Proc. Nat. Acad. Sci. USA
Vol. 69, No. 12, pp. 3745, December 1972

Problems on Periodic Simple Continued Fractions

P. CHOWLA AND S. CHOWLA

Institute for Advanced Study, Princeton, New Jersey 08540; and Pennsylvania State University, University Park, Pa. 16802

Communicated by Deane Montgomery, September 29, 1972

ABSTRACT **Let N be a positive non-square integer and $a_1,a_2,\ldots,a_s$ be the partial denominators in the period of length $s = s(N)$ of the continued fraction for $\sqrt{N}$. Also let**

$$\Sigma_N = a_s - a_{s-1} + - \ldots \pm a_1,$$

and let $h(d)$ be the class-number of $Q(\sqrt{d})$. Hirzebruch (unpublished) recently found the surprising theorem (which is a special case of more general results): If p is a prime $\equiv 3(4)$ and $p > 3$, then $h(p) = 1$ implies that $\Sigma_p = 3h(-p)$.

This result led to related conjectures presented herein.

Let N be a positive non-square integer and $a_1,a_2,\ldots,a_s$ be the partial denominators in the period of length $s = s(N)$ of the continued fraction for $\sqrt{N}$. Also let

$$\Sigma_N = a_s - a_{s-1} + - \ldots \pm a_1,$$

and let $h(d)$ be the class-number of $Q(\sqrt{d})$. Hirzebruch (unpublished) recently found the surprising theorem (which is a special case of more general results): If p is a prime $\equiv 3(4)$ and $p > 3$, then $h(p) = 1$ implies that $\Sigma_p = 3h(-p)$.

This result led us to the following conjectures:

(I) If $N \equiv 3(\text{mod } 4)$, $3 \nmid N$, then $\Sigma_N > 0$.

(II) If $N \equiv 3(\text{mod } 4)$, $3 \nmid N$, then $\Sigma_N \equiv 0(\text{mod } 3)$.

[Conjecture *II* has been proved by A. Schinzel in a recent letter (unpublished) to one of the authors.]

(III) If p is a prime $\equiv 3(4)$, $p > 3$ and if $h(2p) = 1$, then $\Sigma_{2p} = 6h(-p)$.

(IV) If p is a prime $\equiv 3(\text{mod } 4)$, $p > 3$, and if $h(p) = h(2p) = 1$ then $\Sigma_{2p} = 2\Sigma_p$. IV would follow by combining *III* with Hirzebruch's result.

(V) There are infinitely many primes p for which $h(p) = h(2p) = 1$.

[We note that (*IV*) is true for $p = 4519$.]

(VI) Let p be a prime $\equiv 3(\text{mod } 4)$ and q a prime $\equiv 5(\text{mod } 8)$, then the Legendre symbol

$$\left(\frac{p}{q}\right) = (-1)^{\Sigma_N} \quad [N = pq].$$

(VII) If k is any given positive integer, there are infinitely many non-square positive integers N such that $s = s(N) = k$. For "non-square" we can substitute "primes."