ON A CONJECTURE OF MARSHALL HALL

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Let p be a prime ≥ 7 such that the class-number of the imaginary quadratic field $Q(\sqrt{-p})$ is 1. At a seminar on complex multiplication at the Institute for Advanced Study (1957–58),[1] I pointed out that the following theorem is true:

If p is a prime with the property stated above, then the equation

$$x^3 - py^2 = -1728 \tag{1}$$

has a solution with

$$x = \{e^{\pi\sqrt{p}/3}\}, \tag{2}$$

where $\{w\}$ *denotes the integer nearest to* w.

As a matter of fact, this theorem can be read off from Weber's classical *Lehrbuch der Algebra*, volume 3.

In a recent paper (by Birch, Chowla, Hall, and Schinzel[2]), we cited M. Hall's conjecture that if x, y are integers, then

$$|x^3 - y^2| \geq \sqrt{|x|} \ (|x| > x_0), \tag{3}$$

provided $x^3 \neq y^2$.

We wish to point out a connection between our theorem and Hall's conjecture. It is well known that it is still an unsolved problem (P) to find a *determinable* constant c such that $h > 1$ for $p > c$. Here h is the class-number of the imaginary quadratic field $Q(\sqrt{-p})$.

Set $x = p^{-1}u$, $y = p^{-2}v$ in (2). Then (2) becomes

$$u^3 - v^2 = -1728\, p^3. \tag{4}$$

Thus if the class-number h of $Q(\sqrt{-p})$ is 1, we conclude that (4) has a solution in integers u, v and further with

$$u = p\,\{e^{\pi\sqrt{p}/3}\}. \tag{5}$$

Now (4) and (5) contradict (3) for large p. Thus we see that a proof of Hall's conjecture would immediately, in conjunction with our observation, lead to a new proof of the theorem (conjectured by Gauss, proved by Heilbronn) that $h > 1$ for $p > c$. Our theorem also shows that if x_0 in Hall's conjecture is a *determinable* constant, then the c above is also a *determinable* constant, i.e., problem P would then be solved.

If the positive integers x and y tend to infinity through values for which $x^3 \neq y^2$, it is known that

$$|x^3 - y^2| \to \infty. \tag{6}$$

Our method shows, for example, that if we can replace (6) by

$$|x^3 - y^2| > c \log^7 x, \tag{7}$$

where c is a computable positive constant, then the unsolved problem P would be solved.

[1] Professor A. Borel has informed me that the seminar notes will be published by Springer-Verlag.

[2] Birch, B. J., S. Chowla, M. Hall, and A. Schinzel, "On the difference $x^3 - y^2$," *Kgl. Norske Videnskab. Selskabs*, **38**, 65–69 (1965). See also a paper by Davenport in the same volume.