A CONGRUENCE PROPERTY OF RAMANUJAN'S FUNCTION $\tau(n)$

R. P. BAMBAH, S. CHOWLA, AND H. GUPTA

We have recently proved[1] that

$$\tau(n) \equiv \sigma(n) \pmod{8} \tag{1}$$

if $(n, 2) = 1$, where

$$\sum_{1}^{\infty} \tau(n) x^n = x \prod_{1}^{\infty} (1 - x^n)^{24} \qquad (|x| < 1),$$

$$\sigma(n) = \sum_{d|n} d.$$

We give, in this note, a new and short proof of (1).

In the following we write J for any integral power series with integral coefficients.

We have $(1-x)^2 = (1-x^2) + 2J$. Therefore

$$(1 - x)^8 = (1 - x^2)^4 + 8J.$$

Hence

$$\begin{aligned} \sum_{1}^{\infty} \tau(n) x^n &= x\{(1 - x^2)(1 - x^4)(1 - x^6) \cdots \}^{12} + 8J \\ &= x\left\{\sum_{u=0}^{\infty} (-1)^u (2u + 1) x^{u(u+1)}\right\}^4 + 8J \\ &= x\left\{\sum_{u=0}^{\infty} x^{u(u+1)} + 4J\right\}^4 + 8J \\ &= x\left\{\sum_{0}^{\infty} x^{u(u+1)}\right\}^4 + 8J. \end{aligned} \tag{2}$$

But[2]

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[1] Jointly with D. B. Lahiri (Calcutta) in a paper sent to the Quarterly Journal of Mathematics, Oxford.

[2] Whitaker and Watson, *Modern analysis*, 4th ed., p. 479, Example 3; and *Ramanujan* by Hardy, p. 132, for $(1+2x+2x^4+2x^9+\cdots)^4 = 1 + \sum r_4(n)x^n$ and $r_4(n) = 8\sigma^0(n)$ where $\sigma^0(n) = \sigma(n)$ when n is odd and $\sigma^0(n) = 3$ (the sum of the odd divisors of n) when n is even, that is, $\sigma^0(n) = \sigma(n) - 4\sigma(n/4)$ for all n.

$$
\begin{aligned}
16x\left\{\sum_{0}^{\infty} x^{u(u+1)}\right\}^4 &= 16x(1 + x^2 + x^6 + x^{12} + \cdots)^4 \\
&= (1 + 2x + 2x^4 + 2x^9 + \cdots)^4 \\
&\quad - (1 - 2x + 2x^4 - 2x^9 + \cdots)^4 \\
&= 8\sum_{1}^{\infty} \{\sigma(n) - 4\sigma(n/4)\} x^n \qquad (3) \\
&\quad - 8\sum_{1}^{\infty} \{\sigma(n) - 4\sigma(n/4)\}(-1)^n x^n \\
&= 16\sum_{n=1}^{\infty} \sigma(n) x^n \qquad (n \text{ is odd}).
\end{aligned}
$$

Therefore, from (2) and (3), we have

$$\sum_{n=1}^{\infty} \tau(n) x^n = \sum_{n=1}^{\infty} \sigma(n) x^n + 8J \qquad (n, 2) = 1.$$

From the above, comparing coefficients, we have if $(n, 2) = 1$

$$\tau(n) \equiv \sigma(n) \pmod 8$$

and if $2 \mid n$

$$\tau(n) \equiv 0 \pmod 8.$$

The University of the Panjab, Lahore,
Government College, Lahore, and
Government College, Hoshiarpur