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Unitary invariants for Hilbert modules of finite rank

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Abstract. We associate a sheaf model to a class of Hilbert modules satisfying a natural finiteness condition. It is obtained as the dual to a linear system of Hermitian vector spaces (in the sense of Grothendieck). A refined notion of curvature is derived from this construction leading to a new unitary invariant for the Hilbert module. A division problem with bounds, originating in Douady's privilege, is related to this framework. A series of concrete computations illustrate the abstract concepts of the paper.

1. Introduction

A Hilbert module over the ring of polynomials $\mathbb{C}[\underline{z}] := \mathbb{C}[z_1, \dots, z_m]$ is a Hilbert space $\mathscr{H}$ which is a $\mathbb{C}[\underline{z}]$-module, where the multiplication by the polynomial p satisfies a continuity condition of the form

$$\|p \cdot f\| \leqq C_p \|f\|, \quad f \in \mathscr{H},\ p \in \mathbb{C}[\underline{z}],$$

for some positive constant C_p. Thus for any compact set K, we have

$$\|p \cdot f\| \leqq C_{p,K} \|p\|_{\infty,K} \|f\|, \quad f \in \mathscr{H},\ p \in \mathbb{C}[\underline{z}].$$

Extending the product by continuity we find that $\mathscr{H}$ admits a Hilbert module structure over the algebra $\mathscr{O}(\mathbb{C}^m)$ of entire functions. The multiplication M_j by the complex variable $z_j : M_j f = z_j \cdot f$, $1 \leqq j \leqq m$, then defines a commutative tuple $\boldsymbol{M} = (M_1, \dots, M_m)$ of linear bounded operators acting on $\mathscr{H}$ and vice-versa. Any such system of operators induces a topological $\mathscr{O}(\mathbb{C}^m)$-module structure on $\mathscr{H}$.

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The present article has three distinct but interconnected parts: the first deals with the classification up to unitary equivalence of a class of Hilbert $\mathcal{O}(\mathbb{C}^m)$-modules which possess many analytic submodules of finite co-dimension, the second part is devoted to division problems with bounds on classical Hilbert modules of analytic functions and the third part contains explicit computation of unitary invariant for some non-trivial examples (of Hilbert modules) studied in the first and the second part. Throughout the article Ω is assumed to be a bounded domain in $\mathbb{C}^m$.

Definition 1.1. A Hilbert module $\mathcal{H}$ over the polynomial ring $\mathbb{C}[\underline{z}]$ is said to be in the class $\mathrm{B}_n(\Omega)$, $n \in \mathbb{N}$, if

(const) $\dim \mathcal{H}/\mathfrak{m}_w\mathcal{H} = n < \infty$ for all $w \in \Omega$,

(span) $\bigcap_{w \in \Omega} \mathfrak{m}_w\mathcal{H} = \{0\}$,

where $\mathfrak{m}_w$ denotes the maximal ideal in $\mathbb{C}[\underline{z}]$ at w.

Recall that if $\mathfrak{m}_w\mathcal{H}$ has finite co-dimension, then $\mathfrak{m}_w\mathcal{H}$ is a closed subspace of $\mathcal{H}$. Throughout this paper we call $\dim \mathcal{H}/\mathfrak{m}_w\mathcal{H}$ *the rank* of the analytic module at the point w. For any Hilbert module $\mathcal{H}$ in $\mathrm{B}_n(\Omega)$, the analytic localization $\mathcal{O} \hat{\otimes}_{\mathcal{O}(\mathbb{C}^m)} \mathcal{H}$ is a locally free module when restricted to Ω, see for details [19]. Let us denote in short

$$\hat{\mathcal{H}} := \mathcal{O} \hat{\otimes}_{\mathcal{O}(\mathbb{C}^m)} \mathcal{H}|_{\Omega},$$

and let $E_{\mathcal{H}}$ be the associated holomorphic vector bundle. Fix $w \in \Omega$. The last map in Koszul's complex $\delta_1(w) : \mathcal{H} \oplus \cdots \oplus \mathcal{H} \to \mathcal{H}$ is defined by $(f_1, \dots, f_m) \mapsto \sum_{j=1}^{m} (M_j - w_j) f_j$, where M_j is the multiplication operator by the coordinate function z_j, for $1 \leqq j \leqq m$ and $f \in \mathcal{H}$. Then the analytic localization $\hat{\mathcal{H}}_w = \operatorname{coker} \delta_{1w}(w)$ is a locally free $\mathcal{O}_w$ module and the fiber of the associated holomorphic vector bundle $E_{\mathcal{H}}$ is given by

$$E_{\mathcal{H},w} = \hat{\mathcal{H}}_w \otimes_{\mathcal{O}_w} \mathcal{O}_w/\mathfrak{m}_w\mathcal{O}_w,$$

where $\mathcal{O}_w$ denotes the germs of holomorphic functions at w. We identify $E^*_{\mathcal{H},w}$ with $\ker \delta_1(w)^*$. Thus $E^*_{\mathcal{H}}$ is a Hermitian holomorphic vector bundle on $\Omega^* := \{\bar{z} : z \in \Omega\}$. Let $D_{\boldsymbol{M}^*}$ be the commuting m-tuple $(M_1^*, \dots, M_m^*)$ from $\mathcal{H}$ to $\mathcal{H} \oplus \cdots \oplus \mathcal{H}$. Clearly $\delta_1(w)^* = D_{(\boldsymbol{M}-w)^*}$ and $\ker \delta_1(w)^* = \bigcap_{j=1}^{m} \ker(M_j - w_j)^*$ for $w \in \Omega$.

It is easy to see that, within the class $\mathrm{B}_n(\Omega)$, the association $\mathcal{H} \mapsto E^*_{\mathcal{H}}$ provides a complete unitary invariant for $\mathcal{H}$. Thus the problem of classifying these analytic modules is a purely differential geometric one, see [6].

The aim of the present work is to extend the dictionary $\mathcal{H} \mapsto E^*_{\mathcal{H}}$ to analytic Hilbert modules whose rank is finite but non-constant, whence $E^*_{\mathcal{H}}$ is no more a vector bundle but rather a system of Hermitian vector spaces, and to compute differential geometric invariants like the curvature. To be more specific, we will restrict ourselves to the class $\mathfrak{B}_1(\Omega)$ defined below.

Definition 1.2. A Hilbert module $\mathscr{M} \subset \mathcal{O}(\Omega)$ is said to be in the class $\mathfrak{B}_1(\Omega)$ if

(rk) it possesses a reproducing kernel K (we don't rule out the possibility: $K(w,w) = 0$ for w in some closed subset X of Ω) and

(fin) the dimension of $\mathscr{M}/\mathfrak{m}_w\mathscr{M}$ is finite for all $w \in \Omega$.

Most of the examples in $\mathfrak{B}_1(\Omega)$ arise in the form of a submodule of some Hilbert module $\mathscr{H}(\subseteqq \mathcal{O}(\Omega))$ in the Cowen–Douglas class $\mathrm{B}_1(\Omega)$. We don't know of an example which is not of this form.

Note that if $\mathscr{H}$ is a Hilbert module in $\mathrm{B}_1(\Omega)$, restricting Ω to a smaller open set if necessary, the evaluation map $E(w) : \mathscr{H} \mapsto \hat{\mathscr{H}}_w = \mathbb{C}$ is continuous and onto, hence there exists, by Riesz's Lemma, a non-zero vector $K_w \in \mathscr{H}$ so that $E(w)x = \langle x, K_w\rangle$, $x \in \mathscr{H}$. This defines what is commonly called a reproducing kernel $K(z,w) = \langle K_w, K_z\rangle$, $z, w \in \Omega$, for the Hilbert space $\mathscr{H}$. In this case $E^*_{\mathscr{H}} \cong \mathcal{O}_{\Omega^*}$, that is, the associate holomorphic vector bundle is trivial, with K_w as a non-vanishing global section. For modules in $\mathrm{B}_1(\Omega)$, the curvature of the vector bundle $E^*_{\mathscr{H}}$ is a complete invariant.

Denote by $H^2(\mathbb{D}^2)$ the Hardy space of the bidisk. A typical example of a module in the class $\mathfrak{B}_1(\mathbb{D}^2)$, but not in $\mathrm{B}_1(\mathbb{D}^2)$, is $H^2_0(\mathbb{D}^2) := \{f \in H^2(\mathbb{D}^2) : f(0) = 0\}$ (cf. [11]). In this example, we have

$$\dim\ker D_{(\boldsymbol{M}-w)^*} = \dim H^2_0(\mathbb{D}^2) \otimes_{\mathbb{C}[z_1,z_2]} \mathbb{C}_w = \begin{cases} 1, & \text{if } w \neq (0,0), \\ 2, & \text{if } w = (0,0). \end{cases}$$

Here $\mathbb{C}_w$ is the one dimensional module over the polynomial ring $\mathbb{C}[z_1,z_2]$, where the module action is given by the map $(f,\lambda) \mapsto f(w)\lambda$ for $f \in \mathbb{C}[z_1,z_2]$ and $\lambda \in \mathbb{C}_w \cong \mathbb{C}$.

Let us return to a Hilbert module $\mathscr{M}$ in the class $\mathfrak{B}_1(\Omega)$. Assume that $\mathscr{M}$ is a submodule of some Hilbert module $\mathscr{H}$ in $\mathrm{B}_1(\Omega)$ and that $E_{\mathscr{H}}$ is trivial on Ω. Let $\mathscr{S}^{\mathscr{M}}$ be the range of the induced map

$$\mathcal{O} \mathbin{\hat{\otimes}}_{\mathcal{O}(\mathbb{C}^m)} \mathscr{M} \to \mathcal{O} \mathbin{\hat{\otimes}}_{\mathcal{O}(\mathbb{C}^m)} \mathscr{H} \cong \mathcal{O}(\Omega) \tag{1.1}$$

at the level of analytic sheaves. In general, for a Hilbert module $\mathscr{M}$ in $\mathfrak{B}_1(\Omega)$, we give the defintion of the sheaf model $\mathscr{S}^{\mathscr{M}}$ below.

Definition 1.3. Let $\mathscr{S}^{\mathscr{M}}$ be the subsheaf of the sheaf of holomorphic functions $\mathcal{O}(\Omega)$ determined by the stalks

$$\{(f_1)_w\mathcal{O}_w + \cdots + (f_n)_w\mathcal{O}_w : f_1, \ldots, f_n \in \mathscr{M}\} \subseteqq \mathcal{O}_w, \quad w \in \Omega. \tag{1.2}$$

We will prove that $\mathscr{S}^{\mathscr{M}}$ is a coherent analytic sheaf, in particular, its stalk $(\mathscr{S}^{\mathscr{M}})_w$ at a given point $w \in \Omega$ is finitely generated over $\mathcal{O}_w$. The main technical result towards constructing a system of complete unitary invariants for the module $\mathscr{M}$ is formulated as follows.

Theorem 1.4. *Let w_0 be a fixed but arbitrary point in Ω. Suppose $\mathscr{M}$ is in $\mathfrak{B}_1(\Omega)$ and g_i^0, $1 \leqq i \leqq d$, is a minimal set of generators for the stalk $\mathscr{S}_{w_0}^{\mathscr{M}}$. Then*

(i) *there exists an open neighborhood Ω_0 of w_0 such that*

$$K(\cdot, w) := K_w = \overline{g_1^0(w)} K_w^{(1)} + \cdots + \overline{g_d^0(w)} K_w^{(d)}, \quad w \in \Omega_0,$$

for some choice of anti-holomorphic functions $K^{(1)}, \ldots, K^{(d)} : \Omega_0 \to \mathscr{M}$,

(ii) *the vectors $K_w^{(i)}$, $1 \leqq i \leqq d$, are linearly independent in $\mathscr{M}$ for w in some neighborhood of w_0,*

(iii) *the vectors $\{K_{w_0}^{(i)} : 1 \leqq i \leqq d\}$ are uniquely determined by the generators $g_1^0, \ldots, g_d^0$,*

(iv) *the linear span of the set of vectors $\{K_{w_0}^{(i)} : 1 \leqq i \leqq d\}$ in $\mathscr{M}$ is independent of the generators $g_1^0, \ldots, g_d^0$, and*

(v) *$M_p^* K_{w_0}^{(i)} = \overline{p(w_0)} K_{w_0}^{(i)}$ for all i, $1 \leqq i \leqq d$, where M_p denotes the module multiplication by the polynomial p.*

The module map

$$\mathscr{O} \mathbin{\hat{\otimes}}_{\mathscr{O}(\mathbb{C}^m)} \mathscr{M} \to \mathscr{S}^{\mathscr{M}}$$

induced from (1.1) is surjective. This naturally defines a surjective map

$$\mathscr{M}/\mathfrak{m}_{w_0}\mathscr{M} \cong \mathscr{O}_{w_0}/\mathfrak{m}_{w_0}\mathscr{O}_{w_0} \otimes \mathscr{M} \to \mathscr{S}_{w_0}^{\mathscr{M}}/\mathfrak{m}_{w_0}\mathscr{S}_{w_0}^{\mathscr{M}} \tag{1.3}$$

for $w \in \Omega$. In particular,

$$\dim(\mathscr{M}/\mathfrak{m}_{w_0}\mathscr{M}) \geqq \dim(\mathscr{S}_{w_0}^{\mathscr{M}}/\mathfrak{m}_{w_0}\mathscr{S}_{w_0}^{\mathscr{M}}). \tag{1.4}$$

We remark that the map into the Grassmannian manifold $\Gamma_K : \Omega_0^* \to \mathrm{Gr}(\mathscr{M}, d)$ defined by $\Gamma_K(\overline{w}) = (K_w^{(1)}, \ldots, K_w^{(d)})$ is holomorphic. The pull-back of the canonical bundle on $\mathrm{Gr}(\mathscr{M}, d)$ under Γ_K defines a holomorphic Hermitian vector bundle on the open set Ω_0^*. Unfortunately, the decomposition of the reproducing kernel given in Theorem 1.4 is not canonical except when the stalk is singly generated. In this special case, the holomorphic Hermitian bundle obtained in this manner is indeed canonical. However, in general, it is not clear if this vector bundle contains any useful information. Suppose we have equality in (1.4) for a Hilbert module $\mathscr{M}$. Then it is possible to obtain a canonical decomposition following [7], which leads in the same manner as above, to the construction of a Hermitian holomorphic vector bundle in a neighborhood of each point $w \in \Omega$.

For any fixed but arbitrary $w_0 \in \Omega$ and a small enough neighborhood Ω_0 of w_0, the proof of Theorem 2.2 from [7] shows the existence of a holomorphic function $P_{\overline{w_0}} : \Omega_0^* \to \mathscr{L}(\mathscr{M})$ with the property that the operator $P_{\overline{w_0}}$ restricted to the subspace $\ker D_{(\boldsymbol{M}-w_0)^*}$ is invertible. The range of $P_{\overline{w_0}}$ can then be seen to be equal to the kernel of the operator $\mathbb{P}_0 D_{(\boldsymbol{M}-w)^*}$, where $\mathbb{P}_0$ is the orthogonal projection onto $\operatorname{ran} D_{(\boldsymbol{M}-w_0)^*}$.

Lemma 1.5. *The dimension of* $\ker \mathbb{P}_0 D_{(\boldsymbol{M}-w)^*}$ *is constant in a suitably small neighborhood* Ω_0 *of* w_0 *in* Ω.

Let $\{e_0, \dots, e_k\}$ be a basis for $\ker D_{(\boldsymbol{M}-w_0)^*}$. Since $P_{\overline{w_0}}$ is holomorphic on Ω_0^*, it follows that $\gamma_1(\overline{w}) := P_{\overline{w_0}}(\overline{w})e_1, \dots, \gamma_k(\overline{w}) := P_{\overline{w_0}}(\overline{w})e_k$ are holomorphic on Ω_0^*. Thus $\Gamma : \Omega_0^* \to \mathrm{Gr}(\mathscr{M}, k)$, given by $\Gamma(\overline{w}) = \ker \mathbb{P}_0 D_{(\boldsymbol{M}-w)^*}$, $w \in \Omega_0$, defines a holomorphic Hermitian vector bundle $\mathscr{P}_0$ on Ω_0^* of rank k corresponding to the Hilbert module $\mathscr{M}$.

Theorem 1.6. *If any two Hilbert modules* $\mathscr{M}$ *and* $\tilde{\mathscr{M}}$ *belonging to the class* $\mathfrak{B}_1(\Omega)$ *are isomorphic via a unitary module map, then the corresponding vector bundles* $\mathscr{P}_0$ *and* $\tilde{\mathscr{P}}_0$ *on* Ω_0^* *are equivalent as holomorphic Hermitian vector bundles.*

This result shows that complex geometric invariants of the holomorphic Hermitian vector bundle corresponding to Hilbert modules $\mathscr{M}$ and $\tilde{\mathscr{M}}$ in $\mathfrak{B}_1(\Omega)$ would distinguish the unitary orbits of these Hilbert modules. Two examples are included in the last section. The first of the two examples illustrates the computation of these invariants while the second describes the construction of an alternative unitary invariant (see also [3]).

Leaving for the next section the complications related to constructing curvature type invariants, we return to the key extremal case in inequality (1.4). The question of equality in (1.4) is the same as the question of whether the map in (1.3) is an isomorphism and can be interpreted as a global factorization problem. To be more specific, we say that an analytic Hilbert module $\mathscr{M}$ (cf. [5], page 3) possesses *Gleason's property at a point* $w_0 \in \Omega$ if for every element $f \in \mathscr{M}$ vanishing at w_0 there are $f_1, \dots, f_m \in \mathscr{M}$ such that $f = \sum_{i=1}^{m} (z_i - w_{0i}) f_i$. We have generalized the notion of solvability of Gleason's problem for AF-co-submodules (cf. [5], page 38) and will prove in Section 2 that

Proposition 1.7. *Any* AF*-co-submodule* $\mathscr{M}$ *has Gleason's property at* w_0 *if and only if*

$$\dim(\mathscr{M}/\mathfrak{m}_{w_0}\mathscr{M}) = \dim(\mathscr{S}_{w_0}^{\mathscr{M}}/\mathfrak{m}_{w_0}\mathscr{S}_{w_0}^{\mathscr{M}}).$$

This is a special case of a more general division problem for Hilbert modules. To fix ideas, we consider the following setting: let $\mathscr{M}$ be an analytic Hilbert module with the domain Ω disjoint of its essential spectrum, let $A \in M_{p,q}\big(\mathscr{O}(\overline{\Omega})\big)$ be a matrix of analytic functions defined in a neighborhood of $\overline{\Omega}$, where p, q are positive integers, and let $f \in \mathscr{M}^p$. Given a solution $u \in \mathscr{O}(\Omega)^q$ to the linear equation $Au = f$, is it true that $u \in \mathscr{M}^q$? Numerous "hard analysis" questions, such as problems of moduli, or Corona Problem, can be put into this framework.

We study below this very division problem in conjunction with an earlier work of the third author [30] dealing with the "disc" algebra $\mathscr{A}(\Omega)$ instead of Hilbert modules, and within the general concept of "privilege" introduced by Douady more than forty years ago [9], [10].

We only focus on the case of Bergman space below. Specifically, the $\mathscr{A}(\Omega)$-module $\mathscr{N} = \mathrm{coker}(A : \mathscr{M} \otimes_{\mathbb{C}} \mathbb{C}^p \to \mathscr{M} \otimes_{\mathbb{C}} \mathbb{C}^q)$ is called *privileged with respect to the module* $\mathscr{M}$ if it is a Hilbert module in the quotient metric and there exists a resolution

$$(1.5) \qquad 0 \to \mathscr{M} \otimes_{\mathbb{C}} \mathbb{C}^{n_p} \xrightarrow{d_p} \cdots \to \mathscr{M} \otimes_{\mathbb{C}} \mathbb{C}^{n_1} \xrightarrow{d_1} \mathscr{M} \otimes_{\mathbb{C}} \mathbb{C}^{n_0} \to \mathscr{N} \to 0,$$

where $d_q \in M_{n_{q+1},n_q}\big(\mathscr{A}(\Omega)\big)$ and $d_1 = A$. Note that implicitly in the statement it is assumed that the range of the operator A is closed at the level of the Hilbert module $\mathscr{M}$.

An affirmative answer to the division problem is equivalent to the question of "privilege" in case of the Bergman module on a strictly convex bounded domain Ω with smooth boundary.

Theorem 1.8. *Let $\Omega \subset \mathbb{C}^m$ be a strictly convex domain with smooth boundary, let p, q be positive integers and let $A \in M_{p,q}\big(\mathscr{A}(\Omega)\big)$ be a matrix of analytic functions belonging to the disk algebra of Ω. The following assertions are equivalent*:

(a) *The analytic module* $\operatorname{coker}\big(A : L_a^2(\Omega)^p \to L_a^2(\Omega)^q\big)$ *is privileged with respect to the Bergman space.*

(b) *The function* $\zeta \mapsto \operatorname{rank} A(\zeta)$, $\zeta \in \partial\Omega$, *is constant.*

(c) *Let $f \in L_a^2(\Omega)^q$. The equation $Au = f$ has a solution $u \in L_a^2(\Omega)^p$ if and only if it has a solution $u \in \mathcal{O}(\Omega)^p$.*

While we have stated our results for the Bergman space, they remain true for the Hardy space $H^2(\partial\Omega)$, that is, the closure of entire functions in the L^2-space with respect to the surface area measure supported on $\partial\Omega$. Also, the results remain true for the Bergman or Hardy spaces of a poly-domain $\Omega = \Omega_1 \times \cdots \times \Omega_d$, where $\Omega_j \subset \mathbb{C}$, $1 \leqq j \leqq d$, are convex bounded domains with smooth boundary in $\mathbb{C}$. For these Hilbert modules, the notion of the sheaf model from the earlier work of [26], [27] coincides with the sheaf model described here. Details will be given in the third section below.

We finish the introduction by exhibiting a class of Hilbert modules for which the Gleason problem admits a solution.

Theorem 1.9. *If $\mathscr{M}$ is a submodule of an analytic Hilbert module of finite codimension with the zero set $V(\mathscr{M}) \subset \Omega$, then the Gleason problem for the Hilbert module $\mathscr{M}$ admits a solution.*

This theorem isolates a large family of Hilbert modules in $\mathfrak{B}_1(\Omega)$ to which our classification scheme, using the curvature invariant, applies.

Index of notations.

$\mathbb{C}[\underline{z}]$	the polynomial ring $\mathbb{C}[z_1, \dots, z_m]$ of m-complex variables,
$\mathfrak{m}_w$	the maximal ideal of $\mathbb{C}[\underline{z}]$ at the point $w \in \mathbb{C}^m$,
Ω^*	$\{\bar{z} : z \in \Omega\}$ for a bounded domain $\Omega \subseteq \mathbb{C}^m$,
$\mathbb{D}^m$	the unit polydisc in $\mathbb{C}^m$,
M_i	the module multiplication by the co-ordinate function z_i, $1 \leqq i \leqq m$,
M_i^*	the adjoint of the operator M_i, z_i, $1 \leqq i \leqq m$,

$D_{(\boldsymbol{M}-w)^*}$	the operator $\mathscr{M} \to \mathscr{M} \oplus \cdots \oplus \mathscr{M}$ defined by $f \mapsto \big((M_j - w_j)^* f\big)_{j=1}^m$,
$\mathscr{O}(\Omega)$	the sheaf of holomorphic functions on Ω,
$\mathscr{O}_w$	germs of holomorphic functions at the point $w \in \mathbb{C}^m$,
$\hat{\mathscr{H}}$	the analytic localization $\mathscr{O} \hat{\otimes}_{\mathscr{O}(\mathbb{C}^m)} \mathscr{H}$ of the Hilbert module $\mathscr{H}$,
∂^α, $\bar{\partial}^\alpha$	$\partial^\alpha = \dfrac{\partial^{\lvert\alpha\rvert}}{\partial z_1^{\alpha_1} \cdots z_m^{\alpha_m}}$, $\quad \bar{\partial}^\alpha = \dfrac{\partial^{\lvert\alpha\rvert}}{\partial \bar{z}_1^{\alpha_1} \cdots \bar{z}_m^{\alpha_m}}$, $\quad \alpha = (\alpha_1, \dots, \alpha_m) \in \mathbb{Z}^+ \times \cdots \times \mathbb{Z}^+$, $\lvert\alpha\rvert = \sum_{i=1}^m \alpha_i$,
$q(D)$	the differential operator $\sum_\alpha a_\alpha \partial^\alpha$, where $q = \sum_\alpha a_\alpha z^\alpha$,
$\mathrm{B}_n(\Omega)$	Cowen–Douglas class of operators of rank n, also, Hilbert modules such that $\boldsymbol{M}^* = (M_1^*, \dots, M_m^*) \in \mathrm{B}_n(\Omega^*)$,
$\mathscr{S}^{\mathscr{M}}$	the analytic submodule of $\mathscr{O}(\Omega)$, corresponding to $\mathscr{M}$ in $\mathfrak{B}(\Omega)$,
$K(z, w)$	a reproducing kernel,
$E(w)$	the evaluation functional (the linear functional induced by $K(\cdot, w)$),
$\lVert \cdot \rVert_{\bar{\Delta}(0;r)}$	supremum norm,
$\lVert \cdot \rVert_2$	the L^2 norm with respect to the volume measure,
$\mathscr{M}^{(w)}$	the submodule of $\mathscr{M}$ which is of the form $\sum_{j=1}^m (z_j - w_j)\mathscr{M}$,
$V(\mathscr{F})$	$\{z \in \Omega : f(z) = 0 \text{ for all } f \in \mathscr{F}\}$, where $\mathscr{F} \subset \mathscr{O}(\Omega)$,
$\mathbb{V}_w(\mathscr{F})$	$\{q \in \mathbb{C}[\underline{z}] : q(D)f\rvert_w = 0,\ f \in \mathscr{F}\}$ is the characteristic space at w for some set $\mathscr{F}$ of holomorphic functions in a neighborhood of w,
$\tilde{\mathbb{V}}_w(\mathscr{F})$	$\left\{q \in \mathbb{C}[\underline{z}] : \dfrac{\partial q}{\partial z_i} \in \mathbb{V}_w(\mathscr{F}),\ 1 \leqq i \leqq m\right\}$ for some set $\mathscr{F}$ of holomorphic functions in a neighborhood of w,
$[\mathscr{I}]$	the closure of the polynomial ideal $\mathscr{I} \subseteq \mathscr{M}$ in some Hilbert module $\mathscr{M}$,
$\mathscr{A}(\Omega)$	the "disc" algebra over Ω, which is $\mathscr{O}(\Omega) \cap C(\overline{\Omega})$,
$\mathscr{O}(\overline{\Omega})$	the space of germs of analytic functions in a neighborhood of $\overline{\Omega}$,
$\mathbb{P}_0$	the orthogonal projection onto $\operatorname{ran} D_{(\boldsymbol{M}-w_0)^*}$,
$\mathscr{P}_w$	$\ker \mathbb{P}_0 D_{(\boldsymbol{M}-w)^*}$ for $w \in \Omega$.

2. Unitary classification via Hermitian spaces

Throughout this section, the Hilbert module $\mathscr{M}$ is assumed to be in the class $\mathfrak{B}_1(\Omega)$. We prove below a series of technical results culminating with construction of the new curvature invariants for $\mathscr{M}$.

2.1. Coherence of the sheaf $\mathscr{S}^{\mathscr{M}}$.

Proposition 2.1. *For any Hilbert module $\mathscr{M}$ in $\mathfrak{B}_1(\Omega)$, the sheaf $\mathscr{S}^{\mathscr{M}}$ is coherent.*

Proof. The sheaf $\mathscr{S}^{\mathscr{M}}$ is generated by the family $\{f : f \in \mathscr{M}\}$ of global sections of the sheaf $\mathscr{O}(\Omega)$. Let J be a finite subset of $\mathscr{M}$ and $\mathscr{S}_J^{\mathscr{M}} \subseteq \mathscr{O}(\Omega)$ be the subsheaf generated by the sections $f, f \in J$. It follows (see [23], Corollary 9, page 130) that $\mathscr{S}_J^{\mathscr{M}}$ is coherent. The family $\{\mathscr{S}_J^{\mathscr{M}} : J \text{ is a finite subset of } \mathscr{M}\}$ is increasingly filtered, that is, for any two finite subset I and J of $\mathscr{M}$, the union $I \cup J$ is again a finite subset of $\mathscr{M}$ and $\mathscr{S}_I^{\mathscr{M}} \cup \mathscr{S}_J^{\mathscr{M}} \subset \mathscr{S}_{I \cup J}^{\mathscr{M}}$. Also, clearly $\mathscr{S}^{\mathscr{M}} = \bigcup_J \mathscr{S}_J^{\mathscr{M}}$. Using Noether's Lemma [22], page 111, which says that every increasingly filtered family of coherent sheaves must be stationary, we conclude that the analytic sheaf $\mathscr{S}^{\mathscr{M}}$ is coherent. □

Remark 2.2. Let $\mathscr{M}$ be a module in $\mathfrak{B}_1(\Omega)$ with Ω pseudoconvex and a finite set of generators $\{f_1, \dots, f_t\}$. From [5], Lemma 2.3.2, it follows that the associated sheaf $\mathscr{S}^{\mathscr{M}}(\Omega)$ is not only coherent, it has global generators $\{f_1, \dots, f_t\}$, that is, $\{f_{1w}, \dots, f_{tw}\}$ generates the stalk $\mathscr{S}_w^{\mathscr{M}}$ for every $w \in \Omega$. Theorem 2.3.3 of [5] (or equivalently [25], Theorem 7.2.5) is a consequence of the Cartan Theorem B (cf. [25], Theorem 7.1.7) together with the coherence of every locally finitely generated subsheaf of $\mathscr{O}^k$ (cf. [25], Theorem 7.1.8). It is then easy to verify that if $\mathscr{M}$ is any module in $\mathfrak{B}_1(\Omega)$ and if $\{f_1, \dots, f_t\}$ is a finite set of generators for $\mathscr{M}$, then for $f \in \mathscr{M}$, there exist $g_1, \dots, g_t \in \mathscr{O}(\Omega)$ such that

$$f = f_1 g_1 + \cdots + f_t g_t. \tag{2.1}$$

The following lemma isolates a large class of elements from $\mathfrak{B}_1(\Omega)$ which belong to $\mathrm{B}_1(\Omega_0)$ for some open subset $\Omega_0 \subseteqq \Omega$.

Lemma 2.3. *Suppose $\mathscr{M} \in \mathfrak{B}_1(\Omega)$ is the closure of a polynomial ideal $\mathscr{I}$. Then $\mathscr{M}$ is in $\mathrm{B}_1(\Omega)$ if the ideal $\mathscr{I}$ is principal while if $p_1, p_2, \dots, p_t$ $(t > 1)$ is a minimal set of generators for $\mathscr{I}$, then $\mathscr{M}$ is in $\mathrm{B}_1(\Omega \backslash X)$ for $X = \bigcap_{i=1}^{t} \{z : p_i(z) = 0\} \cap \Omega$.*

Proof. The proof is a refinement of the argument given in [13], p. 285. Let γ_w be any eigenvector at w for the adjoint of the module multiplication, that is, $M_p^* \gamma_w = \overline{p(w)} \gamma_w$ for $p \in \mathbb{C}[\underline{z}]$.

First, assume that the module $\mathscr{M}$ is generated by the single polynomial, say p. In this case, $K(z, w) = p(z) \chi(z, w) \overline{p(w)}$ for some positive definite kernel χ on all of Ω. Set $K_1(z, w) = p(z) \chi(z, w)$ and note that $K_1(\cdot, w)$ is a non-zero eigenvector at $w \in \Omega$. We have

$$\langle pq, \gamma_w \rangle = \langle p, M_q^* \gamma_w \rangle = \langle p, \overline{q(w)} \gamma_w \rangle = q(w) \langle p, \gamma_w \rangle.$$

Also, we have

$$p(w) q(w) \langle p, \gamma_w \rangle = \langle pq, K(\cdot, w) \rangle \langle p, \gamma_w \rangle = p(w) \langle pq, \overline{\langle p, \gamma_w \rangle} K_1(\cdot, w) \rangle.$$

The analytic function $q(w) \langle p, \gamma_w \rangle - \langle pq, \overline{\langle p, \gamma_w \rangle} K_1(\cdot, w) \rangle$ on Ω is equal to 0 on $\Omega \backslash \{z : p(z) = 0\}$ and hence is 0 on Ω (as Ω is connected). Thus

$$\langle pq, \gamma_w \rangle = \langle pq, \overline{\langle p, \gamma_w \rangle} K_1(\cdot, w) \rangle.$$

Since vectors of the form $\{pq : q \in \mathbb{C}[\underline{z}]\}$ are dense in $\mathscr{M}$, it follows that $\gamma_w = \overline{\langle p, \gamma_w \rangle} K_1(\cdot, w)$ and the proof is complete in this case.

Now, assume that $p_1, \dots, p_t$ is a set of generators for the ideal $\mathscr{I}$. Then for $w \notin X$, there exists a $k \in \{1, \dots, t\}$ such that $p_k(w) \neq 0$. We note that for any i, $1 \leqq i \leqq m$,

$$p_k(w) \langle p_i, \gamma_w \rangle = \langle p_i, M_{p_k}^* \gamma_w \rangle = \langle p_i p_k, \gamma_w \rangle = \langle p_k, M_{p_i}^* \gamma_w \rangle = p_i(w) \langle p_k, \gamma_w \rangle.$$

Therefore we have

$$\left\langle \sum_{i=1}^{t} p_i q_i, \gamma_w \right\rangle = \sum_{i=1}^{t} \langle p_i, M_{q_i}^* \gamma_w \rangle$$
$$= \sum_{i=1}^{t} q_i(w) \langle p_i, \gamma_w \rangle$$
$$= \sum_{i=1}^{t} \left\langle p_i q_i, \frac{\overline{\langle p_k, \gamma_w \rangle} K(\cdot, w)}{\overline{p_k(w)}} \right\rangle.$$

Setting $c(w) = \dfrac{\langle p_k, \gamma_w \rangle}{p_k(w)}$, we have

$$\left\langle \sum_{i=1}^{t} p_i q_i, \gamma_w \right\rangle = \left\langle \sum_{i=1}^{t} p_i q_i, \overline{c(w)} K(\cdot, w) \right\rangle.$$

Since vectors of the form $\left\{ \sum_{i=1}^{t} p_i q_i : q_i \in \mathbb{C}[\underline{z}], 1 \leqq i \leqq t \right\}$ are dense in $\mathscr{M}$, it follows that $\gamma_w = \overline{c(w)} K(\cdot, w)$ completing the proof of the second half. □

2.2. The proof of the decomposition theorem.

Proof of Theorem 1.4. For simplicity of notation, we assume without loss of generality, that $0 = w_0 \in \Omega$. Let $\{e_n\}_{n=0}^{\infty}$ be an orthonormal basis for $\mathscr{M}$. From the property of reproducing kernels, we have

$$K(z, w) = \sum_{n=0}^{\infty} e_n(z) \overline{e_n(w)}, \quad z, w \in \Omega.$$

It follows from [23], Theorem 2, page 82, that for every element f in $\mathscr{S}_0^{\mathscr{M}}$, and therefore in particular for every e_n, we have

$$e_n(z) = \sum_{i=1}^{d} g_i^0(z) h_i^{(n)}(z), \quad z \in \Delta(0; r),$$

for some holomorphic functions $h_i^{(n)}$ defined on the closed polydisc $\overline{\Delta}(0; r) \subseteqq \Omega$. Furthermore, these functions can be chosen with the bound $\|h_i^{(n)}\|_{\overline{\Delta}(0;r)} \leqq C \|e_n\|_{\overline{\Delta}(0;r)}$ for some positive constant C independent of n. Although, the decomposition is not necessarily with respect to the standard coordinate system at 0, we will be using only a point wise estimate. Consequently, in the equation given above, we have chosen not to emphasize the change of variable involved and we have

$$K(z, w) = \sum_{n=0}^{\infty} \left\{ \sum_{i=1}^{d} \overline{g_i^0(w)} \overline{h_i^{(n)}(w)} \right\} e_n(z) = \sum_{i=1}^{d} \overline{g_i^0(w)} \left\{ \sum_{n=0}^{\infty} \overline{h_i^{(n)}(w)} e_n(z) \right\}.$$

Setting $K_w^{(i)}(z) (= K_i(z, w))$ to be the sum $\sum_{n=0}^{\infty} \overline{h_i^{(n)}(w)} e_n(z)$, we can write

$$K(z, w) = \sum_{i=1}^{d} \overline{g_i^0(w)} K_w^{(i)}(z), \quad w \in \Delta(0; r).$$

The function K_i is holomorphic in the first variable and antiholomorphic in the second by construction. For the proof of part (i), we need to show that $K_w^{(i)} \in \mathscr{M}$ where $w \in \Delta(0;r)$. Or, equivalently, we have to show that $\sum_{n=0}^{\infty} |h_i^{(n)}(w)|^2 < \infty$ for each $w \in \Delta(0;r)$. First, using the estimate on $h_i^{(n)}$, we have

$$|h_i^{(n)}(w)| \leqq \|h_i^{(n)}\|_{\bar{\Delta}(0;r)} \leqq C\|e_n\|_{\bar{\Delta}(0;r)}.$$

We prove below, the inequality $\sum_{n=0}^{\infty} \|e_n\|^2_{\bar{\Delta}(0;r)} < \infty$ completing the proof of part (i). We prove, more generally, that for $f \in \mathscr{M}$,

$$\|f\|_{\bar{\Delta}(0;r)} \leqq C'\|f\|_{2,\bar{\Delta}(0;r)}, \tag{2.2}$$

where $\|\cdot\|_2$ denotes the L^2 norm with respect to the volume measure on $\bar{\Delta}(0;r)$. It is evident from the proof that the constant C' may be chosen to be independent of the functions f.

Any function f holomorphic on Ω belongs to the Bergman space $L_a^2\big(\Delta(0;r+\varepsilon)\big)$ as long as $\Delta(0;r+\varepsilon) \subseteqq \Omega$. We can surely pick $\varepsilon > 0$ small enough to ensure $\Delta(0;r+\varepsilon) \subseteqq \Omega$. Let B be the Bergman kernel of the Bergman space $L_a^2\big(\Delta(0;r+\varepsilon)\big)$. Thus we have

$$|f(w)| = |\langle f, B(\cdot,w)\rangle| \leqq \|f\|_{2,\Delta(0;r+\varepsilon)} B(w,w)^{1/2}, \quad w \in \Delta(0;r+\varepsilon).$$

Since the function $B(w,w)$ is bounded on compact subsets of $\Delta(0;r+\varepsilon)$, it follows that $C'^2 := \sup\{B(w,w) : w \in \bar{\Delta}(0;r)\}$ is finite. We therefore see that

$$\|f\|_{\bar{\Delta}(0;r)} = \sup\{|f(w)| : w \in \bar{\Delta}(0;r)\} \leqq C'\|f\|_{2,\Delta(0;r+\varepsilon)}.$$

Since $\varepsilon > 0$ can be chosen arbitrarily close to 0, we infer the inequality (2.2).

The inequality (2.2) implies, in particular, that

$$\sum_{n=0}^{\infty} \|e_n\|^2_{\bar{\Delta}(0;r)} \leqq C'^2 \sum_{n=0}^{\infty} \int_{\bar{\Delta}(0;r)} |e_n(z)|^2 \, dz_1 \wedge d\bar{z}_1 \wedge \cdots \wedge dz_m \wedge d\bar{z}_m.$$

Since $K_z := K(\cdot,z) = \sum_{n=0}^{\infty} \overline{e_n(z)} e_n$, the function $G(z) := \sum_{n=0}^{\infty} |e_n(z)|^2$ is finite for each $z \in \Omega$. The sequence of positive continuous functions $G_k(z) := \sum_{n=0}^{k} |e_n(z)|^2$ converges uniformly to G on $\bar{\Delta}(0;r)$. To see this, we note that

$$\begin{aligned}\|G_k - G\|^2_{\bar{\Delta}(0;r)} &\leqq C'^2 \int_{\bar{\Delta}(0;r)} |G_k(z) - G(z)|^2 \, dz_1 \wedge d\bar{z}_1 \wedge \cdots \wedge dz_m \wedge d\bar{z}_m \\ &\leqq C'^2 \int_{\bar{\Delta}(0;r)} \left\{ \sum_{n=k+1}^{\infty} |e_n(z)|^2 \right\}^2 dz_1 \wedge d\bar{z}_1 \wedge \cdots \wedge dz_m \wedge d\bar{z}_m,\end{aligned}$$

which tends to 0 as $k \to \infty$. So, by Monotone Convergence Theorem, we can interchange the integral and the infinite sum to conclude

$$\sum_{n=0}^{\infty} \|e_n\|^2_{\bar{\Delta}(0;r)} \leqq C \int_{\bar{\Delta}(0;r)} \sum_{n=0}^{\infty} |e_n(z)|^2 \, dz_1 \wedge d\bar{z}_1 \wedge \cdots \wedge dz_m \wedge d\bar{z}_m < \infty$$

as G is a continuous function on $\bar{\Delta}(0;r)$. This shows that

$$\sum_{n=0}^{\infty} |h_i^{(n)}(w)|^2 \leqq K \sum_{n=0}^{\infty} \|e_n\|^2_{\bar{\Delta}(0;r)} < \infty.$$

Hence $K_w^{(i)} \in \mathcal{M}$, $1 \leqq i \leqq d$.

To prove statement (ii), at 0, we have to show that whenever there exist complex numbers $\alpha_1, \dots, \alpha_d$ such that $\sum_{i=1}^{d} \alpha_i K_i(z,0) = 0$, then $\alpha_i = 0$ for all i. We assume, on the contrary, that there exists some $i \in \{1, \dots, d\}$ such that $\alpha_i \neq 0$. Without loss of generality, we assume $\alpha_1 \neq 0$, then $K_1(z,0) = \sum_{i=2}^{d} \beta_i K_i(z,0)$ where $\beta_i = \frac{\alpha_i}{\alpha_1}$, $2 \leqq i \leqq d$. This shows that $K_1(z,w) - \sum_{i=2}^{d} \beta_i K_i(z,w)$ has a zero at $w = 0$. From [25], Theorem 7.2.9, it follows that

$$K_1(z,w) - \sum_{i=2}^{d} \beta_i K_i(z,w) = \sum_{j=1}^{m} \overline{w}_j G_j(z,w)$$

for some function $G_j : \Omega \times \Delta(0;r) \to \mathbb{C}$, $1 \leqq j \leqq m$, which is holomorphic in the first and antiholomorphic in the second variable. So, we can write

$$\begin{aligned} K(z,w) &= \sum_{i=1}^{d} \bar{g}_i^0(w) K_i(z,w) = \bar{g}_1^0(w) K_1(z,w) + \sum_{i=2}^{d} \bar{g}_i^0(w) K_i(z,w) \\ &= \bar{g}_1^0(w) \left\{ \sum_{i=2}^{d} \beta_i K_i(z,w) + \sum_{j=1}^{m} \overline{w}_j G_j(z,w) \right\} + \sum_{i=2}^{d} \bar{g}_i^0(w) K_i(z,w) \\ &= \sum_{i=2}^{d} \big(\bar{g}_i^0(w) + \beta_i \bar{g}_1^0(w)\big) K_i(z,w) + \sum_{j=1}^{m} \overline{w}_j \bar{g}_1^0(w) G_j(z,w). \end{aligned}$$

For $f \in \mathcal{M}$ and $w \in \Delta(0;r)$, we have

$$\begin{aligned} f(w) &= \langle f, K(\cdot,w) \rangle \\ &= \sum_{i=2}^{d} \big(g_i^0(w) + \bar{\beta}_i g_1^0(w)\big) \langle f, K_i(z,w) \rangle + g_1^0(w) \left\langle f, \sum_{j=1}^{m} \overline{w}_j G_j(z,w) \right\rangle. \end{aligned}$$

We note that $\left\langle f, \sum_{j=1}^{m} \overline{w}_j G_j(z,w) \right\rangle$ is a holomorphic function in w which vanishes at $w = 0$. It then follows that $\left\langle f, \sum_{j=1}^{m} \overline{w}_j G_j(z,w) \right\rangle = \sum_{j=1}^{m} w_j \tilde{G}_j(w)$ for some holomorphic functions $\tilde{G}_j$,

$1 \leqq j \leqq m$, on $\Delta(0;r)$. Therefore, we have

$$f(w) = \sum_{i=2}^{d} \big(g_i^0(w) + \bar{\beta}_i g_1^0(w)\big)\langle f, K_i(z,w)\rangle + \sum_{j=1}^{m} w_j g_1^0(w)\tilde{G}_j(w).$$

Since the sheaf $\mathscr{S}^{\mathscr{M}}|_{\Delta(0;r)}$ is generated by the Hilbert module $\mathscr{M}$, it follows that the set $\{g_2^0 + \bar{\beta}_2 g_1^0, \ldots, g_d^0 + \bar{\beta}_d g_1^0, z_1 g_1^0, \ldots, z_m g_1^0\}$ also generates $\mathscr{S}^{\mathscr{M}}|_{\Delta(0;r)}$. In particular, they generate the stalk at 0. This, we claim, is a contradiction. Suppose $A \subset \mathscr{S}_0^{\mathscr{M}}$ is generated by germs of the functions $g_2^0 + \bar{\beta}_2 g_1^0, \ldots, g_d^0 + \bar{\beta}_d g_1^0$. Let $\mathfrak{m}(\mathscr{O}_0)$ denote the only maximal ideal of the local ring $\mathscr{O}_0$, consisting of the germs of functions vanishing at 0. Then it follows that

$$\mathfrak{m}(\mathscr{O}_0)\{\mathscr{S}_0^{\mathscr{M}}/A\} = \mathscr{S}_0^{\mathscr{M}}/A.$$

Using Nakayama's Lemma (cf. [33], p. 57), we see that $\mathscr{S}_0^{\mathscr{M}}/A = 0$, that is, $\mathscr{S}_0^{\mathscr{M}} = A$. This contradicts the minimality of the generators of the stalk at 0 completing the proof of the first half of (ii).

To prove the slightly stronger statement, namely, the independence of the vectors $K_w^{(i)}$, $1 \leqq i \leqq d$, in a small neighborhood of 0, consider the Grammian $(\!(\langle K_w^{(i)}, K_w^{(j)}\rangle)\!)$. The determinant of this Grammian is nonzero at 0. Therefore it remains non-zero in a suitably small neighborhood of 0 since it is a real analytic function on Ω_0. Consequently, the vectors $K_w^{(i)}$, $i = 1, \ldots, d$ are linearly independent for all w in this neighborhood.

To prove statement (iii), we have to prove that $K_0^{(i)}$ are uniquely determined by the generators g_i^0, $1 \leqq i \leqq d$. We will let g_i^0 denote the germ of g_i^0 at 0 as well. Let $K(z,w) = \sum_{i=1}^{d} \overline{g_i^0(w)}\tilde{K}_w^{(i)}$ be another decomposition. Let $\tilde{K}_w^{(i)} = \sum_{n=0}^{\infty} \overline{\tilde{h}_i^n(w)} e_n$ for some holomorphic functions on some small enough neighborhood of 0. Thus we have

$$\sum_{n=0}^{\infty}\sum_{i=1}^{d} \overline{g_i^0(w)}\{\overline{h_i^n(w)} - \overline{\tilde{h}_i^n(w)}\} e_n = 0.$$

Hence, for each n

$$\sum_{i=1}^{d} g_i^0(z)\{h_i^n(z) - \tilde{h}_i^n(z)\} = 0.$$

Fix n and let $\alpha_i(z) = h_i^n(z) - \tilde{h}_i^n(z)$. In this notation, $\sum_{i=1}^{d} g_i^0(z)\alpha_i(z) = 0$. Now we claim that $\alpha_i(0) = 0$ for all $i \in \{1, \ldots, d\}$. If not, we may assume $\alpha_1(0) \neq 0$. Then the germ of α_1 at 0 is a unit in $\mathscr{O}_0$. Hence we can write, in $\mathscr{O}_0$,

$$g_1^0 = -\left(\sum_{i=2}^{d} g_i^0 \alpha_{i0}\right)\alpha_{10}^{-1},$$

where α_{i0} denotes the germs of the analytic functions α_i at 0, $1 \leqq i \leqq d$. This is a contradiction, as $g_1^0, \ldots, g_d^0$ is a minimal set of generators of the stalk $\mathscr{S}_0^{\mathscr{M}}$ by hypothesis. As a result, $h_i^n(0) = \tilde{h}_i^n(0)$ for all $i \in \{1, \ldots, d\}$ and $n \in \mathbb{N} \cup \{0\}$. This completes the proof of (iii).

To prove statement (iv), let $\{g_1^0, \ldots, g_d^0\}$ and $\{\tilde{g}_1^0, \ldots, \tilde{g}_d^0\}$ be two sets of generators for $\mathscr{S}_0^{\mathscr{M}}$ both of which are minimal. Let $K^{(i)}$ and $\tilde{K}^{(i)}$, $1 \leqq i \leqq d$, be the corresponding vectors that appear in the decomposition of the reproducing kernel K as in (i). It is enough to show that

$$\operatorname{span}_{\mathbb{C}}\{K_i(z,0) : 1 \leqq i \leqq d\} = \operatorname{span}_{\mathbb{C}}\{\tilde{K}_i(z,0) : 1 \leqq i \leqq d\}.$$

There exist holomorphic functions ϕ_{ij}, $1 \leqq i, j \leqq d$, in a small enough neighborhood of 0 such that $\tilde{g}_i^0 = \sum_{j=1}^{d} \phi_{ij} g_j^0$. For w, possibly from an even smaller neighborhood of 0, it follows that

$$\begin{aligned} K(z,w) &= \sum_{i=1}^{d} \bar{\tilde{g}}_i^0(w) \tilde{K}_i(z,w) \\ &= \sum_{i=1}^{d} \left(\sum_{j=1}^{d} \bar{\phi}_{ij}(w) \bar{g}_j^0(w) \right) \tilde{K}_i(z,w) \\ &= \sum_{j=1}^{d} \bar{g}_j^0(w) \left(\sum_{i=1}^{d} \bar{\phi}_{ij}(w) \tilde{K}_i(z,w) \right). \end{aligned}$$

But $K(z,w) = \sum_{j=1}^{d} \bar{g}_j^0(w) K_j(z,w)$ and uniqueness at the point 0 implies that

$$K_j(z,0) = \sum_{i=1}^{d} \bar{\phi}_{ij}(0) \tilde{K}_i(z,0)$$

for $1 \leqq j \leqq d$. So, we have $\operatorname{span}_{\mathbb{C}}\{K_i(z,0) : 1 \leqq i \leqq d\} \subseteqq \operatorname{span}_{\mathbb{C}}\{\tilde{K}_i(z,0) : 1 \leqq i \leqq d\}$. Writing g_j^0 in terms of $\tilde{g}_i^0$, we get the other inclusion.

Finally, to prove statement (v), let us apply M_j^* to both sides of the decomposition of the reproducing kernel K given in part (i) to obtain $\bar{w}_j K(z,w) = \sum_{i=1}^{d} \bar{g}_i^0(w) M_j^* K_i(z,w)$. Substituting K from the first equation, we get

$$\sum_{i=1}^{d} \bar{g}_i^0(w)(M_j - w_j)^* K_i(z,w) = 0.$$

Let $F_{ij}(z,w) = (M_j - w_j)^* K_i(z,w)$. For a fixed but arbitrary $z_0 \in \Omega$, consider the equation $\sum_{i=1}^{d} \bar{g}_i^0(w) F_{ij}(z_0,w) = 0$. Suppose there exists k, $1 \leqq k \leqq d$, such that $F_{kj}(z_0,0) \neq 0$. Then

$$g_k^0 = \{\overline{F_{kj}(z_0,\cdot)}_0\}^{-1} \sum_{i=1,\, i \neq k}^{d} g_i^0 \overline{F_{ij}(z_0,\cdot)}_0.$$

This is a contradiction. Therefore $F_{ij}(z_0,0) = 0$, $1 \leqq i \leqq d$, and for all $z_0 \in \Omega$. So $M_j^* K_i(z,0) = 0$, $1 \leqq i \leqq d$, $1 \leqq j \leqq m$. This completes the proof of the theorem. □

Remark 2.4. Let $\mathscr{I}$ be an ideal in the polynomial ring $\mathbb{C}[\underline{z}]$. Suppose $\mathscr{M} \supset \mathscr{I}$ and that $\mathscr{I}$ is dense in $\mathscr{M}$. Let $\{p_i \in \mathbb{C}[\underline{z}] : 1 \leqq i \leqq t\}$ be a minimal set of generators for the ideal $\mathscr{I}$. Let $V(\mathscr{I})$ be the zero variety of the ideal $\mathscr{I}$. If $w \notin V(\mathscr{I})$, then $\mathscr{S}_w^{\mathscr{M}} = \mathcal{O}_w$. Although $p_1, \dots, p_t$ generate the stalk at every point, they are not necessarily a minimal set of generators. For example, let $\mathscr{I} = \langle z_1(1+z_1), z_1(1-z_2), z_2^2 \rangle \subset \mathbb{C}[z_1, z_2]$. The polynomials $z_1(1+z_1)$, $z_1(1-z_2)$, z_2^2 form a minimal set of generators for the ideal $\mathscr{I}$. Since $1+z_1$ and $1-z_2$ are units in ${}_2\mathcal{O}_0$, it follows that the functions z_1 and z_2^2 form a minimal set of generators for the stalk $\mathscr{S}_0^{\mathscr{M}}$.

For simplicity, we have stated the decomposition theorem for Hilbert modules which consist of holomorphic functions taking values in $\mathbb{C}$. However, all the tools that we use for the proof work equally well in the case of holomorphic functions taking values in $\mathbb{C}^m$. Consequently, we expect it to remain valid in this more general set-up of vector valued holomorphic functions.

2.3. The joint kernel at w_0 and the stalk $\mathscr{S}_{w_0}^{\mathscr{M}}$. Let $g_1^0, \dots, g_d^0$ be a minimal set of generators for the stalk $\mathscr{S}_{w_0}^{\mathscr{M}}$ as before. For $f \in \mathscr{S}_{w_0}^{\mathscr{M}}$, we can find holomorphic functions f_i, $1 \leqq i \leqq d$, on some small open neighborhood U of w_0 such that $f = \sum_{i=1}^{d} g_i^0 f_i$ on U. We write

$$f = \sum_{i=1}^{d} g_i^0 f_i = \sum_{i=1}^{d} g_i^0 \{f_i - f_i(w_0)\} + \sum_{i=1}^{d} g_i^0 f_i(w_0)$$

on U. Let $\mathfrak{m}(\mathcal{O}_{w_0})$ be the maximal ideal (consisting of the germs of holomorphic functions vanishing at the point w_0) in the local ring $\mathcal{O}_{w_0}$ and $\mathfrak{m}(\mathcal{O}_{w_0})\mathscr{S}_{w_0}^{\mathscr{M}} = \mathfrak{m}_{w_0}\mathscr{S}_{w_0}^{\mathscr{M}}$. Thus the linear span of the equivalence classes $[g_1^0], \dots, [g_d^0]$ is the quotient module $\mathscr{S}_{w_0}^{\mathscr{M}} / \mathfrak{m}_{w_0}\mathscr{S}_{w_0}^{\mathscr{M}}$. Therefore we have

$$\dim \mathscr{S}_{w_0}^{\mathscr{M}} / \mathfrak{m}_{w_0}\mathscr{S}_{w_0}^{\mathscr{M}} \leqq d.$$

It turns out that the elements $[g_1^0], \dots, [g_d^0]$ in the quotient module are linearly independent. Then $\dim \mathscr{S}_{w_0}^{\mathscr{M}} / \mathfrak{m}_{w_0}\mathscr{S}_{w_0}^{\mathscr{M}} = d$. To prove the linear independence, let us consider the equation $\sum_{i=1}^{d} \alpha_i [g_i^0] = 0$ for some complex numbers α_i, $1 \leqq i \leqq d$, or equivalently,

$$\sum_{i=1}^{d} \alpha_i g_i^0 \in \mathfrak{m}(\mathcal{O}_w)\mathscr{S}_w^{\mathscr{M}}.$$

Thus there exist holomorphic functions f_i, $1 \leqq i \leqq d$, defined on a small neighborhood of w_0 and vanishing at w_0 such that $\sum_{i=1}^{d} (\alpha_i - f_i) g_i^0 = 0$. Now suppose $\alpha_k \neq 0$ for some k, $1 \leqq k \leqq d$. Then we can write

$$g_k^0 = -\sum_{i \neq k} (\alpha_k - f_k)_0^{-1} (\alpha_i - f_i)_0 g_i^0,$$

which is a contradiction. From the Decomposition Theorem 1.4, it follows that

$$\begin{aligned} (2.3) \qquad \dim \ker D_{(\boldsymbol{M} - w_0)^*} &\geqq \#\{\text{minimal generators for } \mathscr{S}_{w_0}^{\mathscr{M}}\} \\ &\geqq \dim \mathscr{S}_{w_0}^{\mathscr{M}} / \mathfrak{m}_{w_0}\mathscr{S}_{w_0}^{\mathscr{M}}. \end{aligned}$$

We will impose additional conditions on the Hilbert module $\mathscr{M}$, which is always assumed to be in the class $\mathfrak{B}_1(\Omega)$, so as to ensure equality in (2.3) (or (1.4)). One such assumption is that the module $\mathscr{M}$ is finitely generated. Let

$$V(\mathscr{M}) := \{w \in \Omega : f(w) = 0 \text{ for all } f \in \mathscr{M}\}.$$

Then for $w_0 \notin V(\mathscr{M})$, the number of minimal generators for the stalk at w_0 is one, in fact, $\mathscr{S}^{\mathscr{M}}_{w_0} = \mathscr{O}_{w_0}$. Also for $w_0 \notin V(\mathscr{M})$, $\dim \ker D_{(\boldsymbol{M}-w_0)^*} = 1$, following the proof of Lemma 2.3. Therefore, outside the zero set, we have equality in (1.4). For a large class of Hilbert modules we will show, even on the zero set, that the reverse inequality is valid. For instance, for Hilbert modules of rank 1 over $\mathbb{C}[\underline{z}]$, we have equality everywhere. This is easy to see from [15], page 89:

$$1 \geqq \dim \mathscr{M} \otimes_{\mathbb{C}[\underline{z}]} \mathbb{C}_{w_0} = \dim \ker D_{(\boldsymbol{M}-w_0)^*} \geqq \dim \mathscr{S}^{\mathscr{M}}_{w_0}/\mathfrak{m}_{w_0}\mathscr{S}^{\mathscr{M}}_{w_0} \geqq 1.$$

To understand the more general case, consider the map $i_w : \mathscr{M} \to \mathscr{M}_w$ defined by $f \mapsto f_w$, where f_w is the germ of the function f at w. Clearly, this map is a vector space isomorphism onto its image. The linear space

$$\mathscr{M}^{(w)} := \sum_{j=1}^{m} (z_j - w_j)\mathscr{M} = \mathfrak{m}_w \mathscr{M}$$

is closed since $\mathscr{M}$ is assumed to be in $\mathfrak{B}_1(\Omega)$. The map $f \mapsto f_w$ restricted to $\mathscr{M}^{(w)}$ is a linear isomorphism from $\mathscr{M}^{(w)}$ to $(\mathscr{M}^{(w)})_w$. Consider

$$\mathscr{M} \xrightarrow{i_w} \mathscr{S}^{\mathscr{M}}_w \xrightarrow{\pi} \mathscr{S}^{\mathscr{M}}_w/\mathfrak{m}_w\mathscr{S}^{\mathscr{M}}_w,$$

where π is the quotient map. Now we have a map

$$\psi : \mathscr{M}_w/(\mathscr{M}^{(w)})_w \to \mathscr{S}^{\mathscr{M}}_w/\mathfrak{m}_w\mathscr{S}^{\mathscr{M}}_w$$

which is well defined because $(\mathscr{M}^{(w)})_w \subseteqq \mathscr{M}_w \cap \mathfrak{m}_w\mathscr{S}^{\mathscr{M}}_w$. Whenever ψ can be shown to be one-one, equality in (1.4) is forced. To see this, note that $\mathscr{M} \ominus \mathscr{M}^{(w)} \cong \mathscr{M}/\mathscr{M}^{(w)}$ and

$$\ker D_{(\boldsymbol{M}-w)^*} = \bigcap_{j=1}^{m} \{\operatorname{ran}(M_j - w_j)\}^{\perp} = \mathscr{M} \ominus \sum_{j=1}^{m} (z_j - w_j)\mathscr{M} = \mathscr{M} \ominus \mathscr{M}^{(w)}.$$

Hence

$$(2.4) \qquad d \leqq \dim \ker D_{(\boldsymbol{M}-w)^*} = \dim \mathscr{M}/\mathscr{M}^{(w)} \leqq \dim \mathscr{S}^{\mathscr{M}}_w/\mathfrak{m}_w\mathscr{S}^{\mathscr{M}}_w = d.$$

Suppose $\psi(f) = 0$ for some $f \in \mathscr{M}$. Then $f_w \in \mathfrak{m}_w\mathscr{S}^{\mathscr{M}}_w$ and consequently, $f = \sum_{i=1}^{m} (z_i - w_i) f_i$ for holomorphic functions f_i, $1 \leqq i \leqq m$, on some small open set U. The main question is if the functions f_i, $1 \leqq i \leqq m$, can be chosen from the Hilbert module $\mathscr{M}$. We isolate below, a class of Hilbert modules for which this question has an affirmative answer.

Let $\mathscr{H}$ be a Hilbert module in $\mathfrak{B}_1(\Omega) \cap \mathrm{B}_1(\Omega)$. Pick, for each $w \in \Omega$, a $\mathbb{C}$-linear subspace $\mathbb{V}_w$ of the polynomial ring $\mathbb{C}[\underline{z}]$ with the property that it is invariant under the action of the partial differential operators $\left\{\frac{\partial}{\partial z_1},\dots,\frac{\partial}{\partial z_m}\right\}$ (see [5]). Set

$$\mathscr{M}(w)=\{f\in\mathscr{H}: q(D)f|_w=0 \text{ for all } q\in\mathbb{V}_w\}. \tag{2.5}$$

For $f\in\mathscr{M}(w)$ and $q\in\mathbb{V}_w$,

$$q(D)(z_jf)|_w=w_jq(D)f|_w+\frac{\partial q}{\partial z_j}(D)f|_w=0.$$

Now, the assumptions on $\mathbb{V}_w$ ensure that $\mathscr{M}(w)$ is a module. We consider below, the class of (non-trivial) Hilbert modules which are of the form $\mathscr{M}:=\bigcap_{w\in\Omega}\mathscr{M}(w)$. It is easy to see that $w\notin V(\mathscr{M})$ if and only if $\mathbb{V}_w=\{0\}$ and then $\mathbb{V}_w=\{0\}$ if and only if $\mathscr{M}(w)=\mathscr{H}$. Therefore, $\mathscr{M}=\bigcap_{w\in V(\mathscr{M})}\mathscr{M}(w)$. These modules are called AF-co-submodules (see [5], page 38). Let

$$\mathbb{V}_w(\mathscr{M}):=\{q\in\mathbb{C}[\underline{z}]: q(D)f|_w=0 \text{ for all } f\in\mathscr{M}\}.$$

Note that $\mathbb{V}_w(\mathscr{M})=\mathbb{V}_w$. Fix a point in $V(\mathscr{M})$, say w_0. Consider

$$\tilde{\mathbb{V}}_{w_0}(\mathscr{M}):=\left\{q\in\mathbb{C}[\underline{z}]:\frac{\partial q}{\partial z_i}\in\mathbb{V}_{w_0}(\mathscr{M}),\ 1\leqq i\leqq m\right\}.$$

For $w\in V(\mathscr{M})$, let

$$\mathbb{V}_{w_0,w}(\mathscr{M})=\begin{cases}\mathbb{V}_w(\mathscr{M}), & \text{if } w\neq w_0,\\ \tilde{\mathbb{V}}_{w_0}(\mathscr{M}), & \text{if } w=w_0.\end{cases}$$

Now, define $\mathscr{M}^{w_0}(w)$ to be the submodule (of $\mathscr{H}$) corresponding to the $\mathbb{C}$-linear space $\mathbb{V}_{w_0,w}(\mathscr{M})$ (as in (2.5)) and let

$$\mathscr{M}^{w_0}=\bigcap_{w\in V(\mathscr{M})}\mathscr{M}^{w_0}(w).$$

So we have $\mathbb{V}_w(\mathscr{M}^{w_0})=\mathbb{V}_{w_0,w}(\mathscr{M})$. For $f\in\mathscr{M}^{(w_0)}$, we have $f=\sum_{j=1}^m(z_j-w_{0j})f_j$, for some choice of $f_1,\dots,f_m\in\mathscr{M}$. Now for any $q\in\mathbb{C}[\underline{z}]$, following [5], we have

$$\begin{aligned} q(D)f&=\sum_{j=1}^m q(D)\{(z_j-w_{0j})f_j\}\\ &=\sum_{j=1}^m\left\{(z_j-w_{0j})q(D)f_j+\frac{\partial q}{\partial z_j}(D)f_j\right\}.\end{aligned} \tag{2.6}$$

For $w\in V(\mathscr{M})$ and $f\in\mathscr{M}^{(w_0)}$, it follows from the definitions that

$$q(D)f|_w=\begin{cases}\sum_{j=1}^m\left\{(w_j-w_{0j})q(D)f_j|_w+\frac{\partial q}{\partial z_j}(D)f_j|_w\right\}=0, & q\in\mathbb{V}_w(\mathscr{M}),\ w\neq w_0,\\ \sum_{j=1}^m\left\{\frac{\partial q}{\partial z_j}(D)f_j|_{w_0}\right\}=0, & q\in\tilde{\mathbb{V}}_{w_0}(\mathscr{M}),\ w=w_0.\end{cases}$$

Thus $f \in \mathcal{M}^{(w_0)}$ implies that $f \in \mathcal{M}^{w_0}(w)$ for each $w \in V(\mathcal{M})$. Hence $\mathcal{M}^{(w_0)} \subseteqq \mathcal{M}^{w_0}$. Now we describe the Gleason's property for $\mathcal{M}$ at a point w_0.

Definition 2.5. We say that an AF co-submodule $\mathcal{M}$ has the Gleason's property at a point $w_0 \in V(\mathcal{M})$ if $\mathcal{M}^{w_0} = \mathcal{M}^{(w_0)}$.

In analogy with the definition of $\mathbb{V}_{w_0}(\mathcal{M})$ for a Hilbert module $\mathcal{M}$, we define the space $\mathbb{V}_{w_0}(\mathcal{S}_{w_0}^{\mathcal{M}}) = \{q \in \mathbb{C}[\underline{z}] : q(D)f|_{w_0} = 0,\ f_{w_0} \in \mathcal{S}_{w_0}^{\mathcal{M}}\}$. It will be useful to record the relation between $\mathbb{V}_{w_0}(\mathcal{M})$ and $\mathbb{V}_{w_0}(\mathcal{S}_{w_0}^{\mathcal{M}})$ in a separate lemma.

Lemma 2.6. *For any Hilbert module $\mathcal{M}$ in $\mathfrak{B}_1(\Omega)$ and $w_0 \in \Omega$, we have*

$$\mathbb{V}_{w_0}(\mathcal{M}) = \mathbb{V}_{w_0}(\mathcal{S}_{w_0}^{\mathcal{M}}).$$

Proof. Note that the inclusion $\mathbb{V}_{w_0}(\mathcal{S}_{w_0}^{\mathcal{M}}) \subseteqq \mathbb{V}_{w_0}(\mathcal{M})$ follows from $\mathcal{M}_{w_0} \subseteqq \mathcal{S}_{w_0}^{\mathcal{M}}$. To prove the reverse inclusion, we need to show that $q(D)h|_{w_0} = 0$ for $h \in \mathcal{S}_{w_0}^{\mathcal{M}}$, for all $q \in \mathbb{V}_{w_0}(\mathcal{M})$. Since $h \in \mathcal{S}_{w_0}^{\mathcal{M}}$, we can find functions $f_1, \ldots, f_n \in \mathcal{M}$ and $g_1, \ldots, g_n \in \mathcal{O}_{w_0}$ such that $h = \sum_{i=1}^{n} f_i g_i$ in some small open neighborhood of w_0. Therefore, it is enough to show that $q(D)(fg)|_{w_0} = 0$ for $f \in \mathcal{M}$, g holomorphic in a neighborhood, say U_{w_0} of w_0, and $q \in \mathbb{V}_{w_0}(\mathcal{M})$. We can choose U_{w_0} to be a small enough polydisk such that

$$g = \sum_{\alpha} a_\alpha (z - w_0)^\alpha, \quad z \in U_{w_0}.$$

Then $q(D)(fg) = \sum_{\alpha} a_\alpha q(D)\{(z - w_0)^\alpha f\}$ for $z \in U_{w_0}$. Clearly, $(z - w_0)^\alpha f$ belongs to $\mathcal{M}$ whenever $f \in \mathcal{M}$. Hence $q(D)\{(z - w_0)^\alpha f\}|_{w_0} = 0$ and we have $q(D)(fg)|_{w_0} = 0$ completing the proof of the inclusion $\mathbb{V}_{w_0}(\mathcal{M}) \subseteqq \mathbb{V}_{w_0}(\mathcal{S}_{w_0}^{\mathcal{M}})$. □

We will show that we have equality in (1.4) for all AF-co-submodules satisfying Gleason's property. Proposition 1.7 includes this.

Proof of Proposition 1.7. We first show that $\ker(\pi \circ i_{w_0}) = \mathcal{M}^{w_0}$. Showing $\ker(\pi \circ i_{w_0}) \subseteqq \mathcal{M}^{w_0}$ is the same as showing $\mathcal{M}_{w_0} \cap \mathfrak{m}_{w_0} \mathcal{S}_{w_0}^{\mathcal{M}} \subseteqq (\mathcal{M}^{w_0})_{w_0}$. We claim that

$$\mathbb{V}_{w_0}(\mathfrak{m}_{w_0} \mathcal{S}_{w_0}^{\mathcal{M}}) = \mathbb{V}_{w_0, w_0}(\mathcal{M}) (= \tilde{\mathbb{V}}_{w_0}(\mathcal{M})). \tag{2.7}$$

If $f \in \mathfrak{m}_{w_0} \mathcal{S}_{w_0}^{\mathcal{M}}$, then there exists $f_j \in \mathcal{S}_{w_0}^{\mathcal{M}}, 1 \leqq j \leqq m$, such that $f = \sum_{j=1}^{m} (z_j - w_{0j}) f_j$. From equation (2.6), we have

$$q \in \mathbb{V}_{w_0}(\mathfrak{m}_{w_0} \mathcal{S}_{w_0}^{\mathcal{M}}) \quad \text{if and only if} \quad \frac{\partial q}{\partial z_j} \in \mathbb{V}_{w_0}(\mathcal{S}_{w_0}^{\mathcal{M}}) = \mathbb{V}_{w_0}(\mathcal{M}), \quad 1 \leqq j \leqq m.$$

Now, from Lemma 2.6, we find $\frac{\partial q}{\partial z_j} \in \mathbb{V}_{w_0}(\mathcal{M}), 1 \leqq j \leqq m$, if and only if $q \in \tilde{\mathbb{V}}_{w_0}(\mathcal{M})$, which proves our claim. So for $f \in \mathcal{M}$, if $f_{w_0} \in \mathfrak{m}_{w_0} \mathcal{S}_{w_0}^{\mathcal{M}}$, then $f \in \mathcal{M}^{w_0}(w)$ for all $w \in V(\mathcal{M})$. Hence $f \in \mathcal{M}^{w_0}$ and as a result, we have $\mathcal{M}_{w_0} \cap \mathfrak{m}_{w_0} \mathcal{S}_{w_0}^{\mathcal{M}} \subseteqq (\mathcal{M}^{w_0})_{w_0}$.

Now let $f \in \mathscr{M}^{w_0}$. From (2.7) it follows that

$$f \in \{g \in \mathcal{O}_{w_0} : q(D)g|_{w_0} = 0 \text{ for all } q \in \mathbb{V}_{w_0}(\mathfrak{m}_{w_0}\mathscr{S}_{w_0}^{\mathscr{M}})\}.$$

According to [5], Proposition 2.3.1, we have $f \in \mathfrak{m}_{w_0}\mathscr{S}_{w_0}^{\mathscr{M}}$. Therefore $f \in \ker(\pi \circ i_{w_0})$ and $\ker(\pi \circ i_{w_0}) = \mathscr{M}^{w_0}$.

Next we show that the map $\pi \circ i_{w_0}$ is onto. Let $\sum_{i=1}^{n} f_i g_i \in \mathscr{S}_{w_0}^{\mathscr{M}}$, where $f_i \in \mathscr{M}$ and g_i's are holomorphic functions in some neighborhood of w_0, $1 \leqq i \leqq n$. We need to show that there exists $f \in \mathscr{M}$ such that the class $[f]$ is equal to $\left[\sum_{i=1}^{n} f_i g_i\right]$ in $\mathscr{S}_{w_0}^{\mathscr{M}}/\mathfrak{m}_{w_0}\mathscr{S}_{w_0}^{\mathscr{M}}$. Let us take $f = \sum_{i=1}^{n} f_i g_i(w_0)$. Then

$$\sum_{i=1}^{n} f_i g_i - f = \sum_{i=1}^{n} f_i\{g_i - g_i(w_0)\} \in \mathfrak{m}_{w_0}\mathscr{S}_{w_0}^{\mathscr{M}}.$$

This completes the proof of surjectivity.

Suppose Gleason's property holds for $\mathscr{M}$ at w_0. Since $\ker(\pi \circ i_{w_0}) = \mathscr{M}^{w_0}$, it follows from the Gleason's property at w_0 that we have the equality $\ker(\pi \circ i_{w_0}) = \mathscr{M}^{(w_0)}$. The map $\psi : \mathscr{M}/\mathscr{M}^{(w_0)} \to \mathscr{S}_{w_0}^{\mathscr{M}}/\{\mathfrak{m}_{w_0}\mathscr{S}_{w_0}^{\mathscr{M}}\}$ is then one to one. The equality in (1.4) is established as in the equation (2.4).

Now suppose equality holds in (1.4). From the above, it is clear that $\mathscr{M}/\mathscr{M}^{w_0}$ is isomorphic to $\mathscr{S}_{w_0}^{\mathscr{M}}/\mathfrak{m}_{w_0}\mathscr{S}_{w_0}^{\mathscr{M}}$. Thus

$$\dim \mathscr{M}/\mathscr{M}^{w_0} = \dim \mathscr{M}/\mathscr{M}^{(w_0)}.$$

But as $\mathscr{M}^{(w_0)} \subseteqq \mathscr{M}^{w_0}$, we have $\mathscr{M}^{(w_0)} = \mathscr{M}^{w_0}$ and hence Gleason's property holds for $\mathscr{M}$ at w_0. $\square$

A class of examples of Hilbert spaces satisfying Gleason's property can be found in [20]. It was shown in [20] that Gleason's property holds for analytic Hilbert modules ([5], page 3). However it is not entirely clear if it continues to hold for submodules of analytic Hilbert modules. Nevertheless, we will identify here, a class of submodules for which we have equality in (1.4). Let $\mathscr{M}$ be a submodule of an analytic Hilbert module over $\mathbb{C}[\underline{z}]$. Assume that $\mathscr{M}$ is the closure of an ideal $\mathscr{I} \subseteqq \mathbb{C}[\underline{z}]$. From [5], [17], we note that

$$\dim \ker D_{(\boldsymbol{M}-w_0)^*} = \dim \bigcap_{j=1}^{m} \ker(M_j - w_{0j})^* = \dim \mathscr{I}/\mathfrak{m}_{w_0}\mathscr{I}.$$

Therefore from (2.3) we have

$$\dim \mathscr{I}/\mathfrak{m}_{w_0}\mathscr{I} \geqq \dim \mathscr{S}_{w_0}^{\mathscr{M}}/\mathfrak{m}_{w_0}\mathscr{S}_{w_0}^{\mathscr{M}}.$$

So it remains to prove the reverse inequality. Fix a point $w_0 \in \Omega$. Consider the map

$$\mathscr{I} \xrightarrow{i_{w_0}} \mathscr{S}_{w_0}^{\mathscr{M}} \xrightarrow{\pi} \mathscr{S}_{w_0}^{\mathscr{M}}/\mathfrak{m}_{w_0}\mathscr{S}_{w_0}^{\mathscr{M}}.$$

We will show that $\ker(\pi \circ i_{w_0}) = \mathfrak{m}_{w_0}\mathscr{I}$. Let $V(\mathscr{I})$ denote the zero set of the ideal $\mathscr{I}$ and $\mathbb{V}_w(\mathscr{I})$ be its characteristic space at w. We begin by proving that the characteristic space of the ideal coincides with that of the corresponding Hilbert module.

Lemma 2.7. *Assume that $\mathscr{M} = [\mathscr{I}]$ is in $\mathfrak{B}_1(\Omega)$. Then $\mathbb{V}_{w_0}(\mathscr{I}) = \mathbb{V}_{w_0}(\mathscr{M})$ for $w_0 \in \Omega$.*

Proof. Clearly $\mathbb{V}_{w_0}(\mathscr{I}) \supseteqq \mathbb{V}_{w_0}(\mathscr{M})$, so we have to prove $\mathbb{V}_{w_0}(\mathscr{I}) \subseteqq \mathbb{V}_{w_0}(\mathscr{M})$. For $q \in \mathbb{V}_{w_0}(\mathscr{I})$ and $f \in \mathscr{M}$, we show that $q(D)f|_{w_0} = 0$. Now, for each $f \in \mathscr{M}$, there exists a sequence of polynomials $p_n \in \mathscr{I}$ such that $p_n \to f$ in the Hilbert space norm. Recall that if K is the reproducing kernel for $\mathscr{M}$, then

$$(\partial^\alpha f)(w) = \langle f, \bar{\partial}^\alpha K(\cdot, w)\rangle \quad \text{for } \alpha \in \mathbb{Z}_m^+,\ w \in \Omega,\ f \in \mathscr{M}. \tag{2.8}$$

For $w \in \Omega$ and a compact neighborhood C of w, we have

$$\begin{aligned} |q(D)p_n(w) - q(D)f(w)| &= |\langle p_n - f, q(\bar{D})K(\cdot, w)\rangle| \\ &\leqq \|p_n - f\|_{\mathscr{M}} \|q(\bar{D})K(\cdot, w)\|_{\mathscr{M}} \\ &\leqq \|p_n - f\|_{\mathscr{M}} \sup_{w \in C} \|q(\bar{D})K(\cdot, w)\|_{\mathscr{M}}. \end{aligned}$$

Therefore, $q(D)p_n|_{w_0} \to q(D)f|_{w_0}$ as $n \to \infty$. Since $q(D)p_n|_{w_0} = 0$ for all n, it follows that $q(D)f|_{w_0} = 0$. Hence $q \in \mathbb{V}_{w_0}(\mathscr{M})$ and we are done. $\square$

Now let $\mathscr{J} = \mathfrak{m}_{w_0}\mathscr{I}$. Recall (cf. [17], Proposition 2.3) that

$$V(\mathscr{J}) \backslash V(\mathscr{I}) := \{w \in \mathbb{C}^m : \mathbb{V}_w(\mathscr{I}) \subsetneqq \mathbb{V}_w(\mathscr{J})\} = \{w_0\}.$$

Here we will explicitly write down the characteristic space. Let

$$\tilde{\mathbb{V}}_{w_0}(\mathscr{I}) = \left\{q \in \mathbb{C}[\underline{z}] : \frac{\partial q}{\partial z_i} \in \mathbb{V}_{w_0}(\mathscr{I}),\ 1 \leqq i \leqq m\right\}$$

and

$$\mathbb{V}_{w_0, w}(\mathscr{I}) = \begin{cases} \mathbb{V}_w(\mathscr{I}), & w \neq w_0, \\ \tilde{\mathbb{V}}_{w_0}(\mathscr{I}), & w = w_0. \end{cases}$$

Lemma 2.8. *For $w \in \mathbb{C}^m$, $\mathbb{V}_w(\mathscr{J}) = \mathbb{V}_{w_0, w}(\mathscr{I})$.*

Proof. Since $\mathscr{J} \subset \mathscr{I}$, we have $\mathbb{V}_w(\mathscr{I}) \subseteqq \mathbb{V}_w(\mathscr{J})$ for all $w \in \mathbb{C}^m$. Now let $w \neq w_0$. For $f \in \mathscr{I}$ and $q \in \mathbb{V}_w(\mathscr{J})$, we show that $q(D)f|_w = 0$ which implies q must be in $\mathbb{V}_w(\mathscr{I})$.

Note that for any $k \in \mathbb{N}$ and $j \in \{1, \ldots, m\}$,

$$q(D)\{(z_j - w_{0j})^k f\}|_w = 0 \quad \text{as } (z_j - w_{0j})^k f \in \mathscr{J}.$$

This implies $\sum_{l=0}^{k}(w_j - w_{0j})^l \binom{k}{l} \frac{\partial^{k-l} q}{\partial z_j^{k-l}}(D)f|_w = 0$. Hence (inductively) we have

$$(w_j - w_{0j})^k q(D)f|_w = (-1)^k \frac{\partial^k q}{\partial z_j^k}(D)f|_w \quad \text{for all } k \in \mathbb{N} \text{ and } j \in \{1, \dots, m\}.$$

So, if $w \neq w_0$, then there exists $i \in \{1, \dots, m\}$ such that $w_i \neq w_{0i}$. Therefore, by choosing k large enough with respect to the degree of q, we can ensure $(w_i - w_{0i})^k q(D)f|_w = 0$. Thus $q(D)f|_w = 0$. For $w = w_0$, we have $q \in \mathbb{V}_{w_0}(\mathscr{J})$ if and only if $q(D)\{(z_j - w_{0j})f\}|_{w_0} = 0$ for all $f \in \mathscr{I}$ and $j \in \{1, \dots, m\}$ if and only if $\frac{\partial q}{\partial z_j}(D)f|_{w_0} = 0$ for all $f \in \mathscr{I}$ and $j \in \{1, \dots, m\}$ if and only if $\frac{\partial q}{\partial z_j} \in \mathbb{V}_{w_0}(\mathscr{I})$ for all $j \in \{1, \dots, m\}$ if and only if $q \in \tilde{\mathbb{V}}_{w_0}(\mathscr{I})$. This completes the proof of the lemma. $\square$

We have shown that $\mathbb{V}_{w_0}(\mathscr{I}) = \mathbb{V}_{w_0}(\mathscr{M}) = \mathbb{V}_{w_0}(\mathscr{S}_{w_0}^{\mathscr{M}})$. The next lemma provides a relationship between the characteristic space of $\mathscr{J}$ at the point w_0 and the sheaf $\mathscr{S}_{w_0}^{\mathscr{M}}$.

Lemma 2.9. $\mathbb{V}_{w_0}(\mathscr{J}) = \mathbb{V}_{w_0}(\mathfrak{m}_{w_0}\mathscr{S}_{w_0}^{\mathscr{M}})$.

Proof. We have $\mathbb{V}_{w_0}(\mathfrak{m}_{w_0}\mathscr{S}_{w_0}^{\mathscr{M}}) \subseteqq \mathbb{V}_{w_0}(\mathscr{J})$. From the previous lemma, it follows that if $q \in \mathbb{V}_{w_0}(\mathscr{J})$, then $q \in \tilde{\mathbb{V}}_{w_0}(\mathscr{I})$, that is, $\frac{\partial q}{\partial z_j} \in \mathbb{V}_{w_0}(\mathscr{I}) = \mathbb{V}_{w_0}(\mathscr{S}_{w_0}^{\mathscr{M}})$ for all $j \in \{1, \dots, m\}$. From (2.7), it follows that $q \in \mathbb{V}_{w_0}(\mathfrak{m}_{w_0}\mathscr{S}_{w_0}^{\mathscr{M}})$. $\square$

Now, we have all the ingredients to prove that we must have equality in (1.4) for submodules of analytic Hilbert modules which are obtained as closure of some polynomial ideal.

Proposition 2.10. *Let $\mathscr{M} = [\mathscr{I}]$ be a submodule of an analytic Hilbert module over $\mathbb{C}[\underline{z}]$ on a bounded domain Ω, where $\mathscr{I}$ is a polynomial ideal, each of whose algebraic component intersects Ω. Then*

$$\dim \mathscr{M}/\mathfrak{m}_{w_0}\mathscr{M} = \dim \mathscr{S}_{w_0}^{\mathscr{M}}/\mathfrak{m}_{w_0}\mathscr{S}_{w_0}^{\mathscr{M}}, \quad w_0 \in \Omega.$$

Proof. Let $p \in \mathscr{I}$ such that $\pi \circ i_{w_0}(p) = 0$, that is, $p_{w_0} \in \mathfrak{m}_{w_0}\mathscr{S}_{w_0}^{\mathscr{M}}$. The preceding lemma implies $q(D)p|_{w_0} = 0$ for all $q \in \mathbb{V}_{w_0}(\mathscr{J})$. So,

$$p \in \mathscr{J}_{w_0}^e := \{r \in \mathbb{C}[\underline{z}] : q(D)p|_{w_0} = 0 \text{ for all } q \in \mathbb{V}_{w_0}(\mathscr{J})\}.$$

Since each of the algebraic components of $\mathscr{J}$ intersects Ω, therefore, from [5], Corollary 2.1.2, we have $p \in \bigcap_{w \in \Omega} \mathscr{J}_w^e = \mathscr{J}$. Thus $\ker(\pi \circ i_{w_0}) = \mathscr{J} = \mathfrak{m}_{w_0}\mathscr{I}$. Then the map $\pi \circ i_{w_0} : \dim \mathscr{I}/\mathfrak{m}_{w_0}\mathscr{I} \to \dim \mathscr{S}_{w_0}^{\mathscr{M}}/\mathfrak{m}_{w_0}\mathscr{S}_{w_0}^{\mathscr{M}}$ is one-one and we have

$$\dim \mathscr{I}/\mathfrak{m}_{w_0}\mathscr{I} \leqq \dim \mathscr{S}_{w_0}^{\mathscr{M}}/\mathfrak{m}_{w_0}\mathscr{S}_{w_0}^{\mathscr{M}}.$$

Therefore, we have equality in (1.4). $\square$

The proof of the Theorem 1.9 is now immediate.

Proof of Theorem 1.9. From the Rigidity Theorem in [16], it follows that the submodule $\mathscr{M}$ corresponds to an ideal such that $\mathscr{M} = [\mathscr{I}]$. The proof is complete using Propositions 1.7 and 2.10. $\square$

Remark 2.11. In fact, this corollary is valid for all submodules of the form $[\mathscr{I}]$ whenever it is an AF-co-submodule for some polynomial ideal $\mathscr{I}$.

The following corollary to Proposition 2.10 answers, in part, the conjecture of [14], page 262. These answers were found by Duan–Guo earlier in [17].

Corollary 2.12. *Suppose $\mathscr{M}$ is a submodule of an analytic Hilbert module given by closure of a polynomial ideal $\mathscr{I}$ and $w_0 \in V(\mathscr{I})$ is a smooth point then,*

$$\dim \ker D_{(\boldsymbol{M}-w_0)^*} = \textit{co-dimension of } V(\mathscr{I}).$$

Proof. From Remark 2.2, it follows that if $\mathscr{I}$ is generated by $p_1, \dots, p_t$, then $\mathscr{S}_{w_0}^{\mathscr{M}}$ is generated by $p_{1w_0}, \dots, p_{tw_0}$. In the course of the proof in [17], Theorem 2.3, a change of variable arguments is used to show that the stalk $\mathscr{S}_{w_0}^{\mathscr{M}}$ at w_0 is isomorphic to the ideal generated by the co-ordinate functions $z_1 - w_{01}, \dots, z_r - w_{0r}$, where r is the co-dimension of $V(\mathscr{I})$. Therefore, the number of minimal generators for the stalk at a smooth point is equal to r which is the co-dimension of $V(\mathscr{I})$. The proof is completed by Proposition 2.10. $\square$

2.4. Curvature invariants. Let $\mathscr{M}$ be a Hilbert module in $\mathfrak{B}_1(\Omega)$ and $w_0 \in \Omega$ be fixed. The vectors $K_w^{(i)} \in \mathscr{M}$, $1 \leqq i \leqq d$, for w in some small neighborhood, say Ω_0 of w_0, produced in part (ii) of the Decomposition Theorem 1.4 are independent. However, while the choice of these vectors is not canonical, in general, we provide below a recipe for finding the vectors $K_w^{(i)}$, $1 \leqq i \leqq d$, satisfying

$$K(\cdot, w) = \overline{g_1^0(w)} K_w^{(1)} + \dots + \overline{g_d^0(w)} K_w^{(d)}, \quad w \in \Omega_0$$

following [7]. We note that $\mathfrak{m}_w \mathscr{M}$ is a closed submodule of $\mathscr{M}$. We assume that we have equality in (1.4) for the module $\mathscr{M}$ at the point $w_0 \in \Omega$, that is,

$$\operatorname{span}_{\mathbb{C}}\{K_{w_0}^{(i)} : 1 \leqq i \leqq d\} = \ker D_{(\boldsymbol{M}-w_0)^*}.$$

Let $D_{(\boldsymbol{M}-w)^*} = V_{\boldsymbol{M}}(w) |D_{(\boldsymbol{M}-w)^*}|$ be the polar decomposition of $D_{(\boldsymbol{M}-w)^*}$, where $|D_{(\boldsymbol{M}-w)^*}|$ is the positive square root of the operator $(D_{(\boldsymbol{M}-w)^*})^* D_{(\boldsymbol{M}-w)^*}$ and $V_{\boldsymbol{M}}(w)$ is the partial isometry mapping $(\ker D_{(\boldsymbol{M}-w)^*})^\perp$ onto $\operatorname{ran} D_{(\boldsymbol{M}-w)^*}$. Let $Q_{\boldsymbol{M}}(w)$ be the positive operator:

$$Q_{\boldsymbol{M}}(w)|_{\ker D_{(\boldsymbol{M}-w)^*}} = 0 \quad \text{and} \quad Q_{\boldsymbol{M}}(w)|_{(\ker D_{(\boldsymbol{M}-w)^*})^\perp} = \left(|D_{(\boldsymbol{M}-w)^*}| \, |_{(\ker D_{(\boldsymbol{M}-w)^*})^\perp} \right)^{-1}.$$

Let $R_{\boldsymbol{M}}(w) : \mathscr{M} \oplus \dots \oplus \mathscr{M} \to \mathscr{M}$ be the operator $R_{\boldsymbol{M}}(w) = Q_{\boldsymbol{M}}(w) V_{\boldsymbol{M}}(w)^*$. The two equations, involving the operator $D_{(\boldsymbol{M}-w)^*}$, stated below are analogous to the semi-Fredholmness property of a single operator (cf. [6], Proposition 1.11):

$$R_{\boldsymbol{M}}(w) D_{(\boldsymbol{M}-w)^*} = I - P_{\ker D_{(\boldsymbol{M}-w)^*}}, \tag{2.9}$$

$$D_{(\boldsymbol{M}-w)^*} R_{\boldsymbol{M}}(w) = P_{\operatorname{ran} D_{(\boldsymbol{M}-w)^*}}, \tag{2.10}$$

where $P_{\ker D_{(\boldsymbol{M}-w)^*}}$ (resp. $P_{\operatorname{ran} D_{(\boldsymbol{M}-w)^*}}$), for $w \in \Omega_0$, is the orthogonal projection onto $\ker D_{(\boldsymbol{M}-w)^*}$ (resp. $\operatorname{ran} D_{(\boldsymbol{M}-w)^*}$). Consider the operator

$$P(\overline{w}, \overline{w}_0) = I - \{I - R_{\boldsymbol{M}}(w_0) D_{\overline{w}-\overline{w}_0}\}^{-1} R_{\boldsymbol{M}}(w_0) D_{(\boldsymbol{M}-w)^*},$$

$w \in B\big(w_0; \|R(w_0)\|^{-1}\big)$, where $B\big(w_0; \|R(w_0)\|^{-1}\big)$ is the ball of radius $\|R(w_0)\|^{-1}$ around w_0. Using the equations (2.9) and (2.10) given above, we write

$$P(\overline{w}, \overline{w}_0) = \{I - R_{\boldsymbol{M}}(w_0) D_{\overline{w}-\overline{w}_0}\}^{-1} P_{\ker D_{(\boldsymbol{M}-w_0)^*}}, \tag{2.11}$$

where $D_{\overline{w}-\overline{w}_0} f = \big((\overline{w}_1 - \overline{w}_{01}) f_1, \ldots, (\overline{w}_m - \overline{w}_{0m}) f_m\big)$. The details can be found in [7], page 452. From the definition of $P(\overline{w}, \overline{w}_0)$, it follows that $P(\overline{w}, \overline{w}_0) P_{\ker D_{(\boldsymbol{M}-w)^*}} = P_{\ker D_{(\boldsymbol{M}-w)^*}}$ which implies $\ker D_{(\boldsymbol{M}-w)^*} \subset \operatorname{ran} P(\overline{w}, \overline{w}_0)$ for $w \in \Delta(w_0; \varepsilon)$. Consequently $K(\cdot, w) \in \operatorname{ran} P(\overline{w}, \overline{w}_0)$ and therefore

$$K(\cdot, w) = \sum_{i=1}^{d} \overline{a_i(w)} P(\overline{w}, \overline{w}_0) K_{w_0}^{(i)},$$

for some complex valued functions $a_1, \ldots, a_d$ on $\Delta(w_0; \varepsilon)$. We will show that the functions a_i, $1 \leqq i \leqq d$, are holomorphic and their germs form a minimal set of generators for $S_{w_0}^{\mathscr{M}}$. Now

$$R_{\boldsymbol{M}}(w_0) D_{\overline{w}-\overline{w}_0} K(\cdot, w) = R_{\boldsymbol{M}}(w_0) D_{(\boldsymbol{M}-w_0)^*} K(\cdot, w) = (I - P_{\ker D_{(\boldsymbol{M}-w_0)^*}}) K(\cdot, w).$$

Hence we have

$$\{I - R_{\boldsymbol{M}}(w_0) D_{\overline{w}-\overline{w}_0}\} K(\cdot, w) = P_{\ker D_{(\boldsymbol{M}-w_0)^*}} K(\cdot, w).$$

Since $K(\cdot, w) \in \operatorname{ran} P(\overline{w}, \overline{w}_0)$, we also have

$$P(\overline{w}, \overline{w}_0)^{-1} K(\cdot, w) = P_{\ker D_{(\boldsymbol{M}-w_0)^*}} K(\cdot, w).$$

Let $v_1, \ldots, v_d$ be the orthonormal basis for $\ker D_{(\boldsymbol{M}-w_0)^*}$. Let $g_1, \ldots, g_d$ denote the minimal set of generators for the stalk at $\mathscr{S}_{w_0}^{\mathscr{M}}$. Then there exists a neighborhood U, small enough, such that $v_j = \sum_{i=1}^{d} g_i f_i^j$, $1 \leqq j \leqq d$, and for some holomorphic functions f_i^j, $1 \leqq i, j \leqq d$, on U. We then have

$$\begin{aligned} P(\overline{w}, \overline{w}_0)^{-1} K(\cdot, w) &= P_{\ker D_{(\boldsymbol{M}-w_0)^*}} K(\cdot, w) = \sum_{j=1}^{d} \langle K(\cdot, w), v_j \rangle v_j \\ &= \sum_{j=1}^{d} \left\langle K(\cdot, w), \sum_{i=1}^{d} g_i f_i^j \right\rangle v_j = \sum_{i=1}^{d} \sum_{j=1}^{d} \overline{g_i(w) f_i^j(w)} v_j \\ &= \sum_{i=1}^{d} \overline{g_i(w)} \left\{ \sum_{j=1}^{d} \overline{f_i^j(w)} v_j \right\}. \end{aligned}$$

So $K(z,w) = \sum_{i=1}^{d} \overline{g_i(w)} \left\{ \sum_{j=1}^{d} \overline{f_i^j(w)} P(\overline{w}, \overline{w}_0) v_j(z) \right\}$. Let

$$\tilde{K}_w^{(i)} = \sum_{j=1}^{d} \overline{f_i^j(w)} P(\overline{w}, \overline{w}_0) v_j.$$

Since the vectors $K_{w_0}^{(i)}$, $1 \leqq i \leqq d$, are uniquely determined as long as $g_1, \dots, g_d$ are fixed and $P(\overline{w}_0, \overline{w}_0) = P_{\ker D_{(\boldsymbol{M}-w_0)^*}}$, it follows that

$$K_{w_0}^{(i)} = \tilde{K}_{w_0}^{(i)} = \sum_{j=1}^{d} \overline{f_i^j(w_0)} v_j, \quad 1 \leqq i \leqq d.$$

Therefore, the determinant of the $d \times d$ matrix $\left(\overline{f_i^j(w_0)}\right)_{i,j=1}^{d}$ is non-zero. Since $\mathrm{Det}\left(\overline{f_i^j(w)}\right)_{i,j=1}^{d}$ is an anti-holomorphic function, there exists a neighborhood of w_0, say $\Delta(w_0; \varepsilon)$, for some $\varepsilon > 0$, such that

$$\mathrm{Det}\left(\overline{f_i^j(w)}\right)_{i,j=1}^{d} \neq 0, \quad w \in \Delta(w_0; \varepsilon).$$

The set of vectors $\{P(\overline{w}, \overline{w}_0) v_j\}_{j=1}^{n}$ is linearly independent since $P(\overline{w}, \overline{w}_0)$ is injective on $\ker D_{(\boldsymbol{M}-w_0)^*}$. Let $(\alpha_{ij})_{i,j=1}^{d} = \left\{\left(\overline{f_i^j(w_0)}\right)_{i,j=1}^{d}\right\}^{-1}$, in consequence, $v_j = \sum_{l=1}^{d} \alpha_{jl} K_{w_0}^{(l)}$. We then have

$$\begin{aligned} K(\cdot, w) &= \sum_{i=1}^{d} \overline{g_i(w)} \left\{ \sum_{j=1}^{d} \overline{f_i^j(w)} P(\overline{w}, \overline{w}_0) \left(\sum_{l=1}^{d} \alpha_{jl} K_{w_0}^{(l)} \right) \right\} \\ &= \sum_{l=1}^{d} \left\{ \sum_{i,j=1}^{d} \overline{g_i(w)}\, \overline{f_i^j(w)} \alpha_{jl} \right\} P(\overline{w}, \overline{w}_0) K_{w_0}^{(l)}. \end{aligned}$$

Since the matrices $\left(\overline{f_i^j(w)}\right)_{i,j=1}^{d}$ and $(\alpha_{ij})_{i,j=1}^{d}$ are invertible, the functions

$$a_l(z) = \sum_{i,j=1}^{d} g_i(z) f_i^j(z) \alpha_{jl}, \quad 1 \leqq l \leqq d,$$

form a minimal set of generators for the stalk $\mathscr{S}_{w_0}^{\mathscr{M}}$ and hence we have the canonical decomposition,

$$K(\cdot, w) = \sum_{i=1}^{d} \overline{a_i(w)} P(\overline{w}, \overline{w}_0) K_{w_0}^{(i)}.$$

Let $\mathscr{P}_w = \operatorname{ran} P(\overline{w}, \overline{w}_0) P_{\ker D_{(\boldsymbol{M}-w_0)^*}}$ for $w \in B\left(w_0; \|R_{\boldsymbol{M}}(w_0)\|^{-1}\right)$. Since $P(\overline{w}, \overline{w}_0)$ restricted to the $\ker D_{(\boldsymbol{M}-w_0)^*}$ is one-one, and for w in $B\left(w_0; \|R_{\boldsymbol{M}}(w_0)\|^{-1}\right)$, the dimension of $\mathscr{P}_w$ is constant. Thus to prove Lemma 1.5, we will show that $\mathscr{P}_w = \ker \mathbb{P}_0 D_{(\boldsymbol{M}-w)^*}$, where $\mathbb{P}_0$ is the orthogonal projection onto $\operatorname{ran} D_{(\boldsymbol{M}-w_0)^*}$.

Proof of Lemma 1.5. From [7], page 453, it follows that $\mathbb{P}_0 D_{(\boldsymbol{M}-w)^*} P(\overline{w}, \overline{w}_0) = 0$. So, $\mathscr{P}_w \subseteqq \ker \mathbb{P}_0 D_{(\boldsymbol{M}-w)^*}$. Using (2.9) and (2.10), we can write

$$\begin{aligned}\mathbb{P}_0 D_{(\boldsymbol{M}-w)^*} &= D_{(\boldsymbol{M}-w_0)^*} R_{\boldsymbol{M}}(w_0)\{D_{(\boldsymbol{M}-w_0)^*} - D_{(\bar{w}-\bar{w}_0)}\}\\ &= D_{(\boldsymbol{M}-w_0)^*}\{I - P_{\ker D_{(\boldsymbol{M}-w_0)^*}} - R_{\boldsymbol{M}}(w_0) D_{(\bar{w}-\bar{w}_0)}\}\\ &= D_{(\boldsymbol{M}-w_0)^*}\{I - R_{\boldsymbol{M}}(w_0) D_{(\bar{w}-\bar{w}_0)}\}.\end{aligned}$$

Since $\{I - R_{\boldsymbol{M}}(w_0) D_{(\bar{w}-\bar{w}_0)}\}$ is invertible for w in $B(w_0; \|R_{\boldsymbol{M}}(w_0)\|^{-1})$, we have

$$\dim \mathscr{P}_w = \dim \ker D_{(\boldsymbol{M}-w_0)^*} \geqq \dim \ker \mathbb{P}_0 D_{(\boldsymbol{M}-w)^*}.$$

This completes the proof. $\square$

From the construction of the operator $P(\bar{w}, \bar{w}_0)$, it follows that $w \mapsto \mathscr{P}_w$ defines a Hermitian holomorphic vector bundle of rank m over

$$\Omega_0^* = \{\bar{z} : z \in \Omega_0\} \quad \text{where } \Omega_0 = B(w_0; \|R_{\boldsymbol{M}}(w_0)\|^{-1}).$$

Let $\mathscr{P}$ denote this Hermitian holomorphic vector bundle.

Proof of Theorem 1.6. Since $\mathscr{M}$ and $\tilde{\mathscr{M}}$ are equivalent Hilbert modules, there exists a unitary $U : \mathscr{M} \to \tilde{\mathscr{M}}$ intertwining the adjoint of the module multiplication, that is, $UM_j^* = \tilde{M}_j^* U$, $1 \leqq j \leqq m$. Here $\tilde{M}_j$ denotes the multiplication by co-ordinate functions z_j, $1 \leqq j \leqq m$, on $\tilde{\mathscr{M}}$. It is enough to show that

$$UP(\bar{w}, \bar{w}_0) = \tilde{P}(\bar{w}, \bar{w}_0)U \quad \text{for } w \in B(w_0; \|R_{\boldsymbol{M}}(w_0)\|^{-1}).$$

Let $|D_{\boldsymbol{M}^*}| = \left\{\sum_{j=1}^m M_j M_j^*\right\}^{\frac{1}{2}}$, that is, the positive square root of $(D_{\boldsymbol{M}^*})^* D_{\boldsymbol{M}^*}$. We have

$$\sum_{j=1}^m M_j M_j^* = U^*\left(\sum_{j=1}^m \tilde{M}_j \tilde{M}_j^*\right)U = (U^*|D_{\tilde{\boldsymbol{M}}^*}|U)^2.$$

Clearly, $|D_{\boldsymbol{M}^*}| = U^*|D_{\tilde{\boldsymbol{M}}^*}|U$. Similarly, we have $|D_{(\boldsymbol{M}-w_0)^*}| = U^*|D_{(\tilde{\boldsymbol{M}}-w_0)^*}|U$. Let

$$P_i : \mathscr{M} \oplus \mathscr{M} \oplus \cdots \oplus \mathscr{M} \to \mathscr{M}$$

be the orthogonal projection on the ith component. In this notation, for $1 \leqq j \leqq m$, we have $P_j D_{\boldsymbol{M}^*} = M_j^*$. Then,

$$\begin{aligned}\tilde{P}_j D_{(\tilde{\boldsymbol{M}}-w_0)^*} &= UP_j D_{(\boldsymbol{M}-w_0)^*} U^*\\ &= UP_j V_{\boldsymbol{M}}(w_0) U^* U |D_{(\boldsymbol{M}-w_0)^*}| U^*\\ &= UP_j V_{\boldsymbol{M}}(w_0) U^* |D_{(\tilde{\boldsymbol{M}}-w_0)^*}|.\end{aligned}$$

But $\tilde{P}_j D_{(\tilde{\boldsymbol{M}}-w_0)^*} = \tilde{P}_j V_{\tilde{\boldsymbol{M}}}(w_0)|D_{(\tilde{\boldsymbol{M}}-w_0)^*}|$. The uniqueness of the polar decomposition implies that $\tilde{P}_j V_{\tilde{\boldsymbol{M}}}(w_0) = UP_j V_{\boldsymbol{M}}(w_0) U^*$, $1 \leqq j \leqq m$. It follows that $Q_{\tilde{\boldsymbol{M}}}(w_0) = UQ_{\boldsymbol{M}}(w_0)U^*$. Note that $P_j^* : \mathscr{M} \to \mathscr{M} \oplus \cdots \oplus \mathscr{M}$ is given by

$$P_j^* h = (0, \ldots, h, \ldots, 0), \quad h \in \mathscr{M},\ 1 \leqq j \leqq m.$$

So, we have $V_{\tilde{\boldsymbol{M}}}(w_0)^*\tilde{P}_j^* = UV_{\boldsymbol{M}}(w_0)^*P_j^*U^*$, $1 \leqq j \leqq m$. Let $\tilde{D}_{\overline{w}} : \mathscr{M} \to \mathscr{M} \oplus \cdots \oplus \mathscr{M}$ be the operator: $\tilde{D}_{\overline{w}}f = (\overline{w}_1 f, \ldots, \overline{w}_m f)$, $f \in \tilde{\mathscr{M}}$. Clearly, $\tilde{D}_{\overline{w}} = UD_{\overline{w}}U^*$, that is,

$$U^*\tilde{P}_j\tilde{D}_{\overline{w}} = P_jD_{\overline{w}}U^*, \quad 1 \leqq j \leqq m.$$

Finally,

$$\begin{aligned}
R_{\tilde{\boldsymbol{M}}}(w_0)\tilde{D}_{\overline{w}-\overline{w}_0} &= Q_{\tilde{\boldsymbol{M}}}(w_0)V_{\tilde{\boldsymbol{M}}}(w_0)^*\tilde{D}_{\overline{w}-\overline{w}_0}\\
&= Q_{\tilde{\boldsymbol{M}}}(w_0)V_{\tilde{\boldsymbol{M}}}(w_0)^*(\tilde{P}_1\tilde{D}_{\overline{w}-\overline{w}_0}, \ldots, \tilde{P}_m\tilde{D}_{\overline{w}-\overline{w}_0})\\
&= Q_{\tilde{\boldsymbol{M}}}(w_0)V_{\tilde{\boldsymbol{M}}}(w_0)^*\left(\sum_{j=1}^{m}\tilde{P}_j^*\tilde{P}_j\tilde{D}_{\overline{w}-\overline{w}_0}\right)\\
&= Q_{\tilde{\boldsymbol{M}}}(w_0)UV_{\boldsymbol{M}}(w_0)^*\left(\sum_{j=1}^{m}P_j^*U^*\tilde{P}_j\tilde{D}_{\overline{w}-\overline{w}_0}\right)\\
&= UQ_{\boldsymbol{M}}(w_0)V_{\boldsymbol{M}}(w_0)^*\left(\sum_{j=1}^{m}P_j^*P_jD_{\overline{w}-\overline{w}_0}U^*\right)\\
&= UQ_{\boldsymbol{M}}(w_0)V_{\boldsymbol{M}}(w_0)^*D_{\overline{w}-\overline{w}_0}U^*\\
&= UR_{\boldsymbol{M}}(w_0)D_{\overline{w}-\overline{w}_0}U^*.
\end{aligned}$$

Hence $\{R_{\tilde{\boldsymbol{M}}}(w_0)\tilde{D}_{\overline{w}-\overline{w}_0}\}^k = U\{R_{\boldsymbol{M}}(w_0)D_{\overline{w}-\overline{w}_0}\}^kU^*$ for all $k \in \mathbb{N}$. From (2.11), we have

$$P(\overline{w}, \overline{w}_0) = \sum_{k=0}^{\infty}\{R_{\boldsymbol{M}}(w_0)D_{\overline{w}-\overline{w}_0}\}^kP_{\ker D_{(\boldsymbol{M}-w_0)^*}}.$$

Also as U maps $\ker D_{(\boldsymbol{M}-w)^*}$ onto $\ker D_{(\tilde{\boldsymbol{M}}-w)^*}$ for each w, we have in particular, $UP_{\ker D_{(\boldsymbol{M}-w_0)^*}} = P_{\ker D_{(\tilde{\boldsymbol{M}}-w_0)^*}}U$. Therefore,

$$\begin{aligned}
UP(\overline{w}, \overline{w}_0) &= \sum_{k=0}^{\infty}U\{R_{\boldsymbol{M}}(w_0)D_{\overline{w}-\overline{w}_0}\}^kP_{\ker D_{(\boldsymbol{M}-w_0)^*}}\\
&= \sum_{k=0}^{\infty}\{R_{\tilde{\boldsymbol{M}}}(w_0)\tilde{D}_{\overline{w}-\overline{w}_0}\}^kUP_{\ker D_{(\boldsymbol{M}-w_0)^*}}\\
&= \sum_{k=0}^{\infty}\{R_{\tilde{\boldsymbol{M}}}(w_0)\tilde{D}_{\overline{w}-\overline{w}_0}\}^kP_{\ker D_{(\tilde{\boldsymbol{M}}-w_0)^*}}U\\
&= \tilde{P}(\overline{w}, \overline{w}_0)U,
\end{aligned}$$

for $w \in B\big(w_0; \|R_{\boldsymbol{M}}(w_0)\|^{-1}\big)$. $\square$

Remark 2.13. For any commuting m-tuple $\boldsymbol{T} = (T_1, \ldots, T_m)$ of operators on $\mathscr{H}$, the construction given above, of the Hermitian holomorphic vector bundle, provides a unitary invariant, assuming only that $\operatorname{ran} D_{\boldsymbol{T}-w}$ is closed for w in $\Omega \subseteqq \mathbb{C}^m$. Consequently, the class of this Hermitian holomorphic vector bundle is an invariant for any Hilbert module over $\mathbb{C}[\underline{z}]$ of finite rank.

3. Division problems

3.1. Bergman space privilege. Fix two positive integers p, q. The division problem asks if the solution $u \in \mathcal{O}(\Omega)^q$ to the linear equation $Au = f$ must belong to $L_a^2(\Omega)^q$ if $f \in L_a^2(\Omega)^p$ and the matrix $A \in M_{p,q}(\mathcal{O}(\overline{\Omega}))$ of analytic functions defined in a neighborhood of $\overline{\Omega}$ are given. Two independent steps are necessary to understand the nature of the division problem.

First, the solution u may not be unique, simply due to the non-trivial relations among the columns of the matrix A. This difficulty is clarified by homological algebra: at the level of coherent analytic sheaves, $\mathfrak{N} = \operatorname{coker}(A : \mathcal{O}|_{\overline{\Omega}}^p \to \mathcal{O}|_{\overline{\Omega}}^q)$ admits a finite free resolution

$$0 \to \mathcal{O}|_{\overline{\Omega}}^{n_p} \xrightarrow{d_p} \cdots \to \mathcal{O}|_{\overline{\Omega}}^{n_1} \xrightarrow{d_1} \mathcal{O}|_{\overline{\Omega}}^{n_0} \to \mathfrak{N} \to 0, \tag{3.1}$$

where $n_1 = p$, $n_0 = q$ and $d_1 = A$. The existence of such a resolution is assured by the analogue of Hilberts Syzygies Theorem in the analytic context, see for instance [22].

The second step, of circumventing the non-existence of boundary values for Bergman space functions, is resolved by a canonical quantization method, that is, by passing to the algebra of Toeplitz operators with continuous symbol on $L_a^2(\Omega)$. We import below, from the well understood theory of Toeplitz operators on domains of $\mathbb{C}^m$, a crucial criterion for a matrix of Toepliz operators to be Fredholm (cf. [32], [34]).

Assume that the analytic matrix $A(z)$ is defined on a neighborhood of $\overline{\Omega}$. One proves by standard homological techniques that every free, finite type resolution of the analytic coherent sheaf $\mathfrak{N} = \operatorname{coker}(A : \mathcal{O}|_{\overline{\Omega}}^p \to \mathcal{O}|_{\overline{\Omega}}^q)$ induces at the level of the Bergman space $L_a^2(\Omega)$ an exact complex, see [9]. Theorem 1.8 shows that the similarity between the two resolutions given above are not accidental. After understanding the disc-algebra privilege on a strictly convex domain [30], the statement of Theorem 1.8 is not surprising.

Proof of Theorem 1.8. The proof is very similar to the one of the disk algebra case [30], and we only sketch below the main ideas. Assume that the resolution (1.5) exists and that the last arrow has closed range. The exactness at each degree of the resolution is equivalent to the invertibility of the Hodge operator:

$$d_k^* d_k + d_{k+1} d_{k+1}^* : L_a^2(\Omega)^{n_k} \to L_a^2(\Omega)^{n_k}, \quad 1 \leqq k \leqq p,$$

where we put $d_{p+1} = 0$. To be more specific: the condition $\ker[d_k^* d_k + d_{k+1} d_{k+1}^*] = 0$ is equivalent to the exactness of the complex at stage k, implying that $\operatorname{ran}(d_{k+1})$ is closed. In addition, if the range of d_k is closed, then, and only then, the self-adjoint operator $d_k^* d_k + d_{k+1} d_{k+1}^*$ is invertible.

Since the boundary of Ω is smooth, the commutator $[T_f, T_g]$ of two Toeplitz operators acting on the Bergman space and with continuous symbols $f, g \in C(\overline{\Omega})$ is compact, see for details and terminology [4], [32], [34]. Consequently for every k, $d_k^* d_k + d_{k+1} d_{k+1}^*$ is, modulo compact operators, an $n_k \times n_k$ matrix of Toeplitz operators with symbol

$$d_k(z)^* d_k(z) + d_{k+1}(z) d_{k+1}(z)^*, \quad w \in \overline{\Omega},$$

where the adjoint is now taken with respect to the canonical inner product in $\mathbb{C}^{n_k}$. According to a main result of [4], or [34], [32], if the Toeplitz operator $d_k^* d_k + d_{k+1} d_{k+1}^*$ is Fredholm, then its matrix symbol is invertible. Hence

$$\ker[d_k(z)^* d_k(z) + d_{k+1}(z) d_{k+1}(z)^*] = 0, \quad 1 \leqq k \leqq p.$$

Thus, for every $z \in \partial\Omega$,

$$\operatorname{rank} A(z) = \dim \operatorname{coker}\big(d_1(w)\big) = n_0 - n_1 + n_2 - \cdots + (-1)^p n_p.$$

To prove the other implication, we rely on the disk algebra privilege criterion obtained in the note [30]. Namely, in view of [30], Theorem 2.2, if the rank of the matrix $A(z)$ does not jump for z belonging to the boundary of Ω, then there exists a resolution of $\boldsymbol{N} = \operatorname{coker} A : \mathscr{A}(\Omega)^p \to \mathscr{A}(\Omega)^q$ with free, finite type $\mathscr{A}(\Omega)$-modules:

$$0 \to \mathscr{A}(\Omega)^{n_p} \xrightarrow{d_p} \cdots \to \mathscr{A}(\Omega)^{n_1} \xrightarrow{d_1} \mathscr{A}(\Omega)^{n_0} \to \boldsymbol{N} \to 0. \tag{3.2}$$

As before, we denote $d_1 = A$. We have to prove that the induced complex (1.5), obtained after applying (3.2) the functor $\bigotimes_{\mathscr{A}(\Omega)} L_a^2(\Omega)$, remains exact and the boundary operator d_1 has closed range.

For this, we "glue" together local resolutions of $\operatorname{coker} A$ with the aid of Cartan's lemma of invertible matrices, as originally explained in [10], or in [30]. For points close to the boundary of Ω, such a resolution exists by the local freeness assumption, while in the interior, in neighborhoods of the points where the rank of the matrix A may jump, they exist by Douady's privilege on polydiscs. This proves that the Hilbert analytic module $\mathscr{N} = \operatorname{coker}\big(A : L_a^2(\Omega)^p \to L_a^2(\Omega)^q\big)$ is privileged with respect to the Bergman space.

As for assertion (c), we simply remark that it is equivalent to the injectivity of the restriction map

$$\operatorname{coker}\big(A : L_a^2(\Omega)^p \to L_a^2(\Omega)^q\big) \to \operatorname{coker}\big(A : \mathscr{O}(\Omega)^p \to \mathscr{O}(\Omega)^q\big).$$

The last co-kernel is always Hausdorff in the natural quotient topology as the global section space of a coherent analytic sheaf.

The only place in the proof where the convexity of Ω is needed, is to ensure that, if the resolution (1.5) exists, then the induced complex at the level of sheaf models (cf. [19])

$$0 \to \widehat{L_a^2}(\Omega)^{n_p} \xrightarrow{d_p} \cdots \to \widehat{L_a^2}(\Omega)^{n_1} \xrightarrow{d_1} \widehat{L_a^2}(\Omega)^{n_0} \to \hat{\mathscr{N}} \to 0$$

is exact. For a proof see [30]. □

Remark 3.1. It is worth mentioning that for non-smooth domains Ω in $\mathbb{C}^m$ the above result is not true. For instance $\mathscr{A}(\Omega)$-privilege for a poly-domain Ω was fully characterized by Douady [10]. On the other hand, even for smooth boundaries, the privilege with respect to the Fréchet algebra $\mathscr{O}(\Omega) \cap C^\infty(\overline{\Omega})$ seems to be quite intricate and definitely different than the Bergman space or disk algebra privileges, as indicated by an observation of Amar [1].

Corollary 3.2. Coker$[(\varphi_1, \dots, \varphi_p) : L_a^2(\Omega)^p \to L_a^2(\Omega)]$ *is privileged if and only if the analytic functions* $(\varphi_1, \dots, \varphi_p)$ *have no common zero on the boundary.*

Proof. No common zero of the functions $\varphi_1, \dots, \varphi_p$ lies on the boundary of Ω. Therefore, the matrix $(\varphi_1, \dots, \varphi_p)$ is of full rank 1 on the boundary of Ω. $\square$

For many Hilbert modules of finite rank such as the Hardy space on Ω, the result given above, remains true ([9], [10]).

Since the restriction to an open subset $\Omega_0 \subseteqq \Omega$ does not change the equivalence class of a module in $\mathfrak{B}_1(\Omega)$, we can always assume, without loss of generality, that the domain Ω is pseudoconvex in our context. For $w_0 \in \Omega$, the m-tuple $(z_1 - w_{01}, \dots, z_m - w_{0m})$ has no common zero on the boundary of Ω. We have pointed out, in Section 1, that if for $f \in \mathscr{M}$ the equation $f = \sum_{i=1}^{m} (z_i - w_{0i}) f_i$ admits a solution $(f_1, \dots, f_m)$ in $\mathcal{O}(\Omega)^m$ and if the module $\mathscr{M}$ is privileged, then the solution is in $\mathscr{M}^m$. This shows that $f \in \mathscr{M}^{(w_0)}$. Thus for Hilbert modules which are privileged, we have

$$\#\{\text{minimal generators for } \mathscr{S}_w^{\mathscr{M}}\} = \dim \ker D_{(\boldsymbol{M}-w)^*}.$$

In accordance with the terminology of local spectral theory, see [19], we isolate the following observation.

Corollary 3.3. *Assume that the analytic module* $\mathscr{N} = \operatorname{coker}\big(A : L_a^2(\Omega)^p \to L_a^2(\Omega)^q\big)$ *is Hausdorff, where A and* Ω *are as in the theorem. Then* $\mathscr{N}$ *is a Hilbert analytic quasi-coherent module, and for every Stein open subset U of* $\mathbb{C}^m$, *the associated sheaf model is*

$$\begin{aligned} \hat{\mathscr{N}}(U) = \mathcal{O}(U) \,\hat{\otimes}_{\mathscr{A}(\Omega)} \mathscr{N} &= \operatorname{coker}\big(A : \mathscr{H}(U)^p \to \mathscr{H}(U)^q\big) \\ &= \operatorname{coker}\big(z - w : \mathcal{O}(U) \,\hat{\otimes}\, \mathscr{N}^m \to \mathcal{O}(U) \,\hat{\otimes}\, \mathscr{N}\big), \end{aligned}$$

where $\mathscr{H}$ *denotes the sheaf model of the Bergman space.*

Remark 3.4. We recall that (see [19])

$$\mathscr{H}(U) = \{f \in \mathcal{O}(U \cap \Omega) : \|f\|_{2,K} < \infty, K \text{ compact in } U\}.$$

Since $\mathscr{H}|_\Omega = \mathcal{O}|_\Omega$ we infer that the restriction $\hat{\mathscr{N}}|_\Omega$ is a coherent sheaf, with finite free resolution

$$0 \to \mathcal{O}|_\Omega^{n_p} \xrightarrow{d_p} \cdots \to \mathcal{O}|_\Omega^{n_1} \xrightarrow{d_1} \mathcal{O}|_\Omega^{n_0} \to \hat{\mathscr{N}}|_\Omega \to 0.$$

3.2. Coincidence of sheaf models. Besides the expected relaxations of the main result above, for instance from convex to pseudoconvex domains, a natural problem to consider at this stage is the classification of the analytic Hilbert modules

$$\mathscr{N} = \operatorname{coker}\big(A : L_a^2(\Omega)^p \to L_a^2(\Omega)^q\big)$$

appearing in the Theorem 1.8 above. This question fits into the framework of quasi-free Hilbert modules introduced in [12]. That the resulting parameter space is wild, there is no doubt, as all Artinian modules M (over the polynomial algebra) supported by a fix point $w_0 \in \Omega$ enter into our discussion. Specifically, we can take

$$M = \operatorname{coker}\big((\varphi_1, \dots, \varphi_p) : L_a^2(\Omega)^p \to L_a^2(\Omega)\big),$$

where $\varphi_1, \dots, \varphi_p$ are polynomials with the only common zero $\{w_0\}$. Then in virtue of Theorem 1.8, the analytic module M is finite dimensional and privileged with respect to the Bergman space $L_a^2(\Omega)$. An algebraic reduction of the classification of all finite co-dimension analytic Hilbert modules of the Bergman space associated of a smooth, strictly convex domain can be found in [28], [29].

In order to better relate the Cowen–Douglas theory to the above framework, we consider together with the map

$$A : L_a^2(\Omega)^p \to L_a^2(\Omega)^q$$

whose cokernel was supposed to be Hausdorff, the dual, anti-analytic map

$$A^* : L_a^2(\Omega)^q \to L_a^2(\Omega)^p.$$

It is the linear system, in the terminology of Grothendieck [31] or [21], with its associated Hermitian structure induced from the embedding into Bergman space,

$$\ker A^*(z) \subset L_a^2(\Omega)^q, \quad z \in \Omega,$$

which was initially considered in Operator Theory, see [15].

Traditionally one works with the torsion-free module

$$\mathcal{M} = \operatorname{ran}\big(A : L_a^2(\Omega)^p \to L_a^2(\Omega)^q\big),$$

rather than the cokernel $\mathcal{N}$ studied in the previous section. A short exact sequence relates the two modules:

$$0 \to \mathcal{M} \to L_a^2(\Omega)^q \to \mathcal{N} \to 0.$$

Proposition 3.5. *Assume, in the conditions of Theorem* 1.8, *that the range $\mathcal{M}$ of the module map A is closed. Then $\mathcal{M}$ is an analytic Hilbert quasi-coherent module, with associated sheaf model*

$$\hat{\mathcal{M}}(U) = \operatorname{ran}\big(A : \mathcal{H}(U)^p \to \mathcal{H}(U)^q\big),$$

for every Stein open subset U of $\mathbb{C}^m$.

In particular, for every point $w_0 \in \Omega$, there are finitely many elements

$$g_1, \dots, g_d \in \mathcal{M} \subset L_a^2(\Omega)^q,$$

such that the stalk $\hat{\mathcal{M}}_{w_0}$ coincides with the $\mathcal{O}_{w_0}$-module generated in $\mathcal{O}_{w_0}^q$ by $g_1, \dots, g_d$.

Proof. The first assertion follows from the main result of the previous section and the yoga of quasi-coherent sheaves. In particular, we obtain an exact complex of coherent analytic sheaves

$$0 \to \hat{\mathscr{M}}|_\Omega \to \mathscr{O}_\Omega^n \to \hat{\mathscr{N}}|_\Omega \to 0.$$

For the proof of the second assertion, recall that the quasi-coherence of $\mathscr{M}$ yields a finite presentation, derived from the associated Koszul complex

$$\mathscr{O}_{w_0}^m \hat{\otimes} \mathscr{M} \overset{z-w}{\to} \mathscr{O}_{w_0} \hat{\otimes} \mathscr{M} \to \hat{\mathscr{M}}_{w_0} \to 0.$$

By evaluating the presentation at $w = w_0$, we obtain the exact complex

$$\mathscr{M}^m \overset{z-w_0}{\to} \mathscr{M} \to \hat{\mathscr{M}}(w_0) \to 0.$$

Above we denote by $w = (w_1, \dots, w_m)$ the m-tuple of local coordinates in the ring $\mathscr{O}_{w_0}$, while $z = (z_1, \dots, z_m)$ stands for the m-tuple of coordinate functions in the base space of the Hilbert module $L_a^2(\Omega)$.

By coherence, $\dim \hat{\mathscr{M}}(w_0) < \infty$, and it remains to choose the d-tuple of elements $g = (g_1, \dots, g_d)$ as a basis of the ortho-complement of $\operatorname{ran}(z - w_0 : \mathscr{M}^m \to \mathscr{M})$. Then the map

$$\mathscr{O}_{w_0}^m \hat{\otimes} (\mathscr{M} \oplus \mathbb{C}^d) \xrightarrow{z-w,g} \mathscr{O}_{w_0} \hat{\otimes} \mathscr{M}$$

is onto. Consequently, the functions $g_1, \dots, g_d$ generate $\hat{\mathscr{M}}_{w_0}$ as a submodule of $\mathscr{O}_{w_0}^q$. As a matter of fact the same functions will generate $\hat{\mathscr{M}}_w$ for all points w belonging to a neighborhood of w_0. $\square$

Corollary 3.6. *Under the assumptions of the proposition, the restriction to Ω of the sheaf model $\hat{\mathscr{M}} = \widehat{\operatorname{ran} A}$ coincides with the analytic subsheaf of $\mathscr{O}^q$ generated by all functions $f|_\Omega$, $f \in \mathscr{M}$.*

The dual picture emerges easily: let w_0 be a fixed point of Ω, under the assumptions of Theorem 1.8, the map $A_{w_0}(z) := (z_1 - w_{01}, \dots, z_m - w_{0m}) : \mathscr{M}^m \to \mathscr{M}$ has finite dimensional cokernel. Choose a basis $v_1, \dots, v_\ell$ of $\ker A_{w_0}(z)^*$ and denote by P_w the orthogonal projection onto $\ker A_w(z)^*$. Then for w belonging to a small enough open neighborhood Ω_0 of w_0, the elements $P_w(v_1), \dots, P_w(v_\ell)$ generate $\ker A_w(z)^*$ as a vector space, but they need not remain linearly independent on Ω_0. Nevertheless, starting with a module $\mathscr{M}$ in $\mathfrak{B}_1(\Omega)$, we have established the existence of a holomorphic Hermitian vector bundle $E_{\mathscr{M}}$ on Ω_0^* (in Subsection 2.4).

4. Examples

4.1. The (λ, μ) examples. Let $\mathscr{M}$ and $\tilde{\mathscr{M}}$ be two Hilbert modules in $\mathrm{B}_1(\Omega)$ and $\mathscr{I}$, $\mathscr{J}$ be two ideals in $\mathbb{C}[\underline{z}]$. Let $\mathscr{M}_{\mathscr{I}} := [\mathscr{I}] \subseteq \mathscr{M}$ (resp. $\tilde{\mathscr{M}}_{\mathscr{J}} := [\mathscr{J}] \subset \tilde{\mathscr{M}}$) denote the closure of $\mathscr{I}$

in $\mathscr{M}$ (resp. closure of $\mathscr{J}$ in $\tilde{\mathscr{M}}$). Also assume that every algebraic component of $V(\mathscr{I})$ and $V(\mathscr{J})$ intersects Ω and their dimension is at most $m-2$. It is then not hard to see that $\mathscr{M}_{\mathscr{I}}$ and $\tilde{\mathscr{M}}_{\mathscr{J}}$ are equivalent then $\mathscr{I}=\mathscr{J}$ following the argument in the proof of [2], Theorem 2.10, and using the characteristic space theory of [5], Chapter 2 (see [3]).

Although this assertion may appear to be slightly more general than the rigidity theorem of [16], Theorem 3.3, we believe the proof of Theorem 3.3 from [16] works in this case as well.

Assume $\mathscr{M}$ and $\tilde{\mathscr{M}}$ are minimal extensions of the two modules $\mathscr{M}_{\mathscr{I}}$ and $\tilde{\mathscr{M}}_{\mathscr{I}}$ respectively and that $\mathscr{M}_{\mathscr{I}}$ is equivalent to $\tilde{\mathscr{M}}_{\mathscr{I}}$. We ask if these assumptions force the extensions $\mathscr{M}$ and $\tilde{\mathscr{M}}$ to be equivalent. The answer for a class of examples is given below.

For $\lambda, \mu > 0$, let $H^{(\lambda,\mu)}(\mathbb{D}^2)$ be the reproducing kernel Hilbert space on the bi-disc determined by the positive definite kernel

$$K^{(\lambda,\mu)}(z,w) = \frac{1}{(1-z_1\overline{w}_1)^{\lambda}(1-z_2\overline{w}_2)^{\mu}}, \quad z,w \in \mathbb{D}^2.$$

As is well known, $H^{(\lambda,\mu)}(\mathbb{D}^2)$ is in $\mathrm{B}_1(\mathbb{D}^2)$. Let I be the maximal ideal in $\mathbb{C}[z_1,z_2]$ of polynomials vanishing at $(0,0)$. Let $H_0^{(\lambda,\mu)}(\mathbb{D}^2) := [I]$. For any other pair of positive numbers λ', μ', we let $H_0^{(\lambda',\mu')}(\mathbb{D}^2)$ denote the closure of I in the reproducing kernel Hilbert space $H^{(\lambda',\mu')}(\mathbb{D}^2)$. Let $K^{(\lambda',\mu')}$ denote the corresponding reproducing kernel. The modules $H_0^{(\lambda,\mu)}(\mathbb{D}^2)$ and $H_0^{(\lambda',\mu')}(\mathbb{D}^2)$ are in $\mathrm{B}_1\big(\mathbb{D}^2\backslash\{(0,0)\}\big)$ but not in $\mathrm{B}_1(\mathbb{D}^2)$. So, there is no easy computation to determine when they are equivalent. We compute the curvature, at $(0,0)$, of the holomorphic Hermitian bundle $\mathscr{P}$ and $\tilde{\mathscr{P}}$ of rank 2 corresponding to the modules $H_0^{(\lambda,\mu)}(\mathbb{D}^2)$ and $H_0^{(\lambda',\mu')}(\mathbb{D}^2)$ respectively. The calculation of the curvature shows that if these modules are equivalent then $\lambda = \lambda'$ and $\mu = \mu'$, that is, the extensions $H^{(\lambda,\mu)}(\mathbb{D}^2)$ and $H^{(\lambda',\mu')}(\mathbb{D}^2)$ are then equal.

Since $H_0^{(\lambda,\mu)}(\mathbb{D}^2) := \{f \in H^{(\lambda,\mu)}(\mathbb{D}^2) : f(0,0) = 0\}$, the corresponding reproducing kernel $K_0^{(\lambda,\mu)}$ is given by the formula

$$K_0^{(\lambda,\mu)}(z,w) = \frac{1}{(1-z_1\overline{w}_1)^{\lambda}(1-z_2\overline{w}_2)^{\mu}} - 1, \quad z,w \in \mathbb{D}^2.$$

The set $\{z_1^m z_2^n : m,n \geqq 0, (m,n) \neq (0,0)\}$ forms an orthogonal basis for $H_0^{(\lambda,\mu)}(\mathbb{D}^2)$. Also

$$\langle z_1^l z_2^k, M_1^* z_1^{m+1}\rangle = \langle z_1^{l+1} z_2^k, z_1^{m+1}\rangle = 0,$$

unless $l = m$, $k = 0$ and $m > 0$. In consequence,

$$\langle z_1^m, M_1^* z_1^{m+1}\rangle = \langle z_1^{m+1}, z_1^{m+1}\rangle = \frac{1}{(-1)^{m+1}\binom{-\lambda}{m+1}} = \frac{(-1)^m\binom{-\lambda}{m}}{(-1)^{m+1}\binom{-\lambda}{m+1}}\langle z_1^m, z_1^m\rangle.$$

Then

$$\left\langle z_1^l z_2^k, M_1^* z_1^{m+1} - \frac{m+1}{\lambda+m} z_1^m \right\rangle = 0 \quad \text{for all } l,k \geqq 0,\ (l,k) \neq (0,0),$$

where $\binom{-\lambda}{m} = (-1)^m \dfrac{\lambda(\lambda+1)\cdots(\lambda+m-1)}{m!}$. Now, $\langle z_1^l z_2^k, M_1^* z_1 \rangle = \langle z_1^{l+1} z_2^k, z_1 \rangle = 0$, $l,k \geqq 0$ and $(l,k) \neq (0,0)$. Therefore, we have

$$M_1^* z_1^{m+1} = \begin{cases} \dfrac{m+1}{\lambda+m} z_1^m, & m > 0, \\ 0, & m = 0. \end{cases}$$

Similarly,

$$M_2^* z_2^{n+1} = \begin{cases} \dfrac{n+1}{\lambda+n} z_1^n, & n > 0, \\ 0, & n = 0. \end{cases}$$

We easily verify that $\langle z_1^l z_2^k, M_2^* z_1^{m+1} \rangle = \langle z_1^l z_2^{k+1}, z_1^{m+1} \rangle = 0$. Hence $M_2^* z_1^{m+1} = 0 = M_1^* z_2^{n+1}$ for $m,n \geqq 0$. Finally, calculations similar to the one given above, show that

$$M_1^* z_1^{m+1} z_2^{n+1} = \frac{m+1}{\lambda+m} z_1^m z_2^{n+1} \quad \text{and} \quad M_2^* z_1^{m+1} z_2^{n+1} = \frac{n+1}{\mu+n} z_1^{m+1} z_2^n, \quad m,n \geqq 0$$

Therefore we have

$$(M_1 M_1^* + M_2 M_2^*) : \begin{cases} z_1^{m+1} \mapsto \dfrac{m+1}{\lambda+m} z_1^{m+1} & \text{for } m > 0, \\ z_2^{n+1} \mapsto \dfrac{n+1}{\mu+n} z_2^{n+1} & \text{for } n > 0, \\ z_1^{m+1} z_2^{n+1} \mapsto \left(\dfrac{m+1}{\lambda+m} + \dfrac{n+1}{\mu+n} \right) z_1^{m+1} z_2^{n+1} & \text{for } m,n \geqq 0, \\ z_1, z_2 \mapsto 0. \end{cases}$$

Also, since $D_{M^*} f = (M_1^* f, M_2^* f)$, we have

$$D_{M^*} : \begin{cases} z_1^{m+1} \mapsto \left(\dfrac{m+1}{\lambda+m} z_1^m, 0 \right) & \text{for } m > 0, \\ z_2^{n+1} \mapsto \left(0, \dfrac{n+1}{\mu+n} z_2^n \right) & \text{for } n > 0, \\ z_1^{m+1} z_2^{n+1} \mapsto \left(\dfrac{m+1}{\lambda+m} z_1^m z_2^{n+1}, \dfrac{n+1}{\mu+n} z_1^{m+1} z_2^n \right) & \text{for } m,n \geqq 0, \\ z_1, z_2 \mapsto (0,0). \end{cases}$$

It is easy to calculate $V_{\boldsymbol{M}}(0)$ and $Q_{\boldsymbol{M}}(0)$ (see Subsection 2.4) and show that

$$V_{\boldsymbol{M}}(0):\begin{cases} z_1^{m+1} \mapsto \sqrt{\dfrac{m+1}{\lambda+m}}(z_1^m, 0) & \text{for } m > 0,\\ z_2^{n+1} \mapsto \sqrt{\dfrac{n+1}{\mu+n}}(0, z_2^n) & \text{for } n > 0,\\ z_1^{m+1}z_2^{n+1} \mapsto \dfrac{1}{\sqrt{\dfrac{m+1}{\lambda+m}+\dfrac{n+1}{\mu+n}}}\left(\dfrac{m+1}{\lambda+m}z_1^m z_2^{n+1}, \dfrac{n+1}{\mu+n}z_1^{m+1}z_2^n\right) & \text{for } m, n \geqq 0,\\ z_1, z_2 \mapsto (0,0), \end{cases}$$

while

$$Q_{\boldsymbol{M}}(0):\begin{cases} z_1^{m+1} \mapsto \dfrac{1}{\sqrt{\dfrac{m+1}{\lambda+m}}}z_1^{m+1} & \text{for } m > 0,\\ z_2^{n+1} \mapsto \dfrac{1}{\sqrt{\dfrac{n+1}{\mu+n}}}z_2^{n+1} & \text{for } n > 0,\\ z_1^{m+1}z_2^{n+1} \mapsto \dfrac{1}{\sqrt{\dfrac{m+1}{\lambda+m}+\dfrac{n+1}{\mu+n}}}z_1^{m+1}z_2^{n+1} & \text{for } m, n \geqq 0,\\ z_1, z_2 \mapsto 0. \end{cases}$$

Now for $w \in \Delta(0, \varepsilon)$,

$$P(\overline{w}, 0) = \big(I - R_{\boldsymbol{M}}(0)D_{\overline{w}}\big)^{-1}P_{\ker D_{\boldsymbol{M}^*}} = \sum_{n=0}^{\infty}\big(R_{\boldsymbol{M}}(0)D_{\overline{w}}\big)^n P_{\ker D_{\boldsymbol{M}^*}},$$

where $R_{\boldsymbol{M}}(0) = Q_{\boldsymbol{M}}(0)V_{\boldsymbol{M}}(0)^*$. The vectors z_1 and z_2 form a basis for $\ker D_{\boldsymbol{M}^*}$ and therefore define a holomorphic frame: $\big(P(\overline{w}, 0)z_1, P(\overline{w}, 0)z_2\big)$. Recall that

$$P(\overline{w}, 0)z_1 = \sum_{n=0}^{\infty}\big(R_{\boldsymbol{M}}(0)D_{\overline{w}}\big)^n z_1 \quad \text{and} \quad P(\overline{w}, 0)z_2 = \sum_{n=0}^{\infty}\big(R_{\boldsymbol{M}}(0)D_{\overline{w}}\big)^n z_2.$$

To describe these explicitly, we calculate $\big(R_{\boldsymbol{M}}(0)D_{\overline{w}}\big)z_1$ and $\big(R_{\boldsymbol{M}}(0)D_{\overline{w}}\big)z_2$:

$$\begin{aligned}\big(R_{\boldsymbol{M}}(0)D_{\overline{w}}\big)z_1 &= R_{\boldsymbol{M}}(0)(\overline{w}_1, z_1, \overline{w}_2 z_2)\\ &= \overline{w}_1 R_{\boldsymbol{M}}(0)(z_1, 0) + \overline{w}_2 R_{\boldsymbol{M}}(0)(0, z_2)\\ &= \overline{w}_1 Q_{\boldsymbol{M}}(0)V_{\boldsymbol{M}}(0)^*(z_1, 0) + \overline{w}_2 Q_{\boldsymbol{M}}(0)V_{\boldsymbol{M}}(0)^*(0, z_2).\end{aligned}$$

We see that

$$V_{\boldsymbol{M}}(0)^*(z_1,0) = \sum_{l,k \geqq 0,\,(l,k) \neq (0,0)} \left\langle V_{\boldsymbol{M}}(0)^*(z_1,0), \frac{z_1^l z_2^k}{\|z_1^l z_2^k\|} \right\rangle \frac{z_1^l z_2^k}{\|z_1^l z_2^k\|}.$$

Therefore,

$$\langle V_{\boldsymbol{M}}(0)^*(z_1,0), z_1^l z_2^k \rangle = \langle (z_1,0), V_{\boldsymbol{M}}(0)(z_1^l z_2^k) \rangle, \quad l,k \geqq 0,\ (l,k) \neq (0,0).$$

From the explicit form of $V_{\boldsymbol{M}}(0)$, it is clear that the inner product given above is 0 unless $l = 2$, $k = 0$. For $l = 2$, $k = 0$, we have

$$\langle (z_1,0), V_{\boldsymbol{M}}(0) z_1^2 \rangle = \sqrt{\frac{2}{\lambda+1}}\,\|z_1\|^2 = \sqrt{\frac{2}{\lambda+1}}\,\frac{1}{\lambda}.$$

Hence

$$V_{\boldsymbol{M}}(0)^*(z_1,0) = \sqrt{\frac{2}{\lambda+1}}\,\frac{1}{\lambda}\,\frac{z_1^2}{\|z_1^2\|^2} = \sqrt{\frac{2}{\lambda+1}}\,\frac{1}{\lambda}\,\frac{\lambda(\lambda+1)}{2} z_1^2 = \sqrt{\frac{\lambda+1}{2}}\,z_1^2.$$

Again, to calculate $V_{\boldsymbol{M}}(0)^*(0,z_1)$, we note that $\langle V_{\boldsymbol{M}}(0)^*(0,z_1), z_1^l z_2^k \rangle$ is 0 unless $l = 1$, $m = 1$. For $l = 1$, $m = 1$, we have

$$\begin{aligned}
\langle V_{\boldsymbol{M}}(0)^*(0,z_1), z_1 z_2 \rangle &= \langle (0,z_1), V_{\boldsymbol{M}}(0) z_1 z_2 \rangle \\
&= \left\langle \frac{1}{\sqrt{\frac{1}{\lambda}+\frac{1}{\mu}}} \left(\frac{1}{\lambda} z_2, \frac{1}{\mu} z_1 \right), (0,z_1) \right\rangle \\
&= \frac{1}{\sqrt{\frac{1}{\lambda}+\frac{1}{\mu}}}\,\frac{1}{\mu}\,\|z_1\|^2 = \frac{1}{\sqrt{\frac{1}{\lambda}+\frac{1}{\mu}}}\,\frac{1}{\lambda\mu}.
\end{aligned}$$

Thus

$$V_{\boldsymbol{M}}(0)^*(0,z_1) = \langle V_{\boldsymbol{M}}(0)^*(0,z_1), z_1 z_2 \rangle \frac{z_1 z_2}{\|z_1 z_2\|^2} = \frac{1}{\sqrt{\frac{1}{\lambda}+\frac{1}{\mu}}}\,z_1 z_2.$$

Since

$$\begin{aligned}
Q_{\boldsymbol{M}}(0) z_1^2 &= \sqrt{\frac{\lambda+1}{2}}\,z_1^2, \\
Q_{\boldsymbol{M}}(0) z_1 z_2 &= \frac{1}{\sqrt{\frac{1}{\lambda}+\frac{1}{\mu}}}\,z_1 z_2, \\
Q_{\boldsymbol{M}}(0) z_2^2 &= \sqrt{\frac{\mu+1}{2}}\,z_2^2,
\end{aligned}$$

it follows that

$$R_{\boldsymbol{M}}(0)D_{\overline{w}}z_1 = \overline{w}_1\frac{\lambda+1}{2}z_1^2 + \overline{w}_2\frac{\lambda\mu}{\lambda+\mu}z_1z_2.$$

Similarly, we obtain the formula

$$R_{\boldsymbol{M}}(0)D_{\overline{w}}z_2 = \overline{w}_1\frac{\lambda\mu}{\lambda+\mu}z_1z_2 + \overline{w}_2\frac{\mu+1}{2}z_2^2.$$

We claim that

$$\big\langle \big(R_{\boldsymbol{M}}(0)D_{\overline{w}}\big)^m z_i, \big(R_{\boldsymbol{M}}(0)D_{\overline{w}}\big)^n z_j\big\rangle = 0 \quad \text{for all } m \neq n \text{ and } i,j = 1,2. \tag{4.1}$$

This makes the calculation of

$$h(w,w) = \big(\!\big(\langle P(\overline{w},0)z_i, P(\overline{w},0)z_j\rangle\big)\!\big)_{1\leqq i,j\leqq 2}, \quad w \in U \subset \mathbb{D}^2,$$

which is the Hermitian metric for the vector bundle $\mathscr{P}$, on some small open set $U \subseteqq \mathbb{D}^2$ around $(0,0)$, corresponding to the module $H_0^{(\lambda,\mu)}(\mathbb{D}^2)$, somewhat easier.

We will prove the claim by showing that $\big(R_{\boldsymbol{M}}(0)D_{\overline{w}}\big)^n z_i$ consists of terms of degree $n+1$. For this, it is enough to calculate $V_{\boldsymbol{M}}(0)^*(z_1^l z_2^k, 0)$ and $V_{\boldsymbol{M}}(0)^*(0, z_1^l z_2^k)$ for different $l,k \geqq 0$ such that $(l,k) \neq (0,0)$. Calculations similar to that of $V_{\boldsymbol{M}}(0)^*$ show that

$$V_{\boldsymbol{M}}(0)^*(z_1^m,0) = \sqrt{\frac{\lambda+m}{m+1}}\,z_1^{m+1}, \quad V_{\boldsymbol{M}}(0)^*(0,z_2^n) = \sqrt{\frac{\mu+n}{n+1}}\,z_2^{n+1}$$

and

$$V_{\boldsymbol{M}}(0)^*(z_1^m z_2^{n+1},0) = V_{\boldsymbol{M}}(0)^*(0,z_1^{m+1}z_2^n) = \frac{1}{\sqrt{\dfrac{m+1}{\mu+n}+\dfrac{n+1}{\mu+n}}}\,z_1^{m+1}z_2^{n+1}.$$

Recall that $\big(R_{\boldsymbol{M}}(0)D_{\overline{w}}\big)z_i$ is of degree 2. From the equations given above, inductively, we see that $\big(R_{\boldsymbol{M}}(0)D_{\overline{w}}\big)^n z_i$ is of degree $n+1$. Since monomials are orthogonal in $H^{(\lambda,\mu)}(\mathbb{D}^2)$, the proof of claim (4.1) is complete. We then have

$$P(\overline{w},0)z_1 = z_1 + \overline{w}_1\frac{\lambda+1}{2}z_1^2 + \overline{w}_2\frac{\lambda\mu}{\lambda+\mu}z_1z_2 + \sum_{n=2}^{\infty}\big(R_{\boldsymbol{M}}(0)D_{\overline{w}}\big)^n z_1$$

and

$$P(\overline{w},0)z_2 = z_2 + \overline{w}_1\frac{\lambda\mu}{\lambda+\mu}z_1z_2 + \overline{w}_2\frac{\mu+1}{2}z_2^2 + \sum_{n=2}^{\infty}\big(R_{\boldsymbol{M}}(0)D_{\overline{w}}\big)^n z_2.$$

Putting all of this together, we see that

$$h(w,w) = \begin{pmatrix}\lambda & 0\\ 0 & \mu\end{pmatrix} + \sum a_{IJ}w^I\overline{w}^J,$$

where the sum is over all multi-indices I, J satisfying $|I|,|J| > 0$ and $w^I = w_1^{i_1}w_2^{i_2}$, $\overline{w}^J = \overline{w}_1^{j_1}\overline{w}_2^{j_2}$. The metric h is (almost) normalized at $(0,0)$, that is, $h(w,0) = \begin{pmatrix}\lambda & 0\\ 0 & \mu\end{pmatrix}$. The metric h_0 obtained by conjugating the metric h by the invertible (constant) linear transformation $\begin{pmatrix} \sqrt{\lambda} & 0 \\ 0 & \sqrt{\mu} \end{pmatrix}$ induces an equivalence of holomorphic Hermitian bundles. The vector bundle $\mathscr{P}$ equipped with the Hermitian metric h_0 has the additional property that the metric is normalized: $h_0(w,0) = I$. The coefficient of $dw_i \wedge d\overline{w}_j$, $i,j = 1,2$, in the curvature of the holomorphic Hermitian bundle $\mathscr{P}$ at $(0,0)$ is then the Taylor coefficient of $w_i\overline{w}_j$ in the expansion of h_0 around $(0,0)$ (cf. [35], Lemma 2.3).

Thus the normalized metric $h_0(w,w)$, which is real analytic, is of the form

$$
\begin{aligned}
h_0(w,w) &= \begin{pmatrix} \lambda\langle P(\overline{w},0)z_1, P(\overline{w},0)z_1\rangle & \sqrt{\lambda\mu}\langle P(\overline{w},0)z_1, P(\overline{w},0)z_2\rangle \\ \sqrt{\lambda\mu}\langle P(\overline{w},0)z_2, P(\overline{w},0)z_1\rangle & \mu\langle P(\overline{w},0)z_2, P(\overline{w},0)z_2\rangle \end{pmatrix} \\
&= I + \begin{pmatrix} \frac{\lambda+1}{2}|w_1|^2 + \frac{\lambda^2\mu}{(\lambda+\mu)^2}|w_2|^2 & \frac{1}{\sqrt{\lambda\mu}}\left(\frac{\lambda\mu}{\lambda+\mu}\right)^2 w_1\overline{w}_2 \\ \frac{1}{\sqrt{\lambda\mu}}\left(\frac{\lambda\mu}{\lambda+\mu}\right)^2 w_2\overline{w}_1 & \frac{\lambda\mu^2}{(\lambda+\mu)^2}|w_1|^2 + \frac{\mu+1}{2}|w_2|^2 \end{pmatrix} + O(|w|^3),
\end{aligned}
$$

where $O(|w|^3)_{i,j}$ is of degree $\geqq 3$. Explicitly, it is of the form

$$
\sum_{n=2}^{\infty}\left\langle \left(R_{\boldsymbol{M}}(0)D_{\overline{w}}\right)^n z_i, \left(R_{\boldsymbol{M}}(0)D_{\overline{w}}\right)^n z_j\right\rangle.
$$

The curvature at $(0,0)$, as pointed out earlier, is given by $\bar{\partial}\partial h_0(0,0)$. Consequently, if $H_0^{(\lambda,\mu)}(\mathbb{D}^2)$ and $H_0^{(\lambda',\mu')}(\mathbb{D}^2)$ are equivalent, then the corresponding holomorphic Hermitian vector bundles $\mathscr{P}$ and $\tilde{\mathscr{P}}$ of rank 2 must be equivalent. Hence their curvatures, in particular, at $(0,0)$, must be unitarily equivalent. The curvature for $\mathscr{P}$ at $(0,0)$ is given by the 2×2 matrices

$$
\begin{pmatrix} \frac{\lambda+1}{2} & 0 \\ 0 & \frac{\lambda\mu^2}{(\lambda+\mu)^2} \end{pmatrix}, \quad \begin{pmatrix} 0 & \frac{1}{\sqrt{\lambda\mu}}\left(\frac{\lambda\mu}{\lambda+\mu}\right)^2 \\ 0 & 0 \end{pmatrix},
$$

$$
\begin{pmatrix} 0 & 0 \\ \frac{1}{\sqrt{\lambda\mu}}\left(\frac{\lambda\mu}{\lambda+\mu}\right)^2 & 0 \end{pmatrix}, \quad \begin{pmatrix} \frac{\lambda^2\mu}{(\lambda+\mu)^2} & 0 \\ 0 & \frac{\mu+1}{2} \end{pmatrix}.
$$

The curvature for $\tilde{\mathscr{P}}$ has a similar form with λ' and μ' in place of λ and μ respectively. All of them are to be simultaneously equivalent by some unitary map. The only unitary that intertwines the 2×2 matrices

$$
\begin{pmatrix} 0 & \frac{1}{\sqrt{\lambda\mu}}\left(\frac{\lambda\mu}{\lambda+\mu}\right)^2 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & \frac{1}{\sqrt{\lambda'\mu'}}\left(\frac{\lambda'\mu'}{\lambda'+\mu'}\right)^2 \\ 0 & 0 \end{pmatrix}
$$

is aI with $|a| = 1$. Since this fixes the unitary intertwiner, we see that the 2×2 matrices

$$\begin{pmatrix} \frac{\lambda+1}{2} & 0 \\ 0 & \frac{\lambda\mu^2}{(\lambda+\mu)^2} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \frac{\lambda'+1}{2} & 0 \\ 0 & \frac{\lambda'\mu'^2}{(\lambda'+\mu')^2} \end{pmatrix}$$

must be equal. Hence we have $\frac{\lambda+1}{2} = \frac{\lambda+1}{2}$, that is $\lambda = \lambda'$. Consequently,

$$\frac{\lambda\mu^2}{(\lambda+\mu)^2} = \frac{\lambda'\mu'^2}{(\lambda'+\mu')^2} \quad \text{gives} \quad \frac{\mu^2}{(\lambda+\mu)^2} = \frac{\mu'^2}{(\lambda+\mu')^2}$$

and then

$$(\mu - \mu')\{\lambda^2(\mu+\mu') + 2\lambda\mu\mu'\} = 0.$$

We then have $\mu = \mu'$. Therefore, $H_0^{(\lambda,\mu)}(\mathbb{D}^2)$ and $H_0^{(\lambda',\mu')}(\mathbb{D}^2)$ are equivalent if and only if $\lambda = \lambda'$ and $\mu = \mu'$.

We describe below, a second family of examples of Hilbert modules in the class $\mathfrak{B}_1(\mathbb{D}^2)$ which are shown to be inequivalent. For this, we use a somewhat different unitary invariant which is relatively easy to compute.

4.2. The (n,k) examples. For a fixed natural number j, let I_j be the polynomial ideal generated by the set $\{z_1^n, z_1^{k_j} z_2^{n-k_j}\}$, $k_j \neq 0$. Let $\mathcal{M}_j$ be the closure of I_j in the Hardy space $H^2(\mathbb{D}^2)$. We claim that $\mathcal{M}_1$ and $\mathcal{M}_2$ are inequivalent as Hilbert modules unless $k_1 = k_2$. From Lemma 2.3, it follows that both the modules $\mathcal{M}_1$ and $\mathcal{M}_2$ are in $\mathrm{B}_1(\mathbb{D}^2 \backslash X)$, where $X := \{(0,z) : |z| < 1\}$ is the zero set of the ideal I_j, $j = 1, 2$. However, there is a holomorphic Hermitian line bundle corresponding to these modules on the projectivization of $\mathbb{D}^2 \backslash X$ at $(0,0)$ (cf. [14], p. 264). Following the proof of [14], Theorem 5.1, we see that if these modules are assumed to be equivalent, then the corresponding line bundles they determine must also be equivalent. This leads to contradiction unless $k_1 \neq k_2$.

Suppose $L : \mathcal{M}_1 \to \mathcal{M}_2$ is given to be a unitary module map. Let K_j, $j = 1, 2$, be the corresponding reproducing kernel. By our assumption, the localizations of the modules, $M_j(w)$ at the point $w \in \mathbb{D}^2 \backslash X$ are one dimensional and spanned by the corresponding reproducing kernel K_j, $j = 1, 2$. Since L intertwines module actions, it follows that $M_f^* L K_1(\cdot, w) = \overline{f(w)} L K_1(\cdot, w)$. Hence,

$$LK_1(\cdot, w) = \overline{g(w)} K_2(\cdot, w), \quad \text{for } w \notin X. \tag{4.2}$$

We conclude that g must be holomorphic on $\mathbb{D}^2 \backslash X$ since both $LK_1(\cdot, w)$ and $K_2(\cdot, w)$ are anti-holomorphic in w. For $j = 1, 2$, let E_j be the holomorphic line bundle on $\mathbb{P}^1$ whose section on the affine chart $U = \{w_1 \neq 0\}$ is given by

$$\begin{aligned} s_j(\theta) = \lim_{w \to 0, \frac{\bar w_2}{\bar w_1} = \theta} \frac{K_j(z,w)}{\bar w_1^n} &= \frac{z_1^n \bar w_1^n + z_1^{k_j} z_2^{n-k_j} \bar w_1^{k_j} \bar w_2^{n-k_j} + \text{higher order terms}}{\bar w_1^n} \\ &= z_1^n + \theta^{n-k_j} z_1^{k_j} z_2^{n-k_j}. \end{aligned}$$

Consider the co-ordinate change $(w_1, w_2) \to (\rho, \theta)$ where $\bar{w}_1 = \rho$ and $\bar{w}_2 = \rho\theta$ on $\mathbb{D}^2 \backslash X$. Note that

$$\lim_{\frac{\bar{w}_2}{\bar{w}_1} = \theta,\, w \to 0} |g(\rho,\theta)|^2 = \frac{1 + |\theta^{n-k_1}|^2}{1 + |\theta^{n-k_2}|^2}. \tag{4.3}$$

$g(\rho,\theta)$ has a finite limit at $(0,\theta)$, say $g(\theta)$. Then from (4.2), and the expression of $s_j(\theta)$, by a limiting argument, we find that $Ls_1(\theta) = g(\theta)s_2(\theta)$. The unitarity of the map L implies that

$$\|Ls_1(\theta)\|^2 = |g(\theta)|^2 \|s_2(\theta)\|^2$$

and consequently the bundles E_j determined by $\mathscr{M}_j$, $j = 1, 2$, on $\mathbb{P}^1$ are equivalent. We now calculate the curvature to determine when these line bundles are equivalent. Since the monomials are orthonormal, we note that the square norm of the section is given by

$$\|s_1(\theta)\|^2 = 1 + |\theta|^{2(n-k_j)}.$$

Consequently the curvature (actually coefficient of the $(1,1)$ form $d\theta \wedge d\bar{\theta}$) of the line bundle on the affine chart U is given by

$$\begin{aligned}
\mathscr{K}_j(\theta) &= -\partial_\theta \partial_{\bar{\theta}} \log \|s_1(\theta)\|^2 = -\partial_\theta \partial_{\bar{\theta}} \log(1 + |\theta|^{2(n-k_j)}) \\
&= -\partial_\theta \frac{(n-k_j)\theta^{(n-k_j)}\bar{\theta}^{(n-k_j-1)}}{1 + |\theta|^{2(n-k_j)}} \\
&= -\frac{(n-k_j)^2|\theta|^{2(n-k_j-1)}\{1 + |\theta|^{2(n-k_j)}\} - (n-k_j)^2|\theta|^{2(n-k_j)}|\theta|^{2(n-k_j-1)}}{\{1 + |\theta|^{2(n-k_j)}\}^2} \\
&= -\frac{(n-k_j)^2|\theta|^{2(n-k_j-1)}}{\{1 + |\theta|^{2(n-k_j)}\}^2}.
\end{aligned}$$

So if the bundles are equivalent on $\mathbb{P}^1$, then $\mathscr{K}_1(\theta) = \mathscr{K}_2(\theta)$ for $\theta \in U$, and we obtain

$$\begin{aligned}
&(n-k_1)^2\{|\theta|^{2(n-k_1-1)} + 2|\theta|^{2(n-k_2)}|\theta|^{2(n-k_1-1)} + |\theta|^{4(n-k_2)}|\theta|^{2(n-k_1-1)}\} \\
&- (n-k_2)^2\{|\theta|^{2(n-k_2-1)} + 2|\theta|^{2(n-k_1)}|\theta|^{2(n-k_2-1)} + |\theta|^{4(n-k_1)}|\theta|^{2(n-k_2-1)}\} = 0.
\end{aligned}$$

Since the equation given above must be satisfied by all θ corresponding to the affine chart U, it must be an identity. In particular, the coefficient of $|\theta|^{2\{(n-k_1)+(n-k_2)-1\}}$ must be 0 implying $(n-k_1)^2 = (n-k_2)^2$, that is, $k_1 = k_2$. Hence $\mathscr{M}_1$ and $\mathscr{M}_2$ are always inequivalent unless they are equal.

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