have $\Omega_{B'}(G) = 1$ and $S(G, B') \neq 0$ and

$$\deg_Y \Omega_{B'}(J(F,G)) = \deg_Y \Omega_{B'}((FG)_Y) = \deg_Y \Omega_{B'}(F_Y) = D''(B')$$

and

$$\deg_Y \Omega'_{B'}(J(F,G)) = \deg_Y \Omega'_{B'}((FG)_Y) = \deg_Y \Omega'_{B'}(F_Y) = D'(B').$$

(JF1.2) *If $B \in T$ is such that $\Omega_B(G) = 1$ then we have*

$$\deg_Y \Omega_B((FG)_Y) = \deg_Y \Omega_B(F_Y) = D''(B)$$

and

$$\deg_Y \Omega'_B((FG)_Y) = \deg_Y \Omega'_B(F_Y) = D'(B)$$

and

$$\Omega_B((FG)_Y) = \Omega'_B((FG)_Y) \prod_{B' \in \pi(T,B)} \Omega'_{B'}((FG)_Y)$$

where for every $B' \in \pi(T,B)$ *we have* $\Omega_{B'}(G)=1$ *and*

$$\deg_Y \Omega_{B'}((FG)_Y)=\deg_Y \Omega_{B'}(F_Y)=D''(B')$$

and

$$\deg_Y \Omega'_{B'}((FG)_Y)=\deg_Y \Omega'_{B'}(F_Y)=D'(B').$$

Remark (JF2). The jacobian factorization (JF1.1) was based on (SP80), and it invoked the jacobian estimates (JE1) to (JE3) of § 5 only in the special case when $\deg_Y \mathrm{minco}_X [z,V,W](G)=0$. Elsewhere we shall discuss a more refined Jacobian factorization based on (SP30) by invoking the general case of (JE1) to (JE3).

Example (JF3). Now let us illustrate Theorem (JF1.1) by the example

$$F=F(X,Y)=Y^n+X^e \in R^\natural \text{ where } 0\neq e\in \mathbb{Z} \text{ with } \mathrm{GCD}(n,e)=1$$

considered in (DF3) of § 8. For $G=X$ we have $J(F,G)=F_Y$ and we are reduced to (DF3)

Example (JF4). Next let us illustrate Theorem (JF1.1) by the example

$$F=F(X,Y)=(Y^2-X^{2a+1})^2-X^{3a+b+2}Y \in R^\natural \text{ with } a\in\mathbb{Z} \text{ and } 0\leq b\in\mathbb{Z}$$

considered in (DF4) of § 8. Again, for $G=X$ we have $J(F,G)=F_Y$ and we are reduced to (DF4). Moreover, for

$$\hat{G}=\hat{G}(X,Y)=Y\in R^\natural$$

we have

$$J(F,\hat{G})=F_X=-(4a+2)X^{2a}\hat{F}$$

with

$$\hat{F}=\hat{F}(X,Y)=Y^2+(3a+b+2)(4a+2)^{-1}X^{a+b+1}-X^{2a+1}.$$

By (TR3) and (TR4) of § 6, it follows that $h(T(F\hat{G}))=2$ with $l_0(T(F\hat{G}))=-\infty$ and

$$l_1(T(F\hat{G}))=(2a+1)/2 \text{ and } l_2(T(F\hat{G}))=(4a+2b+3)/4$$

and upon letting

$$\begin{cases} \hat{B}_0\in \hat{R}^\flat \text{ with } \sigma(\hat{B}_0)=\{F,\hat{G}\} \text{ and } \lambda(\hat{B}_0)=l_0(T(F\hat{G})),\\ \text{and } \hat{B}_1\in \hat{R}^\flat \text{ with } \sigma(\hat{B}_1)=\{F,\hat{G}\} \text{ and } \lambda(\hat{B}_1)=l_1(T(F\hat{G})),\\ \text{and } \hat{B}_2\in \hat{R}^\flat \text{ with } \sigma(\hat{B}_2)=\{F\} \text{ and } \lambda(\hat{B}_2)=l_2(T(F\hat{G})),\\ \text{and } \hat{B}'_2\in \hat{R}^\flat \text{ with } \sigma(\hat{B}'_2)=\{\hat{G}\} \text{ and } \lambda(\hat{B}'_2)=l_2(T(F\hat{G})),\end{cases}$$

we have

$$T(F\hat{G})=\{\hat{B}_0,\hat{B}_1,\hat{B}_2,\hat{B}'_2\}$$

with $D'(\hat{B}_0)=0$ and

$$D'(\hat{B}_1)=2 \text{ and } D'(\hat{B}_2)=2 \text{ and } D'(\hat{B}'_2)=0.$$

By (TR3) and (TR4) we also see that $\hat{F} \in R^{\natural}$ with

$$\mathrm{noc}(\hat{G}, \hat{F}) = (2a+1)/4 \text{ and } \mathrm{noc}(F, \hat{F}) = (4a+2b+3)/4$$

and hence

$$\Omega'_B(J(F,\hat{G})) = \begin{cases} \hat{F} & \text{if } B = \hat{B}_2 \\ 1 & \text{if } B = \hat{B}_0 \text{ or } B = \hat{B}_1 \text{ or } B = \hat{B}'_2 \end{cases}$$

in accordance with Theorem (JF1.1). Likewise, for

$$\tilde{G} = \tilde{G}(X,Y) = Y^2 - X^{2a+1} \in R^{\natural}$$

we have

$$J(F,\tilde{G}) = J(-X^{3a+b+2}Y, \tilde{G}) = \begin{cases} -(6a+2b+4)X^{3a+b+1}\tilde{F} & \text{if } 3a+b+2 \neq 0 \\ -(2a+1)X^{2a}\tilde{F} & \text{if } 3a+b+2 = 0 \end{cases}$$

with

$$\tilde{F} = \tilde{F}(X,Y) = \begin{cases} Y^2 + (2a+1)(6a+2b+4)^{-1}X^{2a+1} & \text{if } 3a+b+2 \neq 0 \\ 1 & \text{if } 3a+b+2 = 0. \end{cases}$$

By (TR3) and (TR4) of § 6, it follows that $h(T(F\tilde{G})) = 2$ with $l_0(T(F\tilde{G})) = -\infty$ and

$$l_1(T(F\tilde{G})) = (2a+1)/2 \text{ and } l_2(T(F\tilde{G})) = (4a+2b+3)/4$$

and upon letting

$$\begin{cases} \tilde{B}_0 \in \hat{R}^{\flat} \text{ with } \sigma(\tilde{B}_0) = \{F, \tilde{G}\} \text{ and } \lambda(\tilde{B}_0) = l_0(T(F\tilde{G})), \\ \text{and } \tilde{B}_1 \in \hat{R}^{\flat} \text{ with } \sigma(\tilde{B}_1) = \{F, \tilde{G}\} \text{ and } \lambda(\tilde{B}_1) = l_1(T(F\tilde{G})), \\ \text{and } \tilde{B}_2 \in \hat{R}^{\flat} \text{ with } \sigma(\tilde{B}_2) = \{F, \tilde{G}\} \text{ and } \lambda(\tilde{B}_2) = l_2(T(F\tilde{G})), \end{cases}$$

we have

$$T(F\tilde{G}) = \{\tilde{B}_0, \tilde{B}_1, \tilde{B}_2\}$$

with $D'(\tilde{B}_0) = 0$ and

$$D'(\tilde{B}_1) = 3 \text{ and } D'(\tilde{B}_2) = 4.$$

Thus the stem of every bud of $T(F\tilde{G})$ contains F as well as $\tilde{G}$, and hence Theorem (JF1.1) does not predict any factors of $J(F,\tilde{G})$. This is quite satisfactory when $3a+b+2=0$ because then $J(F,\tilde{G}) = -(2a+1)X^{2a+1}$ and so $J(F,\tilde{G})$ has no factor involving Y. A particularly interesting case of $3a+2b+2=0$ is the pure meromorphic case when $(a,b) = (-1,1)$. In that case, as noted in (TR4) of § 6, we have $F(X,Y) = \Phi(X^{-1}, Y)$ where $\Phi(X,Y) \in k[X,Y]$ is the variable $\Phi(X,Y) = (Y^2-X)^2 - Y$; indeed, then $k[X,Y] = k[\Phi,\Psi]$ where $\Psi(X,Y) = Y^2 - X$ with $\tilde{G}(X,Y) = \Psi(X^{-1}, Y)$.

Example (JF5). Finally let us illustrate Theorem (JF1.1) by the example

$$F = F(X,Y) = \Phi(X^{-1}, Y)$$

where

$$\Phi(X,Y) = Y^{n+2} + \kappa' XY^n + \hat{\kappa}Y + \kappa + \sum_{3\le i\le n} \kappa_i Y^{n+2-i} \in k[X,Y]$$

with $n > 1$ and

$$0 \neq \kappa' \in k \text{ and } 0 \neq \hat{\kappa} \in k \text{ and } \kappa \in k \text{ and } \kappa_i \in k \text{ for } 3 \le i \le n$$

considered in (DF5) of § 8 (where we used the notation $\hat{\Phi}$ and $\hat{F}$ for the special case of $\kappa = 0$). Once again, for $G = -X$ we have $J(F,G) = F_Y$ and we are reduced to (DF5). Note that the affine plane curve $\Phi = 0$ is nonsingular (at finite distance) for every κ; equivalently, Φ_X and Φ_Y have no solution in the affine plane $k \times k$. Moreover, Φ is irreducible for every $\kappa \neq 0$, but reducible for $\kappa = 0$. In (DF5) we have shown that Φ has two or three places at ∞, i.e., F has two or three factors in $R^\natural$, according as $\kappa \neq 0$ or $\kappa = 0$. Further interest in this nice family of bivariate polynomials Φ lies in the fact that it provides a convenient testing ground for the trivariate Jacobian conjecture. To elucidate this, given any $H_1 \in k[X_1,\ldots,X_r]$ where r is any positive integer, let us say that H_1 is a variable in $k[X_1,\ldots,X_r]$ to mean that $k[X_1,\ldots,X_r] = k[H_1,\ldots,H_r]$ for some $H_2,\ldots,H_r$ in $k[X_1,\ldots,X_r]$, and let us say that H_1 is a weak variable in $k[X_1,\ldots,X_r]$ to mean that $0 \neq J(H_1,\ldots,H_r) \in k$ for some $H_2,\ldots,H_r$ in $k[X_1,\ldots,X_r]$ where $J(H_1\ldots,H_r)$ is the jacobian of $H_1,\ldots,H_r$ with respect to $X_1,\ldots,X_r$. The reducibility of Φ when $\kappa = 0$ shows that Φ is not a variable in $k[X,Y]$. The reducibility of Φ when $\kappa = 0$ also shows that Φ is not a variable in $k[X,Y,Z]$. It can be shown that Φ is not a weak variable in $k[X,Y]$, at least when $n+1$ is a prime number. However, as was pointed out to us by Ignacio Luengo, it is not known whether Φ is or is not a weak variable in $k[X,Y,Z]$, even when $n = 2$. To see that Φ is not a weak variable in $k[X,Y]$, first note that by the automorphism $(X,Y) \mapsto ((X - Y^2)/\kappa', Y)$ we can send Φ to the polynomial

$$XY^n + \hat{\kappa}Y + \kappa + \sum_{3\le i\le n} \kappa_i Y^{n+2-i} \in k[X,Y]$$

whose degree is $n+1$ and whose degree form XY^n has two coprime factors. On the other hand, it can easily be shown that if $H \in k[X,Y]$ is a weak variable in $k[X,Y]$ of prime degree then its degree form must be a power of a linear form.

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