ON THE CLASS NUMBER OF REAL QUADRATIC FIELDS

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THEOREM.—Let \( p \) be a prime \( \equiv 1 \pmod{4} \), \( h \) the class number of the real quadratic field \( \mathbb{Q}(\sqrt{p}) \), \( \epsilon = \left[ \frac{(t + u \sqrt{p})/2}{2} \right] > 1 \) its fundamental unit. Then,

\[
\left( \frac{p - 1}{2} \right)_! \equiv (-1)^{(p+1)/2} \frac{p}{2} \pmod{p}.
\]

(1)

Proof: From Dirichlet’s classical formula we derive

\[
\sqrt{p} \; \epsilon^h = \prod_{\mathbb{Z}} \left( 1 - \theta^n \right),
\]

(2)

where \( \theta = e^{2\pi i/p} \) and \( n \) runs over the numbers with \( 0 < n < p \) and \( (n/p) = -1 \), where \( (x/p) \) is Legendre’s symbol of quadratic residuacity. Working with integers of \( \mathcal{R}(\theta) \), we note

\[
\sqrt{p} = \left( \frac{p - 1}{2} \right)_! \left( 1 - \theta^{(p-1)/2} \right) \pmod{1 - \theta^{(p+1)/2}},
\]

(3)

\[
\prod_{\mathbb{Z}} \left( 1 - \theta^n \right) = (1 - \theta^{(p-1)/2}) \pmod{1 - \theta^{(p+1)/2}}.
\]

(4)

(1) follows from (2), (3), and (4). Our result supplements formulae of Ankeny- Artin-Chowla in *Ann. Math.*, 1952, p. 479.

I am grateful to Professor A. Selberg for noticing an error in my original argument. Paromita Chowla has checked the formula in many special cases. See L. J. Mordell, *Amer. Math. Monthly* (Feb. 1961) for a similar result for primes \( p \equiv 3 \pmod{4} \).