Some Calculations for Hilbert Modules

R. G. Douglas and G. Misra

1 Introduction

In a series of papers [3], [4], [5], the first author has studied a new approach to the study of bounded linear operators on Hilbert space based on the notion of Hilbert modules for function algebras. Although one can reformulate many single operator results in this new language, it's real promise lies in the possibilities for the case of several variables and the connections which are made between operator theory and algabric geometry. Before this can be realized, however, ground work must be laid and examples worked out. Unfortunately, all examples, no matter how elementary, depend on rather messy and nontrivial calculations. In this note we record some such calculations obtained during the summer, 1986, while the second author was supported as a postdoctral fellow on the National Science Foundation Grant of the first. Although, we will try to put these calculations in context we will not try to draw any final conclusions from them. This note is a preliminary report on work in progress.

2 Higher Multiplicity Localisations

In [3] the good class of Silov modules is introduced and arbitrary modules are to be studied via Silov resolutions, that is by resolutions of the given module in terms of Silov modules. For modules over the disk algebra one should recover the canonical model of Sz-Nagy and Foise and in [4] it is shown how to do that. A localisation technique is introduced based on the module tensor product. The resolution is tensored with the local module $\mathbb{C}_w$ which represent point evaluation at w in the open unit disk. For more general algebras one will have to consider higher multiplicity localisation. Although we work this out for the disk algebra, most of the calculations extend to other function algebras.

For w_1 and w_2 in the open unit disk, let $\mathbb{C}^{\alpha}_{w_1,w_2}$ denote the Hilbert module $\mathbb{C}^2$ defined for $\mathcal{A} = A(\mathbb{D})$ such that

$$r\begin{pmatrix} \lambda \\ \mu \end{pmatrix} = \begin{pmatrix} r(w_1) & \alpha\frac{\Delta r}{\Delta w} \\ 0 & r(w_2) \end{pmatrix}\begin{pmatrix} \lambda \\ \mu \end{pmatrix},$$

where $\frac{\Delta r}{\Delta w} = \frac{r(w_1)-r(w_2)}{w_1-w_2}$ and $\alpha \geq 0$. We begin by calculating an orthonormal basis for $\mathcal{M} \otimes_{\mathcal{A}} \mathbb{C}^{\alpha}_{w_1,w_2}$. For $\mathcal{M}$ a Hilbert module over $\mathcal{A}(\mathbb{D})$, let h, k be any two vectors in $\mathcal{M}$

and $\binom{\lambda}{\mu}$ be in in $\mathbb{C}^2$. The subspace $\mathcal{N}$ of $\mathcal{M}$ is defined as

$$\begin{aligned}\mathcal{N} = \Big\{ & r\cdot h\otimes\begin{pmatrix}\lambda\\0\end{pmatrix} + r\cdot k\otimes\begin{pmatrix}0\\\mu\end{pmatrix} - h\otimes r\cdot\begin{pmatrix}\lambda\\0\end{pmatrix} - k\otimes r\cdot\begin{pmatrix}0\\\mu\end{pmatrix}: \\ & \qquad h \text{ and } k \text{ are in } \mathcal{M}, \begin{pmatrix}\lambda\\\mu\end{pmatrix} \text{ is in } \mathbb{C}^2 \text{ and } r \text{ is in } \mathcal{A}(\mathbb{D})\Big\} \\ = & \Big\{\lambda(r-r(w_1))\cdot h\otimes\begin{pmatrix}1\\0\end{pmatrix} + \mu((r-r(w_2))\cdot k)\otimes\begin{pmatrix}0\\1\end{pmatrix} - \mu\alpha\frac{\Delta r}{\Delta w}k\otimes\begin{pmatrix}1\\0\end{pmatrix}: \\ & \qquad h \text{ and } k \text{ are in } \mathcal{M}, \begin{pmatrix}\lambda\\\mu\end{pmatrix} \text{ is in } \mathbb{C}^2 \text{ and } r \text{ is in } \mathcal{A}(\mathbb{D})\Big\}\end{aligned}$$

If the module map for $\mathcal{M}$ is defined for r in the disk algebra $\mathcal{A}(\mathbb{D})$ and f in $\mathcal{M}$ as $r\cdot f = r(T)f$, where T is in $B_1(\mathbb{D})$ introduced in [2] and $\gamma(w_i),\ i = 1,2$ are eigen vectors for T then, it is clear that $\gamma(w_2)\otimes\binom{0}{1}$ is orthogonal to $\mathcal{N}$. An element $sf\otimes\binom{0}{1} + tg\otimes\binom{1}{0}$ is orthogonal to $\mathcal{N}$ if and only if

$$\begin{aligned}& \mu s\langle(r-r(w_2))\cdot k, f\rangle \\ & \quad + t\lambda\langle(r-r(w_1))\cdot h, g\rangle - \mu\alpha\frac{\Delta r}{\Delta w}t\langle k, g\rangle = 0 \text{ for all } \begin{pmatrix}\lambda\\\mu\end{pmatrix}\in\mathbb{C}^2.\end{aligned}$$

Equivalently we must have $\langle(r-r(w_1))\cdot h, g\rangle = 0$, for all h in $\mathcal{M}$, which implies $g = \gamma(w_1)$ and $s\langle(r-r(w_2))\cdot k, f\rangle - \alpha\frac{\Delta r}{\Delta w}t\langle k, g\rangle = 0$, for all k in $\mathcal{M}$, which implies $s = \alpha, t = 1$ and $f = (w_1 - w_2)^{-1}\gamma(w_1)$. Thus,

$$\left\{\gamma(w_1)\otimes\begin{pmatrix}1\\0\end{pmatrix} + \gamma(w_1)\otimes\begin{pmatrix}0\\\alpha(w_1-w_2)^{-1}\end{pmatrix}, \gamma(w_2)\otimes\begin{pmatrix}0\\1\end{pmatrix}\right\}$$

is a basis for $\mathcal{M}\otimes_{\mathcal{A}}\mathbb{C}^{\alpha}_{w_1,w_2}$.

We orthogonalise to obtain

$$\begin{aligned}\varepsilon_1 &= \gamma(w_2)\otimes\begin{pmatrix}0\\1\end{pmatrix}\|\gamma(w_2)\otimes\begin{pmatrix}0\\1\end{pmatrix}\|^{-1} \\ &= \frac{\gamma(w_2)}{\|\gamma(w_2)\|}\otimes\begin{pmatrix}0\\1\end{pmatrix} \\ e_2 &= \gamma(w_1)\otimes\begin{pmatrix}1\\0\end{pmatrix} + \gamma(w_1)\otimes\begin{pmatrix}0\\\alpha(w_1-w_2)^{-1}\end{pmatrix} \\ & \qquad - \frac{\langle\gamma(w_1),\gamma(w_2)\rangle}{\|\gamma(w_2)\|}\frac{\gamma(w_2)}{\|\gamma(w_2)\|}\otimes\begin{pmatrix}0\\\alpha(w_1-w_2)^{-1}\end{pmatrix}.\end{aligned}$$

Set $\gamma(w_1,w_2) = \langle\gamma(w_1),\gamma(w_2)\rangle$,

$$e_2 = \gamma(w_1)\otimes\begin{pmatrix}1\\0\end{pmatrix} + \alpha(w_1-w_2)^{-1}\left(\gamma(w_1) - \gamma(w_2)\frac{\gamma(w_1,w_2)}{\|\gamma(w_2)\|^2}\right)\otimes\begin{pmatrix}0\\1\end{pmatrix}$$

and $\varepsilon_2 = e_2/\|e_2\|$, where

$$\|e_2\| = \left(\|\gamma(w_1)\|^2 + \frac{|\alpha|^2}{|(w_1 - w_2)|^2} (\|\gamma(w_1)\|^2 - \frac{|\gamma(w_1, w_2)|^2}{\|\gamma(w_2)\|^2}) \right)^{1/2}.$$

Thus $\{\varepsilon_1, \varepsilon_2\}$ is an orthonormal basis for $\mathcal{M} \otimes_{\mathcal{A}} \mathbb{C}^{\alpha}_{w_1,w_2}$.

We now obtain a projection formula for $pr : \mathcal{M} \otimes C^{\alpha}_{w_1,w_2} \longrightarrow \mathcal{M} \otimes_{\mathcal{A}} \mathbb{C}^{\alpha}_{w_1,w_2}$

$$\begin{aligned} pr : h \otimes \begin{pmatrix} \lambda \\ 0 \end{pmatrix} + k \otimes \begin{pmatrix} 0 \\ \mu \end{pmatrix} &\longrightarrow (\langle \varepsilon_1, h \otimes \begin{pmatrix} \lambda \\ 0 \end{pmatrix} \rangle + \langle \varepsilon_1, k \otimes \begin{pmatrix} 0 \\ \mu \end{pmatrix} \rangle) \varepsilon_1 \\ &\qquad + (\langle \varepsilon_2, h \otimes \begin{pmatrix} \lambda \\ 0 \end{pmatrix} \rangle + \langle \varepsilon_2, k \otimes \begin{pmatrix} 0 \\ \mu \end{pmatrix} \rangle) \varepsilon_2 \\ = \quad & \left\{ \frac{\mu\alpha}{\|e_2\|(\overline{w}_1 - \overline{w}_2)} (\langle k, \gamma(w_1) \rangle - \frac{\gamma(w_1, w_2) \langle k, \gamma(w_2) \rangle}{\|\gamma(w_2)\|^2}) + \frac{\lambda}{\|e_2\|} \langle h, \gamma(w_1) \rangle \right\} \varepsilon_2 \\ &\qquad + \frac{\mu}{\|\gamma(w_2)\|} \langle k, \gamma(w_2) \rangle \varepsilon_1. \end{aligned}$$

Let $\tilde{\mathcal{M}}$ be another Hilbert module over $\mathcal{A}(\mathbb{D})$ and the module map $r.f = r(\tilde{T})f$ for some $\tilde{T}$ in $B_1(\mathbb{D})$. Let $X : \mathcal{M} \longrightarrow \tilde{\mathcal{M}}$ be a module map and $\{\tilde{\varepsilon}_1, \tilde{\varepsilon}_2\}$ be an orthonormal basis for $\tilde{\mathcal{M}} \otimes_{\mathcal{A}} \mathbb{C}^{\alpha}_{w_1,w_2}$.

Set $\lambda = \|e_2\| \langle h, \gamma(w_1) \rangle^{-1}$, $\mu = 0$, then we have

$$\varepsilon_2 \xrightarrow{pr^{-1}} h \otimes \begin{pmatrix} \lambda \\ 0 \end{pmatrix} \xrightarrow{X \otimes_{\mathcal{A}} \mathrm{Id}} Xh \otimes \begin{pmatrix} \lambda \\ 0 \end{pmatrix} \xrightarrow{pr} \frac{\lambda}{\|\tilde{e}_2\|} \langle Xh, \tilde{\gamma}(w_1) \rangle \tilde{\varepsilon}_2 = \|e_2\| \|\tilde{e}_2\|^{-1} \tilde{\varepsilon}_2.$$

Again, set $\mu = \|\gamma(w_2)\| \langle k, \gamma(w_2) \rangle^{-1}$ and

$$\lambda = -\frac{\mu\alpha}{\|e_2\|(\overline{w}_1 - \overline{w}_2)} (\langle k, \gamma(w_1) \rangle - \frac{\overline{\gamma(w_1, w_2)}}{\|\gamma(w_2)\|^2} \langle k, \gamma(w_2) \rangle) \frac{\|e_2\|}{\langle h, \gamma(w_1) \rangle}.$$

Then we have

$$\begin{aligned} &\varepsilon_1 \xrightarrow{\mathrm{pr}^{-1}} h \otimes \begin{pmatrix} \lambda \\ 0 \end{pmatrix} + k \otimes \begin{pmatrix} 0 \\ \mu \end{pmatrix} \xrightarrow{X \otimes_{\mathcal{A}} \mathrm{Id}} Xh \otimes \begin{pmatrix} \lambda \\ 0 \end{pmatrix} + Xk \otimes \begin{pmatrix} 0 \\ \mu \end{pmatrix} \xrightarrow{pr} \\ &\quad \mu \frac{\langle Xk, \tilde{\gamma}(w_2) \rangle}{\|\tilde{\gamma}(w_2)\|} \tilde{\varepsilon}_1 + \left\{ \frac{\mu_\alpha}{\|\tilde{e}_2\|(\overline{w}_1 - \overline{w}_2)} \left(\langle Xk, \tilde{\gamma}(w_1) \rangle - \frac{\overline{\gamma(w_1, w_2)}}{\|\tilde{\gamma}(w_2)\|^2} \langle Xk, \tilde{\gamma}(w_2) \rangle \right) \right. \\ &\qquad \left. + \frac{\lambda}{\|\tilde{e}_2\|} \langle Xh, \tilde{\gamma}(w_1) \rangle \right\} \tilde{\varepsilon}_2. \\ &\quad = \frac{\|\gamma(w_2)\|}{\|\tilde{\gamma}(w_2)\|} \tilde{\varepsilon}_1 + \frac{\|\gamma(\omega e_2)\|\alpha}{\|\tilde{e}_2\|(\overline{w}_1 - \overline{w}_2)} \left(\frac{\overline{\gamma(w_1, w_2)}}{\|\gamma(w_2)\|^2} - \frac{\overline{\tilde{\gamma}(w_1, w_2)}}{\|\tilde{\gamma}(w_2)\|^2} \right) \tilde{\varepsilon}_2. \end{aligned}$$

Thus $X \otimes_{\mathcal{A}} \mathrm{Id}$ can be written down in a two by two matrix form.

Although at first glance, it might appear that these matrix entries would have some intrinsic meaning one must be careful. As a matrix $X \otimes_{\mathcal{A}} \mathrm{Id}$ between two Hilbert spaces which are isometarically isomorphic to $\mathbb{C}^2$ (but not identical), only symmetric combinations of the eigen values of $(X \otimes_{\mathcal{A}} \mathrm{Id})(X \otimes_{\mathcal{A}} \mathrm{Id})^*$ are well defined. If $\alpha > 0$ then the absolute values of the matrix entries of $X \otimes_{\mathcal{A}} \mathrm{Id}$ can be shown to be well defined. If the matrix for $X \otimes_{\mathcal{A}} \mathrm{Id}$ is $\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}$, then the first approach yields $\{|a|^2 \cdot |c|^2, |a|^2 + |b|^2 + |c|^2\}$, while the second yields $\{|a|, |b|, |c|\}$. Thus we are interested in the trace and determinant of $(X \otimes_{\mathcal{A}} \mathrm{Id})(X \otimes_{\mathcal{A}} Id)^*$ to be denoted as Tr and det.

$$\begin{aligned}
\mathrm{Tr} &= \frac{\|e\|^2}{\|\tilde{e}\|^2} + \frac{\|\gamma(w_2)\|^2}{\|\tilde{\gamma}(w_2)\|^2} + \frac{\|\gamma(w_2)\|^2\alpha^2}{|(w_1-w_2)|^2}\left|\frac{\overline{\gamma(w_1,w_2)}}{\|\gamma(w_2)\|^2} - \frac{\overline{\tilde{\gamma}(w_1,w_2)}}{\|\tilde{\gamma}(w_2)\|^2}\right|^2 \frac{1}{\|\tilde{e}_2\|^2} \\
&= (\|\tilde{e}_2\|\,|(w_1-w_2)|\,\|\tilde{\gamma}(w_2)\|)^{-2}\{(\alpha^2+|(w_1-w_2)|^2)(\|\gamma(w_1)\|^2\|\tilde{\gamma}(w_2)\|^2 \\
&\quad + \|\tilde{\gamma}(w_1)\|^2\|\gamma(w_2)\|^2) - 2\alpha^2\mathrm{Re}\overline{\gamma(w_1,w_2)}\tilde{\gamma}(w_1,w_2)\} \\
\det &= \frac{\|e_2\|}{\|\tilde{e}_2\|}\frac{\|\gamma(w_2)\|}{\|\tilde{\gamma}(w_2)\|} \\
&= \left\{\frac{\|\gamma(w_1)\|^2\|\gamma(w_2)\|^2|(w_1-w_2)|^2 + \alpha^2(\|\gamma(w_1)\|^2\|\gamma(w_2)\|^2 - |\gamma(w_1,w_2)|^2)}{\|\tilde{\gamma}(w_2)\|^2\|\tilde{\gamma}(w_2)\|^2|(w_1-w_2)|^2 + \alpha^2(\|\tilde{\gamma}(w_1)\|^2\|\tilde{\gamma}(w_2)\|^2 - |\tilde{\gamma}(w_1,w_2)|^2)}\right\}
\end{aligned}$$

One could of course do all those calculations, when $w = w_1 = w_2$. In this case the module $\mathbb{C}^\alpha_{w,w}$ is defined by the multiplication

$$r.\begin{pmatrix}\lambda \\ \mu\end{pmatrix} = \begin{pmatrix} r(w) & \alpha r'(w) \\ 0 & r(w)\end{pmatrix}\begin{pmatrix}\lambda \\ \mu\end{pmatrix}$$

and we obtain

$$\begin{aligned}
\mathrm{Tr}_w &= \{2\|\tilde{\gamma}(w)\|^2\|\gamma(w)\|^2 + \alpha^2(\|\tilde{\gamma}(w)\|^2\|\gamma'(w)\|^2 + \|\tilde{\gamma}'(w)\|^2\|\gamma(w)\|^2) \\
&\quad - 2\mathrm{Re}\langle\tilde{\gamma}'(w),\tilde{\gamma}(w)\rangle\langle\gamma(w),\gamma'(w)\rangle\}(\|\tilde{\gamma}(w)\|\,\|\tilde{e}_2\|)^{-2}. \\
\det_w &= \frac{\|\gamma(w)\|^2}{\|\tilde{\gamma}(w)\|^2}\left(\frac{1-\alpha^2\mathcal{K}_T(w)}{1-\alpha^2\mathcal{K}_{\tilde{T}}(w)}\right)^{1/2},
\end{aligned}$$

where

$$\mathcal{K}_T(w) = -\frac{\partial^2}{\partial w\partial\overline{w}}\log\|\gamma(w)\|^2,$$

denotes the curvature of the operator T.

3 Non uniqueness for the Annulus Algebra

In the case of Silov resolution for the disk algebra, there is little difficulty in obtaining invariants for the given module in terms of the resolution due to the fact that the resolution is unique. The problem is finding useful invariants. We attempt to consider the case of a simple example for the annulus algebra. We let H^2_α denote the Hardy space defined by holomorphic sections of the line bundle over the annulus with twist $e^{i\alpha}$ for α in $\mathbb{R}$. We find that if

$$0 \longleftarrow \mathbb{C}_a \longleftarrow H^2_{\alpha+\beta} \xleftarrow{X} H^2_\beta \longleftarrow 0$$

is a Silov resolution for the module $\mathbb{C}_a$ defined by evaluation at a in $\boldsymbol{A}$ then the above resolution is not unique. In fact, for each β in $(0, 2\pi)$, we get a distinct resolution.

There is a holomorphic vector bundle defined by H^2_α for all α, whose curvature $\mathcal{K}_\alpha(w)$ can be calculated using any holomorphic cross-section. Let $K_\alpha(z,w)$ be the reproducing kernel for $H^2_\alpha(\boldsymbol{A})$. It is then easily verified that $\overline{w} \longrightarrow K_\alpha(\cdot,w)$ is a holomorphic cross-section for H^2_α, any other holomorphic cross-section is of the from $\varphi(w)K(\cdot,w)$, where $\varphi(w)$ is a non-vanishing bounded analytic function on the annulus $\boldsymbol{A}$.

Theorem 3.1 *The quantity $\mathcal{K}_{\alpha+\beta}(a)/K_\beta(a,a)$ is independent of β.*

Proof : Let $H_\alpha(\cdot,a)$ denote the unique (upto a scalar of absolute value 1) inner function on $\boldsymbol{A}$, which vanishes at a and is smooth on closure of $\boldsymbol{A}$. In the Silov resolution discussed above let X denote multiplication by $H(\cdot,a)$. Since $f \longrightarrow H_\alpha(\cdot,a)f$ is a unitary map of H^2_β onto a subspace of codimension 1 in $H^2_{\alpha+\beta}$, we have the identity

$$K_{\alpha+\beta}(z,w) = \frac{K_{\alpha+\beta}(z,a)\overline{K_{\alpha+\beta}(w,a)}}{K_{\alpha+\beta}(a,a)} + H_\alpha(z,a)\overline{H_\alpha(w,a)}K_\beta(z,w).$$

Set,

$$\tilde{K}_{\alpha+\beta}(z,w) = \frac{K_{\alpha+\beta}(a,a)K_{\alpha+\beta}(z,w)}{K_{\alpha+\beta}(z,a)\overline{K_{\alpha+\beta}(w,a)}}$$

and

$$\tilde{K}_\beta(z,w) = \frac{K_{\alpha+\beta}(a,a)K_\beta(z,w)}{K_{\alpha+\beta}(z,a)\overline{K_{\alpha+\beta}(w,a)}}.$$

Thus,

$$\tilde{K}_{\alpha+\beta}(w,w) = 1 + |H(w,a)|^2\tilde{K}_\beta(w,w).$$

To compute curvature, note

$$\begin{aligned}
&-\frac{\partial^2}{\partial w\partial\overline{w}}\log\frac{\tilde{K}_{\alpha+\beta}(w,w)}{\tilde{K}_\beta(w,w)} \\
&= \mathcal{K}_{\alpha+\beta}(w) - \mathcal{K}_\beta(w) \\
&= -\frac{\partial^2}{\partial w\partial\overline{w}}\log(\frac{1}{\tilde{K}_\beta(w,w)} + |H_\alpha(w,a)|^2) \\
&= \frac{\partial}{\partial\overline{w}}\frac{\frac{\partial}{\partial w}\tilde{K}_\beta(w,w) - \overline{H_\alpha(w,a)}\left(\frac{\partial}{\partial w}H_\alpha(w,a)\right)\left(\tilde{K}_\beta(w,w)\right)^2}{\tilde{K}_\beta(w,w)(1+\tilde{K}_\beta(w,w)|H_\alpha(w,a)|^2)}
\end{aligned}$$

After differentiating once again and simplifying we obtain the formula

$$\mathcal{K}_{\alpha+\beta}(w) - \mathcal{K}_\beta(w) = -\frac{\mathcal{K}_\beta(w)}{\tilde{K}_{\alpha+\beta}(w,w)} - \frac{(|\frac{d}{dw}H(w,a)\tilde{K}_\beta(w,w) + H(w,a)\frac{\partial}{\partial w}\tilde{K}_\beta(w,w)|^2)}{\tilde{K}_\beta(w,w)(\tilde{K}_{\alpha+\beta}(w,w))^2}.$$

To evaluate this difference at a, observe that if $G(w,a)$ is the Green's function for $\boldsymbol{A}$ with singularity at a and $G^*(w,a)$, the multivalued harmonic conjugate then

$$H(w,a) = exp(g(w,a) + ig^*(w,a)).$$

Note that $\frac{d}{dw}(g(w,a)+ig^*(w,a))$ is holomorphic in the annulus except for a pole of order 1 at a. Thus $\frac{d}{dw}H(w,a)|_{w=a}$ is not zero. Therefore we have

$$\mathcal{K}_{\alpha+\beta}(a)-\mathcal{K}_\beta(a)=-(\mathcal{K}_\beta(a)+\frac{d}{dw}H(w,a)|_{w=a}K_\beta(a,a)).$$

In otherwords,

$$\frac{\mathcal{K}_{\alpha+\beta}(a)}{K_\beta(a,a)}=-\frac{d}{dw}H(w,a)|_{w=a},$$

which completes the proof.

Although, we have stated the theorem in the context of the algebra of clos Rat ($\boldsymbol{A}$), it clearly remains true for other function algebras.

While it is not clear immediately, whether $\mathcal{K}_{\alpha+\beta}(w)-\mathcal{K}_\beta(w)$ is independent of β even at $w=a$ the ratio $\mathcal{K}_{\alpha+\beta}(a)/K_\beta(a,a)$ may be related to the absolute value of the $(1,1)$ entry in $(X\otimes_{\mathcal{A}}\mathrm{Id})(X\otimes_{\mathcal{A}}\mathrm{Id})^*$ found in the previous section. Of course using the same sort of technique as in the theorem we can write down the trace and determinant of $(X\otimes_{\mathcal{A}}\mathrm{Id})(X\otimes_{\mathcal{A}}\mathrm{Id})^*$ explicitly.

If β_0 is chosen so that $T_{\alpha+\beta_0}$ is the extremal operator (c.f. [6]) at a and $\hat{K}(w,w)$ is the Szego kernel for $\boldsymbol{A}$ then

$$-\hat{K}(a,a)^2=-K_{\beta_0}(a,a)\frac{d}{dw}H(w,a)|_{w=a}.$$

4 Computation of two dimensional modules for $\mathcal{A}(\mathbb{D}^2)$

In [4] it is pointed out how earlier work of Misra and Paulsen allowed one to decide for which $\alpha>0$, the modules $C^\alpha_{w_1,w_2}$ and $C^\alpha_{w,w}$ are contractive or completely contractive over $\mathcal{A}(\mathbb{D})$. In this section we consider the analogous problem for modules over the bidisk algebra $\mathcal{A}(\mathbb{D}^2)$. Let p be a polynomial in $\mathcal{A}(\mathbb{D}^2)$, define the module map $\mathcal{A}(\mathbb{D}^2)\times\mathbb{C}^2\to\mathbb{C}^2$ by

$$\begin{aligned} p\cdot\begin{pmatrix}\lambda_1\\ \lambda_2\end{pmatrix} &= p(T_1,T_2)\begin{pmatrix}\lambda_1\\ \lambda_2\end{pmatrix}, \text{ where}\\ T_1 &= \begin{pmatrix} w_1 & r\\ 0 & w_2\end{pmatrix} \quad\text{and}\quad T_2=\begin{pmatrix} w_2 & s\\ 0 & w_2\end{pmatrix}\end{aligned}$$

Let $\mathbb{C}^2_{r,s}(w_1,w_2)$ denote this module and set

$$\lambda_{r,s}(w_1,w_2)=\max\left\{\frac{|r|^2}{1-|w_1|^2},\frac{|s|^2}{1-|w_2|^2}\right\}.$$

Proposition 4.1 *The two dimensional module $\mathbb{C}^2_{r,s}(w_1,w_2)$ is contractive if and only if $\lambda_{r,s}(w_1,w_2)\le 1$.*

Proof: The module $\mathbb{C}^2_{r,s}(w_1, w_2)$ is contractive if $\|p(T_1, T_2)\| \leq 1$, for all polynomials $p \in \mathcal{A}(\mathbb{D}^2)$ of norm at most 1. We note that

$$p(T_1, T_2) = \begin{pmatrix} p(w_1, w_2) & (\partial_{(r,s)} p)(w_1, w_2) \\ 0 & p(w_1, w_2) \end{pmatrix},$$

where $(\partial)_{(r,s)} p)(w_1, w_2) = r \frac{\partial p}{\partial z_1}|_{(w_1,w_2)} + s \frac{\partial p}{\partial z_2}|_{(w_1,w_2)}$. It is easy to see that $\|p(T_1, T_2)\| \leq 1$ if and only if $|(\partial_{(r,s)} p)(w_1, w_2)| \leq 1 - |p(w_1, w_2)|^2$. If we set $q = \varphi \circ p$, where $\varphi(p(w_1, w_2)) = 0$ then

$$\begin{aligned} |(\partial_{(r,s)} p)(w_1, w_2) &= \varphi'(p(w_1, w_2))(\partial_{(r,s)} p)(w_1, w_2) \\ &= \frac{(\partial_{(r,s)} p)(w_1, w_2)}{1 - |p(w_1, w_2)|^2}. \end{aligned}$$

Thus we see that if $|(\partial_{(r,s)} p)(w_1, w_2)| \leq 1$, for all p in $\mathcal{A}(\mathbb{D}^2)$ with $p(w_1, w_2) = 0$ then

$$|(\partial_{(r,s)} p)(w_1, w_2)| \leq 1 - |p(w_1, w_2)|^2 \text{ for all } p \text{ in } \mathcal{A}(\mathbb{D}^2).$$

So, $\|p(T_1, T_2)\| \leq 1$ if and only if

$$\begin{aligned} 1 &\geq \sup \left\{ |(\partial_{(r,s)} f)(w_1, w_2)| : f \text{ is analytic on } \mathbb{D}^2 \text{ and } f(w_1, w_2) = 0 \right\} \\ &= \max \left\{ \frac{|r|}{1 - |w_1|^2}, \frac{|s|}{1 - |w_2|^2} \right\}. \end{aligned}$$

The last equality is well known (c.f. [1]). This completes the proof of the proposition.

Of course, it is possible to obtain similar results for the ball and other homogeneous domains. Sastry and the second author have been able to obtain similar conditions to determine when $\mathbb{C}^{n+1}$ is contractive module under a somewhat different module multiplication, by introducing an analogous extremal problem (cf. [10], [11]).

5 Silov resolutions in bidisk algebra

Since the main interest in the Hilbert module approach to operator theory lies in the several variables case, we would being studying examples. We concentrate on quotient modules obtained from the Hardy module $H^2(\mathbb{D}^2)$. In particular, we set $\mathcal{S}_0 = H^2(\mathbb{D}^2)$ and let $\mathcal{S}_1$ be the closure of an ideal $\mathcal{I}$ in $\mathbb{C}[w, z]$. If $\mathcal{Z}(\mathcal{I})$ denote the set of common zeros of the polynomials in $\mathcal{I}$ then the case when $\mathcal{Z}(\mathcal{I})$ is discrete and finite can be analyzed since the quotient module is finite-dimensional (c.f. [4]). Here we want to consider the case when $\mathcal{I}$ is the principal ideal in $\mathbb{C}[w, z]$ generated by a polynomial $p(w, z)$. If $\mathcal{M}_p$ denote the quotient module then we are interested in describing the properties of $\mathcal{M}_p$ in terms of those of p. Here we examine some very simple examples.

Let $p(w, z) = w - z$ and set, $\mathcal{M}_0 = \mathcal{M}_{w-z}$. For each $k \geq 0$, let $\mathcal{P}_k$ denote the closed subspace of homogeneous polynomials in $H^2(\mathbb{D}^2)$ spanned by $\{w^{k-j} z^j\}_{j=0}^k$. Then we have $H^2(\mathbb{D}^2) = \bigoplus_{k=0}^{\infty} \mathcal{P}_k$ and $\mathcal{M}_0 = \bigoplus_{k=0}^{\infty} (\mathcal{P}_k \cap \mathcal{M}_0)$. Moreover, each $\mathcal{P}_k \cap \mathcal{M}_0$ is one-dimensional and is spanned by the polynomial $e_k = \sum_{j=0}^{k} w^{k-j} z^j$. The set $\{e_k\}_{k=0}^{\infty}$ form an orthogonal basis for $\mathcal{M}_0$ and $\|e_k\| = \sqrt{k+1}$. Therefore, we have the orthonormal basis

$$\left\{ \frac{1}{\sqrt{k+1}} e_k \right\}_{k=0}^{\infty} \quad \text{for} \quad \mathcal{M}_0.$$

What about the module multiplication on $\mathcal{M}_0$? It is enough to calculate the action of w since it is identical to that of z on $\mathcal{M}_0$. But

$$w \cdot \frac{1}{\sqrt{k+1}} e_k = \sqrt{\frac{k+1}{k+2}} \frac{1}{\sqrt{k+2}} e_{k+1}$$

in $\mathcal{M}_0$ and hence $\mathcal{M}_0$ is isomorphic to the Bergman module on which the $\mathcal{A}(\mathbb{D}^2)$ action is pulled back via the map $\mathbb{D} \longrightarrow \mathbb{D}^2$ defined by $z \longrightarrow (z, z)$. In particular, the operator action is essentially normal and hence the module $\mathcal{M}_0$ is what we called an essentially reductive Silov module for $\mathcal{A}(\mathbb{D}^2)$. Hence, we have the Chern character $Ch_\partial[\mathcal{M}_0]$ in $K^1(\boldsymbol{T}^2) = \boldsymbol{Z} \oplus \boldsymbol{Z}$. With the standard orientation, we see that $Ch_\partial[\mathcal{M}_0] = 1 \oplus 1$.

Not all quotient modules for $H^2(\mathbb{D}^2)$ are essentially reductive. The principal ideal in $\mathbb{C}[w, z]$ generated by $p(w, z) = w^2$ does not yield an essentially reductive module since multiplication by w is not essentially normal. Understanding why this is so is not obvious. Is it the fact that w^2 is not prime? Let us consider the case of $(w - z)^2$.

Let $\mathcal{M}_1$ be the quotient module for $\mathcal{S}_0 = H^2(\mathbb{D}^2)$ and $\mathcal{S}_2$ be the closure of the principal ideal generated by $(w - z)^2$. Observe that we have $\mathcal{M}_1 = \oplus_{k=0}^{\infty} (\mathcal{P}_k \cap \mathcal{M}_1)$ and, in fact, we have $\mathcal{P}_k \cap \mathcal{M}_0 \subset \mathcal{P}_k \cap \mathcal{M}_1$. Moreover, if we ignore the anamolous cases $\mathcal{P}_0$ and $\mathcal{P}_1$ which are contained in $\mathcal{M}_1$, we see that

$$\dim(\mathcal{P}_k \cap \mathcal{M}_1) \ominus (\mathcal{P}_k \cap \mathcal{M}_0) = 1 \quad \text{for} \quad k \geq 2.$$

Thus we have the orthogonal basis for $\mathcal{P}_k \cap \mathcal{M}_1$ consisting of e_k and f_k, where

$$f_k = \begin{cases} m w^k + (m-1) w^{k-1} z + \cdots + w^{m+1} z^{m-1} \\ \qquad - (w^{m-1} z^{m+1} + 2 w^{m-2} z^{m+2} + \cdots + m z^k) & k = 2m \geq 2 \\ k w^k + (k-2) w^{k-1} + \cdots + w^{m+1} z^m \\ \qquad - (w^m z^{m+1} + 3 w^{m-1} z^{m+2} + \cdots + k z^k) & k = 2m+1 \geq 3 \end{cases}$$

It can be checked that e_k and f_k are orthogonal and in turn are orthogonal to $\mathcal{S}_2$. Finally we have

$$\|f_k\|^2 = \begin{cases} \frac{2m}{3}(m+1)(2m+1) & k = 2m \geq 2 \\ \frac{4(m+1)}{3}(2m+1)(2m+3) & k = 2m+1 \geq 3. \end{cases}$$

Now the action of z again agrees with that of w and since $w\mathcal{P}_k \subset \mathcal{P}_{k+1}$, we have

$$w \cdot \frac{e_k}{\sqrt{k+1}} = \alpha_k \frac{e_{k+1}}{\sqrt{k+2}} + \beta_k \frac{f_{k+1}}{\|f_{k+1}\|}$$

$$w \cdot \frac{f_k}{\|f_k\|} = \gamma_k \frac{e_{k+1}}{\sqrt{k+2}} + \delta_k \frac{f_{k+1}}{\|f_{k+1}\|}$$

Calculating we have the matrix,

$$\begin{pmatrix} \alpha_k & \gamma_k \\ \beta_k & \delta_k \end{pmatrix} = \begin{pmatrix} \sqrt{\frac{k+1}{k+2}} & 0 \\ \sqrt{\frac{3}{2}\frac{1}{(k+2)(k+3)}} & \sqrt{\frac{k}{k+3}} \end{pmatrix} \qquad \text{for } k \geq 2.$$

Therefore, the module multiplication on $\mathcal{M}_1$ is essentially normal and hence $\mathcal{M}_1$ is an essentially reductive module since modulo compacts, multiplication by w is the unilateral shift of multiplicity two, and have $Ch_\partial[\mathcal{M}_1] = 2 \oplus 2 = 2Ch_\partial[\mathcal{M}_0]$.

But there is much more information to obtain about these modules since each has a kernel function. Since the e_k's form an orthogonal basis for $\mathcal{M}_0$, and $\|e_k\| = \sqrt{k+1}$, the kernel function for $\mathcal{M}_0$ is

$$K_{\mathcal{M}_0}(\underline{z}, \underline{w}) = \sum_{n=0}^{\infty} e_n(\underline{z})\overline{e_n(\underline{w})}(n+1)^{-1}$$

where $\underline{z} = (z_1, z_2)$ and $\underline{w} = (w_1, w_2)$. Thus,

$$\begin{aligned} K_{\mathcal{M}_0}(\underline{w}, \underline{w}) &= \sum_{n=0}^{\infty} |e_n(w_1, w_2)|^2/(n+1) \\ &= \sum_{n=0}^{\infty} \left(\sum_{j=0}^{n} |w_1^{n-j} w_2^j|^2 \right) /(n+1) \\ &= \sum_{n=0}^{\infty} \frac{|w_1^{n+1} - w_2^{n+1}|^2}{(|w_1 - w_2|^2)(n+1)} \end{aligned}$$

Since this series is absolutely convergent, we can sum term by term to obtain

$$\begin{aligned} &K_{\mathcal{M}_0}(\underline{w}, \underline{w}) \\ &= \frac{1}{|w_1 - w_2|^2} \Big(-\log(1 - |w_1|^2) + \log(1 - \overline{w}_1 w_2) + \log(1 - w_2 \overline{w}_1) - \log(1 - |w_2|^2 \Big) \\ &= \frac{1}{|w_1 - w_2|^2} \log \left\{ \frac{|1 - w_1 \overline{w}_2|^2}{(1 - |w_1|^2)(1 - |w_2|^2)} \right\} \end{aligned}$$

D.N. Clark has also obtained the same kernel function by different methods (cf. [2]). Some of his results are however more general. As we have pointed out $\mathcal{M}_0$ is equivalent to the Bergman module and now, we observe that

$$\lim_{w_1 \to w_2} K_{\mathcal{M}_0}(\underline{w}, \underline{w}) = \frac{1}{(1 - |w_2|^2)^2},$$

which is the norm of Bergman kernel in $\mathcal{A}^2(I\!\!D)$.

$$\begin{aligned} &K_{\mathcal{S}_1}(\underline{w}, \underline{w}) = K_{\mathcal{S}_0}(\underline{w}, \underline{w}) - K_{\mathcal{M}_0}(\underline{w}, \underline{w}) \\ &= \frac{1}{(1 - |w_1|^2)(1 - |w_2|^2)} - \frac{1}{|w_1 - w_2|^2} \log \left(1 + \frac{|(w_1 - w_2)|^2}{(1 - |w_1|^2)(1 - |w_2|^2)} \right) \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{(1-|w_1|^2)(1-|w_2|^2)} \\
&\qquad -\frac{1}{|w_1-w_2|^2}\left(\frac{|(w_1-w_2)|^2}{(1-|w_1|^2)(1-|w_2|^2)} - \frac{1}{2}\frac{|w_1-w_2|^4}{((1-|w_2|^2)(1-|w_2|^2))^2} + \cdots\right) \\
&= \frac{|(w_1-w_2)|^2}{(1-|w_1|^2)(1-|w_2|^2)}\left(\frac{1}{2}\frac{1}{(1-|w_1|^2)(1-|w_2|^2)} - \frac{|(w_1-w_2)|^2}{3(1-|w_1|^2)^2(1-|w_2|^2)^2} + \cdots\right)
\end{aligned}
$$

Thus on the zero set,

$$\frac{K_{\mathcal{S}_1}(w_1,w_2)}{|(w_1-w_2)|^2} = \frac{1}{2}\frac{1}{(1-|w_2|^2)^4}.$$

We obtain

$$\left(\lim_{w_1\to w_2}\frac{K_{\mathcal{S}_1}(w_1,w_2)}{|(w_1-w_2)|^2}\right)(-\mathcal{K}_{\mathcal{S}_0}(w_2,w_2))^{-1} = \frac{1}{4(1-|w_2|^2)^2},$$

which is a scalar multiple of the kernel function of the Bergman space. Therefore, in this particular case

$$\frac{K_{\mathcal{S}_1}(w_1,w_2)}{|(w_1-w_2)|^2}(-\mathcal{K}_{\mathcal{S}_0}(w_2,w_2))^{-1}$$

recaptures the inner product of the quotient space upto a scalar multiple.

We have made some progress on these and related questions which is reported in [6].

References

[1] J. Burbea, *On the Hessian of the Carathéodory metric, Rockey Mountain* , J. of Math. 8(1978), 555-559.

[2] D. N. Clark, *Restrictions of H^p functions in the polydisk*, Amer J. Math., 110(1988), 1119 - 1152.

[3] M. J. Cowen and R. G. Douglas, *Complex geometry and operator theory*, Acta Math, 141 (1978), 1987-261.

[4] R. G. Douglas, *Hilbert modules for function algebras*, Operator theory: Advances and applications, Vol.17 (1986), 125-139.

[5] R. G. Douglas, *Hilbert module for function algebras*, Szechuan Lectures, Fall, 1986.

[6] R. G. Douglas and G. Misra, *Geometric invariants for resolutions of Hilbert modules*, preprint, 1995.

[7] R. G. Douglas and V. I. Paulsen, *Hilbert Modules over Function Algebras*, Longman Research Notes, 217, 1989.

[8] R. G. Douglas, *On Silov resolutions of Hilbert modules*, in preparation.

[9] G. Misra, *Extremal properties of bundle shifts and curvature inequalities*, J. Operator Th., 11 (1984), 305-317.

[10] G. Misra and N. S. N. Sastry, *Contractive Modules, Extremal Problems, and Curvature Inequalities*, J. Functional Analysis, 88 (1990), 118-134.

[11] G. Misra and N. S. N. Sastry, *Completely Contractive Modules, and Associated Extremal Problems*, J. Functional Analysis, 91 (1990), 213-220.

SUNY at Stony Brook
Stony Brook, NY 11794

Indian Statistical Institute
R.V. Colleg Post, Bangalore 560 059