IMPROVEMENT OF A THEOREM OF LINNIK AND WALFISZ

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1. On the basis of the hitherto unproved "extended Riemann hypothesis", Littlewood (1) proved that there are infinitely many k such that

$$L(1) = \sum_1^\infty \frac{\chi(n)}{n} < \frac{\{1+o(1)\}}{\log\log k}\frac{\pi^2}{6}e^{-C}, \tag{1}$$

where $\chi(n)$ denotes a real primitive character (mod k), and C is Euler's constant.

Independently of each other, and almost simultaneously, Linnik (2), Walfisz (3) and I (4) proved the following results without assuming any hypothesis:

(I) There are infinitely many k such that

$$\sum_1^\infty \frac{\chi(n)}{n} < \frac{A}{\sqrt{(\log\log k)}},$$

where A is a certain absolute positive constant, and $\chi(n)$ is a real *primitive* character (mod k).

(II) There are infinitely many k such that

$$\sum_1^\infty \frac{\chi(n)}{n} < \epsilon,$$

where ϵ is an arbitrary positive number, and $\chi(n)$ is a real primitive character (mod k).

Of these results (II) was proved by me; the sharper result (I) is due to Linnik and Walfisz. I now find that a simple sharpening of my method used to prove (II) will prove Littlewood's result without assuming "the extended Riemann hypothesis". In fact, all we have to do is to replace the number g of my paper (4) by

$$\left[\frac{\log x}{(\log\log x)^2}\right],$$

where $[t]$ denotes the greatest integer contained in t.

As my paper (4) contains misprints (nor is it easily available) I develop the whole argument without any reference to this paper. We actually prove somewhat more than Littlewood's conjecture, namely, theorems 1 and 2 of § 11 (towards the end of this paper).

2. *Definitions.* p_m denotes the mth odd prime,

$$a = p_1 p_2 p_3 \cdots p_g, \tag{2}$$

where g is defined by (3) below; b is a positive integer such that $(b/p_r) = -1$ for $1 \leqq r \leqq g$, $b \equiv 5 \pmod 8$ and $1 < b < 8a$; x is a sufficiently large positive integer,

$$g = \left[\frac{\log x}{(\log\log x)^2}\right]; \tag{3}$$

(n/m) is the Jacobi symbol if m is odd and prime to n, but is 0 in all other cases (i.e. when m is even or when m and n have a common factor). We write

$$T(x) = \sum_{x<n\leqq 2x} \sum_{m=1}^{\infty} \frac{1}{m}\left(\frac{8an+b}{m}\right) \tag{4}$$

and

$$S(x) = \sum_{x<n\leqq 2x} \sum_{m=1}^{x^{\frac{1}{4}}} \frac{1}{m}\left(\frac{8an+b}{m}\right). \tag{5}$$

We observe (for further reference) the fact that we may write

$$\sum_{m=1}^{\infty} \frac{1}{m}\left(\frac{m}{8an+b}\right) = \sum_{1}^{\infty} \frac{\chi(m)}{m},$$

where $\chi(m)$ is a real primitive character $(\operatorname{mod} k)$ if $8an+b$ is *quadratfrei* and $k = 8an+b$. We also note that

$$\left(\frac{m}{8an+b}\right) = \left(\frac{8an+b}{m}\right),$$

provided m is odd, which is the reciprocity law for Jacobi's symbol.

3. We first prove that $\quad T(x) = S(x) + O(x^{\frac{3}{4}}).$ (6)

The proof of (6) needs

LEMMA 1. *If $\chi(n)$ is a non-principal character* $(\operatorname{mod} k)$, *then*

$$\sum_{n=u}^{v} \chi(n) = O(\sqrt{k}\log k).$$

This is a well-known result when $\chi(n)$ is a primitive character $(\operatorname{mod} k)$; the extension (due to Davenport) to non-principal characters $\chi(n)$ is easily made.

LEMMA 2. *We have* $a < x^{\frac{1}{30}}$ $(x > x_0)$.

Proof. For $x > x_0$, $\log a < \vartheta(p_g) < 2p_g < 3g \log g < 3 \log x/\log\log x < \frac{1}{30} \log x$.

Now, using lemma 1,

$$T(x) - S(x) = \sum_{x<n\leqq 2x} \sum_{m>x^{3/4}} \frac{1}{m}\left(\frac{8an+b}{m}\right)$$

$$= O\left\{x\frac{\sqrt{(ax)}\log(ax)}{x^{3/4}}\right\} = O(x^{4/5}),$$

by lemma 2 and the fact that $\left(\dfrac{8an+b}{m}\right)$ is a character (mod $2(8an+b)$) (since the symbol is 0, by definition, whenever m is even). Thus (6) is proved.

4. We also need the following two lemmas:

LEMMA 3. *The number of quadratfrei integers* $8an+b$ $(x < n \leqq 2x)$ *is* $\{1+o(1)\}x$.

Proof. The number $8an+b$ cannot be divisible by p_r^2 when $1 \leqq r \leqq g$. Now the number of numbers $8an+b$ $(x<n\leqq 2x)$ which are divisible by p_r^2 $(r>g)$ is clearly of the order

$$\sum_{r>g} \frac{x}{p_r^2} = O\left(\frac{x}{p_g}\right) = O\left(\frac{x}{g\log g}\right)$$

$$= o(x).$$

Hence we obtain lemma 3.

LEMMA 4. *Let* $F(y)$ *denote the number of positive integers* m *such that*

(i) $$m \leqq y,$$

(ii) $$m = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_g^{\alpha_g} \quad (\alpha\text{'s} \geqq 0).$$

Then $$F(y) < \sqrt[4]{y}, \quad \textit{when} \quad y > x^{1/4},$$

where g *is as defined above.*

Proof. The number of positive integers of the form p^t (p fixed $\geqq 2$; $t = 0, 1, 2, 3, \dots$) and not exceeding y is clearly less than $2(\log y/\log p) < 3 \log y$, whenever $p \leqq p_g$ and $x > x_0$. Hence, for $x > x_0$,

$$\log F(y) < g \log(3 \log y).$$

Now $$g = \frac{\log x}{(\log\log x)^2} = O\left(\frac{\log y}{(\log\log y)^2}\right)$$

since $x < y^4$; hence $$\log F(y) = O(\log y/\log\log y), \tag{7}$$

and lemma 4 follows at once.

5. It is clear from (5) that

$$S(x) = S_1(x) + S_2(x) + S_3(x), \tag{8}$$

where, for $r = 1, 2, 3$,

$$S_r(x) = \sum_{x<n\leqq 2x} \sum_{m\leqq x^{\frac{1}{2}}} \frac{1}{m}\left(\frac{8an+b}{m}\right), \tag{9}$$

and m runs through different sets of values (described below) in the 3 sums:

(i) In $S_1(x)$, m takes all values ($\leqq x^{\frac{1}{2}}$) of the form $p_1^{\alpha_1} p_2^{\alpha_2} \dots p_g^{\alpha_g}$ (α's $\geqq 0$), i.e. m is not divisible by any prime greater than p_g.

(ii) In $S_2(x)$, m takes only values of the form $m = m_1 m_2$, where $m_1 = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_g^{\alpha_g}$ (α's $\geqq 0$), while $m_2 = Q^2 M$, where $(m_2, a) = 1$, and M is *quadratfrei* and greater than 1 (so that m and m_2 cannot be perfect squares).

(iii) In $S_3(x)$, m takes only values of the form $m_1 Q^2$, where

$$m_1 = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_g^{\alpha_g} \quad (\alpha\text{'s} \geqq 0) \quad \text{and} \quad (Q, a) = 1, \; Q > 1.$$

It is clear that these three types of m are non-overlapping and exhaust all positive integers m, and so (8) is rendered obvious.

6. In $S_1(x)$ we clearly have

$$\left(\frac{8an+b}{m}\right) = \left(\frac{b}{m}\right) = \lambda(m),$$

where $\lambda(m)$ is Liouville's function (Landau, *Handbuch der Primzahlen*, 2, (1909), 617) defined as follows:

$$\lambda(1) = 1, \quad \lambda(n) = (-1)^{\beta_1+\beta_2+\beta_3+\dots+\beta_t},$$

where $n = q_1^{\beta_1} q_2^{\beta_2} \dots q_t^{\beta_t}$, and the q's are distinct primes (β's > 0). Hence

$$S_1(x) = x \sum_{m\leqq x^{\frac{1}{2}}} \frac{\lambda(m)}{m}, \tag{10}$$

where m runs only through positive integers of the form $p_1^{\alpha_1} p_2^{\alpha_2} \dots p_g^{\alpha_g}$ (each $\alpha \geqq 0$).

7. In $S_2(x)$ we have $m = m_1 m_2$ and so

$$\left(\frac{8an+b}{m}\right) = \left(\frac{8an+b}{m_1}\right)\left(\frac{8an+b}{m_2}\right)$$

$$= \lambda(m_1)\left(\frac{8an+b}{m_2}\right),$$

since m_1 is not divisible by any prime greater than p_g. Since m_2 is not a perfect square, we have

$$\sum \left(\frac{8an+b}{m_2}\right) = 0,$$

where n runs through a complete set of residues $(\bmod\, m_2)$; and hence

$$\sum_{n=u}^{v}\left(\frac{8an+b}{m_2}\right) = O(m_2) = O(m),$$

$$\sum_{n=u}^{v}\left(\frac{8an+b}{m}\right) = O(m).$$

Hence
$$S_2(x) = O\left(\sum_{m\leq x^{1/4}} \frac{m}{m}\right) = O(x^{1/4}). \tag{11}$$

8. In $S_3(x)$, m runs through numbers of the type $m_1 Q^2$ where m_1 is of the form $p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_g^{\alpha_g}$ while $(Q, a) = 1$, $Q > 1$. Hence

$$\begin{aligned}
|S_3(x)| &= \left|\sum_{x<n\leq 2x}\sum_{m\leq x^{1/4}} \frac{1}{m}\left(\frac{8an+b}{m}\right)\right| \leq \sum_{x<n\leq 2x}\sum_{m\leq x^{1/4}}\frac{1}{m} \\
&= x\sum_{m\leq x^{1/4}}\frac{1}{m} < x\left\{\prod_{r=1}^{g}\left(1+\frac{1}{p_r}+\frac{1}{p_r^2}+\frac{1}{p_r^3}+\ldots\right)\right\}\sum_{\substack{(n,a)=1\\ n>1}}\frac{1}{n^2} \\
&= O(x\log p_g)\sum_{n>p_g}\frac{1}{n^2} = O\left(\frac{x\log(g\log g)}{p_g}\right) = O\left(\frac{x}{g}\right) \\
&= O\left(\frac{x(\log\log x)^2}{\log x}\right),
\end{aligned}$$

and so
$$S_3(x) = O\left(\frac{x(\log\log x)^2}{\log x}\right). \tag{12}$$

9. From (6), (8), (10), (11), (12), we get

$$\begin{aligned}
T(x) &= \sum_{x<n\leq 2x}\sum_{m=1}^{\infty}\frac{1}{m}\left(\frac{8an+b}{m}\right) \\
&= x\sum_{m\leq x^{1/4}}\frac{\lambda(m)}{m} + O\left(\frac{x(\log\log x)^2}{\log x}\right).
\end{aligned} \tag{13}$$

10. We now proceed to consider the sum

$$\sum_{m\leq x^{1/4}}\frac{\lambda(m)}{m},$$

which occurs in (13) above. We have

$$\begin{aligned}
\sum_{m\leq x^{1/4}}\frac{\lambda(m)}{m} &= \sum_{m}\frac{\lambda(m)}{m} - \sum_{m>x^{1/4}}\frac{\lambda(m)}{m} \\
&= \alpha(x) - \beta(x),
\end{aligned} \tag{14}$$

where, in $\alpha(x)$, m runs over *all* positive integers of the form $p_1^{\alpha_1} p_2^{\alpha_2} \dots p_g^{\alpha_g}$ (each $\alpha \geqq 0$); in $\beta(x)$, m runs over all such values which are, in addition, greater than $x^{\frac{3}{4}}$. Now

$$\alpha(x) = \prod_{r=1}^{g} \left(1 - \frac{1}{p_r} + \frac{1}{p_r^2} - \frac{1}{p_r^3} + \dots\right)$$

$$= \prod_{r=1}^{g} \frac{(1-1/p_r)}{(1-1/p_r^2)} \sim \frac{2e^{-C}}{\log p_g} \frac{\pi^2}{8}$$

$$\sim \frac{\pi^2}{4} \frac{e^{-C}}{\log\log x}. \tag{15}$$

Again,

$$\left.\begin{aligned} \beta(x) &= O\left(\sum_{m > x^{\frac{3}{4}}} \frac{1}{m}\right), \\ \sum_{m > x^{\frac{3}{4}}} \frac{1}{m} &= \sum_{q > x^{\frac{3}{4}}} \frac{F(q) - F(q-1)}{q}, \end{aligned}\right\} \tag{16}$$

where q runs over all positive integers ($> x^{\frac{3}{4}}$), and $F(q)$ is as in lemma 4. By partial summation, from (16), using $F(y) < y^{\frac{1}{4}}$, we have

$$\beta(x) = O((x^{\frac{3}{4}})^{\frac{1}{4}} (x^{\frac{3}{4}})^{-1}) = O(x^{-\frac{9}{16}}) \tag{17}$$

by lemma 4.

From (14), (15), (16), (17) we get

$$\sum_{m \leq x^{\frac{3}{4}}} \frac{\lambda(m)}{m} \sim \frac{\pi^2 e^{-C}}{4 \log\log x}. \tag{18}$$

11. From (13) and (18) we get

$$\sum_{x < n \leqq 2x} \sum_{m=1}^{\infty} \frac{1}{m} \left(\frac{8an+b}{m}\right) \sim \frac{\pi^2}{4} \frac{e^{-C} x}{\log\log x}. \tag{19}$$

Now, from the reciprocity theorem for Jacobi's symbol, we have

$$\left(\frac{8an+b}{m}\right) = \left(\frac{m}{8an+b}\right) \tag{20}$$

if m is odd; and, by definition,

$$\left(\frac{8an+b}{m}\right) = 0$$

if m is even. It now follows, from (19) and lemma 3, that there exists an integer n with $x < n \leqq 2x$ and such that $8an+b$ is *quadratfrei*, and further

$$\sum_{m \text{ odd}} \frac{1}{m} \left(\frac{m}{8an+b}\right) > \frac{\pi^2}{4} e^{-C} \frac{\{1 + o(1)\}}{\log\log x}. \tag{21}$$

Now $$\sum_{m=1}^{\infty}\frac{1}{m}\left(\frac{m}{8an+b}\right)=\left(1-\frac{1}{2}+\frac{1}{2^2}-\frac{1}{2^3}+\dots\right)\sum_{m\text{ odd}}\frac{1}{m}\left(\frac{m}{8an+b}\right)$$
$$=\frac{2}{3}\sum_{m\text{ odd}}\frac{1}{m}\left(\frac{m}{8an+b}\right), \tag{22}$$

since $\left(\frac{2}{8an+b}\right)=-1$ [using $b\equiv 5 \pmod 8$]. From (21) and (22) we get the following result:

THEOREM 1. *For $x>x_0$, there exists a positive integer n satisfying*

(i) $x<n\leqq 2x$,

(ii) $8an+b$ *is quadratfrei, and*

(iii) $$\sum_{m=1}^{\infty}\frac{1}{m}\left(\frac{m}{8an+b}\right)>\frac{\pi^2}{6}e^{-C}\frac{\{1+o(1)\}}{\log\log(8an+b)},$$

since $\log\log(8an+b)\sim\log\log x$.

Again, since $\left(\frac{m}{8an+b}\right)$ is a real primitive character (mod $(8an+b)$), when m runs through all positive integral values (because $8an+b$ is *quadratfrei*) we can write theorem 1 as follows:

THEOREM 2. *There exist infinitely many k such that*

$$L(1)=\sum_{1}^{\infty}\frac{\chi(n)}{n}>\{1+o(1)\}\frac{\pi^2}{6}e^{-C}\frac{1}{\log\log k},$$

where $\chi(n)$ denotes a real primitive character (mod k). *In fact such a k exists between x and $2x$ for all large x.*

References

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