CONGRUENCE PROPERTIES OF RAMANUJAN'S FUNCTION $\tau(n)$

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Introduction. With Ramanujan we define $\tau(n)$ by

$$\sum_1^\infty \tau(n)x^n = x\prod_1^\infty (1-x^n)^{24} \qquad (|x|<1).$$

Write $\sigma_k(n)$ for the sum of the kth powers of the divisors of n; $\sigma(n)=\sigma_1(n)$. It is known that[1]

$$\tau(n) \equiv n\sigma(n) \pmod 5,$$

$$\tau(n) \equiv \sigma(n) \pmod 3 \qquad \text{if } (n, 3) = 1.$$

The object of this note is to give proofs of the much stronger results:

(A) $$\tau(n) \equiv 5n^2\sigma_7(n) - 4n\sigma_9(n) \pmod{5^3}$$

when n is prime to 5;

(B) $$\tau(n) \equiv (n^2+k)\sigma_7(n) \pmod{3^4}$$

when n is prime to 3 and where $k=0$ if $n\equiv 1(3)$, $k=9$ if $n\equiv 2(3)$.

1. Some lemmas.

LEMMA 1. *We have*

$$\sum u\sigma_3(u)\sigma_5(v) \equiv \sum \sigma(u)\sigma(v) - P(n) \pmod 5$$

where

$$P(n) = \sum_{u\equiv 0 \pmod 5} \sigma(u)\sigma(v)$$

where $u+v=n$; $u, v\geq 1$ *in all three sums* ($\sum$).

PROOF. We have

(1) $$u\sigma_3(u)\sigma_5(v) \equiv 0 \pmod 5 \quad \text{when} \quad u\equiv 0(5);$$

when $(u, 5)=1$ we have

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[1] The first of these is proved in Hardy's *Ramanujan* (Cambridge, 1940); the second by Gupta in J. Indian Math. Soc. vol. 9 (1945) pp. 59–60. In what follows we refer to Ramanujan's *Collected papers* (Cambridge, 1927) by the letters RCP. We have also proved that $\tau(n)\equiv\sigma_{11}(n) \pmod{2^8}$ if n is odd; this result has been accepted for publication in J. London Math. Soc.

$$u\sigma_3(u) = u\sum_{d|u} d^3 \equiv u\sum_{d|u} d^{-1} = \sum_{d|u} \frac{u}{d} = \sigma(u),$$

so that

(2) $$u\sigma_3(u) \equiv \sigma(u) \pmod 5 \qquad \text{when } (u, 5) = 1.$$

Similarly

(3) $$\sigma_5(v) \equiv \sigma(v) \pmod 5.$$

From (1), (2), (3):

$$\sum u\sigma_3(u)\sigma_5(v) \equiv \sum_{(u,5)=1} \sigma(u)\sigma(v) \equiv \sum \sigma(u)\sigma(v) - P(n) \pmod 5.$$

LEMMA 2. *If* $(n, 5) = 1$ *we have*

$$\sum uv\sigma_3(u)\sigma_3(v) \equiv \sum \sigma(u)\sigma(v) - 2P(n) \pmod 5$$

where, as in Lemma 1, *the conditions*

$$u + v = n, \qquad u, v \geqq 1$$

are understood in every $\sum$.

PROOF. If u or $v \equiv 0(5)$, $uv\sigma_3(u)\sigma_3(v) \equiv 0(5)$. From this and (1) we get since $(n, 5) = 1$,

$$\sum uv\sigma_3(u)\sigma_3(v) \equiv \sum_{(u,5)=1,(v,5)=1} \sigma(u)\sigma(v)$$
$$\equiv \sum \sigma(u)\sigma(v) - 2P(n) \pmod 5,$$

the desired result.

2. **Proof of (A).** Write, for x numerically less than unity,

$$P = 1 - 24\sum_1^\infty \sigma(n)x^n,$$

$$Q = 1 + 240\sum_1^\infty \sigma_3(n)x^n,$$

$$R = 1 - 504\sum_1^\infty \sigma_5(n)x^n.$$

Then (44), p. 144 of RCP, is

(4) $$Q^3 - R^2 = 1728\sum_1^\infty \tau(n)x^n$$

and we deduce from relations 5 and 2, Table II, p. 142 of RCP, that

$$1584\sum_1^\infty n\sigma_9(n)x^n$$
$$= 3(Q^3 - R^2) - 5R(PQ - R)$$
$$= 5184\sum_1^\infty \tau(n)x^n - 5\left(1 - 504\sum_1^\infty \sigma_5(n)x^n\right)(PQ - R)$$
$$= 5184\sum_1^\infty \tau(n)x^n - 5\left(1 - 504\sum_1^\infty \sigma_5(n)x^n\right)720\sum_1^\infty n\sigma_3(n)x^n.$$

Comparing coefficients of x^n and using Lemma 1 we have

$$(5)\qquad 1584n\sigma_9(n) = 5184\tau(n) - 3600n\sigma_3(n) + 5\cdot 504\cdot 720\sum u\sigma_3(u)\sigma_5(v)$$

(where $u+v=n$ ($u, v \geqq 1$) in the $\sum$ sum),

$$(6)\qquad \begin{aligned} 84n\sigma_9(n) &\equiv 59\tau(n) + 25n\sigma_3(n) \\ &\quad + 25\sum\sigma(u)\sigma(v) - 25P(n) \pmod{125}. \end{aligned}$$

Again, relations 4, Table III, and 2, Table II, p. 142 of RCP give us

$$(6')\qquad \begin{aligned} 8640\sum_1^\infty n^2\sigma_7(n)x^n &= 5(Q^3 - R^2) + 9(PQ - R)^2 \\ &= 8640\sum_1^\infty \tau(n)x^n + 9\cdot 720^2\left\{\sum_1^\infty n\sigma_3(n)x^n\right\}^2. \end{aligned}$$

Comparing the coefficients of x^n here we get

$$(7)\qquad n^2\sigma_7(n) = \tau(n) + 135\cdot 4\sum u\sigma_3(u)v\sigma_3(v)$$

or

$$(7')\qquad 15n^2\sigma_7(n) \equiv 15\tau(n) - 25\sum uv\sigma_3(u)\sigma_3(v) \pmod{125}.$$

From (7′) and Lemma 2 we get

$$(8)\qquad 15n^2\sigma_7(n) \equiv 15\tau(n) - 25\sum\sigma(u)\sigma(v) + 50P(n) \pmod{125}.$$

Eliminating $P(n)$ from (6) and (8) we get

$$168n\sigma_9(n) + 15n^2\sigma_7(n) \equiv 133\tau(n) + 50n\sigma_3(n) + 25\sum\sigma(u)\sigma(v) \pmod{125},$$

or

$$(9)\qquad \begin{aligned} 8\tau(n) &\equiv 43n\sigma_9(n) + 15n^2\sigma_7(n) - 50n\sigma_3(n) \\ &\quad - 25\sum\sigma(u)\sigma(v) \pmod{125}. \end{aligned}$$

Again (relation 1, Table IV, p. 146 of RCP)

$$\sum \sigma(u)\sigma(v) = \frac{5\sigma_3(n) - 5n\sigma(n)}{12} - \frac{(n-1)\sigma(n)}{12} \tag{10}$$
$$\equiv 2(n-1)\sigma(n) \pmod 5.$$

From (9) and (10) we obtain

$$8\tau(n) \equiv 43n\sigma_9(n) + 15n^2\sigma_7(n) - 50n\sigma_3(n) - 50(n-1)\sigma(n) \pmod{5^3}.$$

Hence, multiplying by 47,

$$\begin{aligned} \tau(n) &\equiv 21n\sigma_9(n) - 45n^2\sigma_7(n) + 25n\sigma_3(n) \\ &\quad + 25(n-1)\sigma(n) \pmod{5^3} \\ &\equiv 5n^2\sigma_7(n) - 4n\sigma_9(n) \pmod{5^3} \end{aligned} \tag{11}$$

for

$$\begin{aligned} 25n\sigma_9(n) - 50n^2\sigma_7(n) + 25n\sigma_3(n) + 25(n-1)\sigma(n) &= 50\{n\sigma_9(n) - n^2\sigma_7(n)\} \\ &\quad + 25n\{\sigma(n) - \sigma_9(n)\} + 25\{n\sigma_3(n) - \sigma(n)\} \\ &\equiv 0 \pmod{125}, \end{aligned}$$

since the terms inside each set of braces are a multiple of 5 provided $(n, 5)=1$. Thus (A) is proved by (11).

3. **Proof of (B).** We shall need the following results:

$$\sigma(3t+2) \equiv 0\ (3), \tag{12}$$

$$\sigma_3(3t+2) \equiv 0 \pmod 9 \tag{13}$$

where t is 0 or a positive integer. To prove (13) we observe that to every divisor $3m+1$ of $3t+2$, there corresponds another $3n+2 = (3t+2)/(3m+1)$, and

$$(3m+1)^3 + (3n+2)^3 \equiv 0(9);$$

while (12) is proved still more simply. We next prove the following lemma.

Lemma 3. *If $n \equiv 1(3)$, we have*

$$\sum uv\sigma_3(u)\sigma_3(v) \equiv 0(3)$$

where (in the summation $\sum$) $u+v=n$ and $u, v \geqq 1$.

Proof. Since $n\equiv 1(3)$ and $u+v=n$, we have the 3 cases:

$$u \equiv 0(3), \qquad v \equiv 1(3),$$
$$u \equiv 1(3), \qquad v \equiv 0(3),$$
$$u \equiv 2(3), \qquad v \equiv 2(3),$$

so that $uv\sigma_3(u)\sigma_3(v) \equiv 0(3)$ in each case on account of (13). Hence the lemma is proved.

LEMMA 4. *If* $n \equiv 2(3)$, *we have*

$$\sum_{u+v=n,u,v\geqq 1} uv\sigma_3(u)\sigma_2(v) \equiv \frac{\sigma_7(n) - \sigma_3(n)}{120} \pmod 3.$$

PROOF. If $u+v=n$, $n\equiv 2(3)$, we have 3 cases:

(i) $\quad u \equiv 0(3), \qquad v \equiv 2(3),$

(ii) $\quad u \equiv 2(3), \qquad v \equiv 0(3),$

(iii) $\quad u \equiv 1(3), \qquad v \equiv 1(3).$

In the first two cases

$$uv\sigma_3(u)\sigma_3(v) \equiv 0 \pmod 3;$$

while in the third case

$$uv\sigma_3(u)\sigma_3(v) \equiv \sigma_3(u)\sigma_3(v) \pmod 3.$$

Hence we have (in the sums $u+v=n$; u, $v\geqq 1$), using (13),

$$\sum uv\sigma_3(u)\sigma_3(v) \equiv \sum_{u,v\equiv 1(3)} \sigma_3(u)\sigma_3(v) \pmod 3$$
$$\equiv \sum \sigma_3(u)\sigma_3(v) \equiv \frac{\sigma_7(n) - \sigma_3(n)}{120} \pmod 3$$

since (relation 3, Table IV of RCP, p. 146)

$$\sum \sigma_3(u)\sigma_3(v) = \frac{\sigma_7(n) - \sigma_3(n)}{120} \tag{14}$$

where, in the $\sum$, $u+v=n$ $(u, v\geqq 1)$.

We are now ready to prove (B). Comparing the coefficients of x^n in (6′) we obtain

$$27.320n^2\sigma_7(n) = 27.320\tau(n) + 27^2\cdot 80^2 \sum_{u+v=n,u,v\geqq 1} uv\sigma_3(u)\sigma_3(v). \tag{15}$$

We, therefore, have

$$\tau(n) \equiv n^2\sigma_7(n) \pmod{3^3}. \tag{16}$$

Case 1. $n \equiv 1(3)$. In this case (15) and Lemma 3 give

$$\tau(n) \equiv n^2\sigma_7(n) \pmod{3^4}. \tag{17}$$

Case 2. $n \equiv 2 \pmod 3$. In this case (15) and Lemma 4 give

$$\tau(n) \equiv n^2\sigma_7(n) - \frac{27\cdot 20}{120}\left\{\sigma_7(n) - \sigma_3(n)\right\} \pmod{3^4}$$

or

$$\begin{aligned} \tau(n) &\equiv (n^2 + 36)\sigma_7(n) \pmod{3^4} \\ &\equiv (n^2 + 9)\sigma_7(n) \pmod{3^4} \end{aligned} \tag{18}$$

since, when $n \equiv 2(3)$, we have

$$\begin{aligned} \sigma_7(n) &\equiv \sigma(n) \equiv 0(3), \\ \sigma_3(n) &\equiv 0(9), \end{aligned}$$

from (12) and (13).

(17) and (18) together give (B).

Mordell proved Ramanijan's conjecture

$$\tau(mn) = \tau(m)\tau(n) \qquad \text{if } (m, n) = 1.$$

From this result or directly we can prove that

$$\tau(n) \equiv 16n\sigma_9(n) \pmod{5^3} \qquad \text{if } n \equiv 0(5), \tag{C}$$

$$\tau(n) \equiv n^2\sigma_7(n) \pmod{3^4} \qquad \text{if } n \equiv 0(3). \tag{D}$$

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