A REMARK ON A CONJECTURE OF C. L. SIEGEL

BY S. CHOWLA

DEPARTMENT OF MATHEMATICS, PENNSYLVANIA STATE UNIVERSITY

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In his beautiful memoir "Über einige Anwendungen diophantischer Approximationen" [*Abhandlungen der Preussichen Akademie der Wissenschaften, Phys. Math. Klasse*, 1–70 (1929)], Siegel made the following conjecture: Let $f(x,y) = 0$ be the equation of an algebraic curve of genus $p > 0$, whose coefficients belong to an algebraic number field K. "Die Untersuchungen der vorangehenden Paragraphen geben die Möglichkeit, eine Schranke für die Anzahl der Lösungen der diophantischen Gleichung $f(x,y) = 0$ als Funktion der Koeffizienten von f explicit aufzustellen, falls diese Gleichung nur endlich viele Lösungen besitzt. *Man kann nun vermuten, dass sich sogar eine Schranke finden lässt, die nur von der Anzahl der Koeffizienten abhängt;* doch dürfte dies recht schwer zu beweisen sein. Eine Stütze für diese Vermutung bilden die im folgenden entwickelten, allerdings sehr speziellen Resultate." Serge Lang in his paper, "Some theorems and conjectures in diophantine equations" [*Bull. Amer. Math. Soc.*, **66**, 240–249 (1960)], draws attention to this "remarkable conjecture of Siegel" (p. 247).

As supports for the conjecture, Siegel mentions the results of Delaunay and Nagell that a cubic form with coefficients in Z and negative discriminant assumes the value 1 at most 5 times. He proceeds to prove that the equation[1] $\phi(x,y) = k$ has at most 18 solutions, if the *positive* discriminant d exceeds a certain limit depending on k (in fact, if $d > \gamma_7\, k^{33}$, $d \geq \gamma_3^2\, k^4$ where γ_3, γ_7 are certain absolute constants).

We remark, however, that the conjecture is, as it stands, false. For consider the equation $x^3 + y^3 = n$. I proved (Jubilee Commemoration Volume of the *J. Indian Math. Soc.*, vol. 20, 1934) that there are infinitely many n, and an absolute constant $c > 0$ such that the above equation has more than $c \log \log n$ solutions in *positive* integers. On the other hand the number of solutions is clearly finite. This example contradicts Siegel's conjecture. That the conjecture is false can also be seen directly by starting say from the equation $x^3 - y^3 = 7$ which has the solution

$x = 2, y = 1$. Using the tangent-chord process (of Euler) we obtain a sequence of rational solutions $x = p_m/q_m$, $y = r_m/s_m$ $(1 \leq m \leq t)$ of $x^3 - y^3 = 7$, where t is as large as we please. Then $7 \prod_{m=1}^{t} (q_m, s_m)^3$ has at least t representations by the form $x^3 - y^3$. The notation (a, b) means the least common multiple of a and b.

[1] Here $\phi(x, y)$ is a cubic form (coefficients in Z).