AN EXTENSION OF HEILBRONN'S CLASS-NUMBER THEOREM

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LET $h(d)$ denote the number of primitive classes of binary quadratic forms of negative discriminant d. Heilbronn* has recently proved that

THEOREM I. $\quad h(d) \to \infty \quad as \quad -d \to \infty.$

By a slight modification of Heilbronn's argument I show that

THEOREM II. $\quad \dfrac{h(d)}{2^t} \to \infty \quad as \quad -d \to \infty,$

where t is the number of different prime factors of d.

Both these results were conjectured by Gauss.† Theorem II is equivalent to

THEOREM III. $\quad p(d) \to \infty \quad as \quad -d \to \infty,$

where $p(d)$ is the number of (primitive) classes in the principal genus.

We shall write $\quad h(d) = H, \quad p(d) = P.$

We assume, with Heilbronn, that there is‡ an $m > 0$ and a character $\chi(n)$ $(\operatorname{mod} m)$ such that

$$L_0(s) = L(s, \chi) = \sum_{n=1}^{\infty} \chi(n) n^{-s}$$

vanishes for $s = \rho$, where

$$\rho = \theta + i\phi \qquad (\tfrac{1}{2} < \theta < 1). \tag{1}$$

We follow the notation of Heilbronn except that the constants implied in our O-symbols are independent of H and t.

* See above, pp. 150–60. This paper is hereafter referred to as 'Heilbronn'. I am very much indebted to Dr. Heilbronn for an advance copy of his manuscript.

† *Disquisitiones Arithmeticae* (1801), Art. 303.

‡ If there is no such m, then, by a theorem of Hecke, we have

$$h(d) > \frac{c\sqrt{|d|}}{\log|d|} \qquad (d < -1);$$

so that Theorem II is then certainly true.

LEMMA I. *If* $a > 1, \quad (a,d) = 1, \quad a|(x^2-d),$ (2)

and if $a_s x^2 + b_s xy + c_s y^2 \quad (1 \leqslant s \leqslant P)$ (3)

are the P classes in the principal genus, then there is an s $(1 \leqslant s \leqslant P)$ such that

$$a^{2P} = a_s x^2 + b_s xy + c_s y^2 \qquad (y \neq 0). \tag{4}$$

Proof. Now a^{2P} can only be represented by the P classes of the principal genus and not by any of the other $H-P$ classes. The number of representations of a^{2P} by these P forms is, by a well-known theorem,* not less than

$$2(2P+1) = 4P+2.$$

Now $a_s x^2 + b_s xy + c_s y^2$ can represent a^{2P} with $y = 0$ in at most two ways. Hence the P classes (3) can represent a^{2P} in at most $2P$ ways with $y = 0$. It follows that a^{2P} must have a representation by

$$a_s x^2 + b_s xy + c_s y^2$$

with $y \neq 0$ for some s in $1 \leqslant s \leqslant P$.

LEMMA II. *If* $(a,d) = 1, \quad a|(x^2-d),$ (5)

then $a^{2P} \geqslant \sqrt{(\tfrac{3}{16}|d|)}.$ (6)

Proof. From Lemma I there is an s $(1 \leqslant s \leqslant P)$ such that

$$a^{2P} = a_s x^2 + b_s xy + c_s y^2 \qquad (y \neq 0)$$

or

$$4a_s a^{2P} = (2a_s x + b_s y)^2 - dy^2 \qquad (y \neq 0). \tag{7}$$

Further $1 \leqslant a_s \leqslant \sqrt{(\tfrac{1}{3}|d|)}.$ (8)

From (7) and (8) we obtain (6).

In what follows, a runs through the minima of the H forms of discriminant d.

LEMMA III. †*If $a|d^k$ for some $k > 0$, then $\mu(a) \neq 0$ and $a|d$.* (9)

LEMMA IV. *If* $a \nmid d,$ (10)

then $|a|^{2P} \geqslant \sqrt{(\tfrac{3}{16}|d|)}.$ (11)

Proof. From (10) and (9), $a \nmid d^k$ for any $k > 0$. Hence $(a_1, d) = 1$ for some a_1 such that $a_1 > 1$. Further, since $a_1|(x^2-d)$, we obtain from Lemma II

$$a_1^{2P} \geqslant \sqrt{(\tfrac{3}{16}|d|)}. \tag{12}$$

(11) follows from (12) since $|a| \geqslant |a_1|$.

* See, for example, Satz 204 of Landau, *Vorlesungen über Zahlentheorie.*

† Lemma XI of Heilbronn.

LEMMA V.* *If* $A > 0, \quad A|d, \quad \mu(A) \neq 0,$
then there is at most one form of discriminant d with minimum A.

LEMMA VI.† *If under the assumptions of Lemma V,*

$$A \leqslant \sqrt{(\tfrac{1}{3}|d|)},$$

then there is at least one form of discriminant d with minimum A.

LEMMA VII. *For* $\sigma > \frac{1}{2}$,

$$\Bigl|\sum_a \chi(a)a^{-s}\Bigr| \geqslant \frac{1}{4H^2}+O(H|d|^{-\sigma/4P}). \tag{13}$$

Proof. By (9),

$$\Bigl|\sum_a \chi(a)a^{-s}-\sum_{n|d}\chi(n)\mu^2(n)n^{-s}\Bigr| = \Bigl|\sum_{a\nmid d}\chi(a)a^{-s}-\sum_{\substack{n|d\\ n\neq a}}\chi(n)\mu^2(n)n^{-s}\Bigr|. \tag{14}$$

Now by (10) and (11),

$$\sum_{\substack{a\\ a\nmid d}}\chi(a)a^{-s} = O(H|d|^{-\sigma/4P}). \tag{15}$$

By Lemmas V and VI,

$$\begin{aligned}\sum_{\substack{n|d\\ n\neq a}}\chi(n)\mu^2(n)n^{-s} &= O(2^t|d|^{-\frac{1}{4}\sigma})\\ &= O(H|d|^{-\frac{1}{4}\sigma}).\end{aligned} \tag{16}$$

Further, as in the proof of Lemma XV of Heilbronn,

$$\Bigl|\sum_{n|d}\chi(n)\mu^2(n)n^{-s}\Bigr| \geqslant \frac{1}{4H^2}. \tag{17}$$

Then (13) follows from (14), (15), (16), (17).

LEMMA VIII. *For* $\sigma_m < \sigma < 2$, $s \neq 1$,

$$L_0(s)L_2(s) = \zeta(2s)\prod_{p|m}(1-p^{-2s})\sum_a \chi(a)a^{-s}+O\Bigl\{\Bigl(|s|+\frac{1}{|s-1|}\Bigr)H|d|^{\frac{1}{2}-\frac{1}{4}\sigma}\Bigr\}. \tag{18}$$

Proof. When $\sigma_m < \sigma < 2$, $s \neq 1$ we have for $\phi(s)$ the expression on the top of page 156 of Heilbronn,

$$\phi(s) = O\Bigl\{\Bigl(|s|+\frac{1}{|s-1|}\Bigr)(|a^\sigma d^{-\sigma}|+|a^{\sigma-1}d^{\frac{1}{2}-\sigma}|)\Bigr\}. \tag{19}$$

Now
$$|a^\sigma d^{-\sigma}| = \Bigl|\frac{a^2}{d}\Bigr|^{\frac{1}{2}\sigma}|d|^{-\frac{1}{2}\sigma}, \tag{20}$$

$$|a^{\sigma-1}d^{\frac{1}{2}-\sigma}| = a^{-\frac{1}{2}}\Bigl|\frac{a^2}{d}\Bigr|^{\frac{1}{2}\sigma-\frac{1}{4}}|d|^{\frac{1}{4}-\frac{1}{2}\sigma}, \tag{21}$$

$$a \geqslant 1, \qquad 3a^2 \leqslant |d|. \tag{22}$$

* Lemma XII of Heilbronn. † Lemma XIII of Heilbronn.

Then (18) follows from (19), (20), (21), (22), and the proof of Lemma X in Heilbronn.

Proof of the main result.

We put $s = \rho$ in Lemma VIII. Then we get

$$0 = \zeta(2\rho)\prod_{p|m}(1-p^{-2\rho})\sum_a \chi(a)a^{-\rho}+O(H|d|^{\frac{1}{4}-\frac{1}{2}\theta}). \tag{23}$$

From Lemma VII,

$$\Big|\sum_a \chi(a)a^{-\rho}\Big| \geqslant \frac{1}{4H^2}+O(H|d|^{-\theta/4P}). \tag{24}$$

Now

$$\zeta(2\rho)\prod_{p|m}(1-p^{-2\rho})$$

is absolutely greater than a positive constant independent of d. Hence, unless we assume

$$P \to \infty \quad \text{as} \quad -d \to \infty, \tag{25}$$

(23) and (24) contradict each other for $-d \to \infty$.* Hence (25) is true. This proves our result for negative discriminants d, where d or $\frac{1}{4}d$ is *quadratfrei*. The result for general $d < 0$ follows now from Lemma I of Heilbronn.

* For $2^{t-1} = H/P = O(|d|^{\epsilon})$ for every $\epsilon > 0$.