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Estimation of parameters of gravitational waves from coalescing binaries

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Abstract. In this paper we deal with the measurement of parameters of the gravitational wave signal emitted by a coalescing binary system of compact stars. We present the results of Monte Carlo simulations carried out for initial LIGO, incorporating the first post-Newtonian corrections to the waveform. Using the parameters so determined, we estimate the direction to the source. We stress the use of the time-of-coalescence rather than the time-of-arrival of the signal to determine the direction of the source. We show that this can considerably reduce the errors in the determination of the direction of the source.

Keywords. Gravitational waves; coalescing binaries; data analysis.

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1. Introduction and summary

A major source for the planned interferometric gravitational wave detectors like LIGO [1] and VIRGO [2] is the radiation emitted by a binary system of compact stars, e.g. neutron stars or black holes, during the last few minutes of its evolution just before the two stars coalesce. The interferometers are expected to have high enough sensitivity to detect sources at cosmological distances. Since the waveform emitted during a binary coalescence, often called the chirp, can be modeled fairly accurately it is possible to enhance the visibility or the 'signal-to-noise' ratio henceforth denoted as SNR, (defined later in the text in (eq. 9)) of the signal in noisy data by employing powerful data analysis techniques such as Weiner or matched filtering [3, 4]. A filter, in the Fourier domain, is a copy of the expected signal weighted by the noise power spectral density. The process of filtering involves obtaining correlations of the detector output with a set of templates which together span the relevant range of the parameters of the waveform. On filtering the detector output for signal of known shape one enhances the raw SNR by comparing the signal energy with that of the noise as opposed to the case of a burst signal where one has to contend with the signal power vis-a-vis noise power [4]. The fact that the signal waveform can be predicted very well also implies that the parameters of the waveform can be estimated to a high accuracy. For instance, by tracking the radiation emitted by a neutron star-neutron star (In this paper a neutron star is considered to have a mass of $1{\cdot}4\, M_{\odot}$, and a black hole will be considered to have a mass of $10\, M_{\odot}$.) (NS–NS) binary starting from 10 Hz to say 1 kHz, the masses can be determined to an accuracy better than a few per cent [5–9]. Observation of several coalescence events could be potentially used to determine the Hubble parameter to an accuracy of $\sim 10\%$ [10]. Once in a while, when an event produces a very high SNR ($\geqslant 40$) it would be possible to infer the presence of such non-linear effects as tails of gravitational waves in the signal waveform and test general relativity in the strongly non-linear regime [11, 12].

When a signal is present in the detector output, amongst all filters the filter matched to that particular signal will, on the average, obtain the largest possible correlation with the detector output, thus enabling the detection as well as the estimation of parameters. The parameters of the template that obtains the maximum correlation are unbiased estimates of the actual signal parameters. Of course, the detector output also contains noise which can affect the correlation thereby giving rise to spurious maxima even when the parameters of the template and those of the signal are mismatched. Thus, the measured parameters are in error. These errors reduce with increasing SNR. In the limit of high SNR the covariance matrix gives errors in the estimation of parameters and the covariances among them. Based on the computation of the covariance matrix (for the advanced LIGO), it is now generally agreed that the chirp mass, which is a combination of the masses of the binary stars (see below for its definition), can be determined at a relative accuracy $\sim 0.1\% - 1\%$ at a SNR of 10. The reduced mass, as it turns out, will have a much larger error (few $- 50\%$) especially if the spins of the bodies are large [6]. The Cramer–Rao theorem asserts that the covariance matrix is only a lower bound on the errors in the estimation of parameters [13]. Consequently, a more rigorous or a different method of computing the variances and covariances is in order. Alternatively one can determine the errors through the Monte Carlo simulation method. The basic idea here is to mimic the actual detection problem on a computer by adding numerous realizations of the noise to the signal and then passing the resultant time series through a bank of filters each matched to a gravitational wave signal from a coalescing binary system with a particular set of parameters. The errors involved in the measurement can be computed by using the ensemble of measured values of the signal parameters. Such a method is robust in the sense that it does not make any assumptions about the SNR and is therefore much closer to the actual detection problem. It also squarely addresses the difficulties faced in a realistic data analysis exercise such as the finite sampling rate, the discreteness of the set of filters, etc. For details on the Monte Carlo method see Balasubramanian *et al* [14].

In this communication we summarize the outcome of the first in a series of Monte Carlo simulations which suggests that the covariance matrix grossly underestimates the errors in the estimation of the parameters by over a factor of 3 at an SNR of 10. The standard method to deduce the direction to the source of the gravitational radiation has been to use the differences in the times-of-arrival in a network of three or more detectors. We point out that since the instant of coalescence of a binary system can be measured to a much greater accuracy than the time-of-arrival, the former can be employed to infer the direction of the source at a much lower uncertainty than has been so far thought.

2. Basic equations and formalism

In the point mass approximation the restricted first-order post-Newtonian gravitational radiation emitted by a binary system of stars induces a strain in the detector which can be written as

$$h(t) = A[\pi f(t)]^{2/3}\cos(\varphi(t)], \tag{1}$$

where $f(t)$ is the instantaneous gravitational wave frequency, which at this level of approximation is equal to twice the orbital frequency and is implicitly given by

$$t - t_a = \tau_0\left[1-\left(\frac{f}{f_a}\right)^{-8/3}\right] + \tau_1\left[1-\left(\frac{f}{f_a}\right)^{-2}\right] \tag{2}$$

and $\varphi(t)$ is the phase of the signal given by

$$\varphi(t) = \frac{16\pi f_a\tau_0}{5}\left[1-\left(\frac{f}{f_a}\right)^{-5/3}\right] + 4\pi f_a\tau_1\left[1-\left(\frac{f}{f_a}\right)^{-1}\right] + \Phi \tag{3}$$

where t_a, the time-of-arrival, is the time at which the signal reaches a frequency f_a; and Φ is the phase of the signal at t_a a constant depending on the relative orientations of the binary orbit and the arms of the interferometer; τ_0 and τ_1 are constants, having dimensions of time, depending on the masses m_1 and m_2 of the binary system. They are referred to as Newtonian and post-Newtonian chirp time, respectively, and are given by,

$$\tau_0 = \frac{5}{256}\mathcal{M}^{-5/3}(\pi f_a)^{-8/3}, \quad \tau_1 = \frac{5}{192\mu(\pi f_a)^2}\left(\frac{743}{336} + \frac{11}{4}\frac{\mu}{M}\right). \tag{4}$$

Here μ is the reduced mass, M is the total mass, and $\mathcal{M} = \mu^{3/5}M^{2/5}$ is the chirp mass. In this level of approximation the post-Newtonian waveform is characterized by a set of five parameters: The amplitude parameter A, the time-of-arrival t_a, the phase at the time of arrival Φ, and the two masses m_1 and m_2. We shall find below a more convenient parametrization of the waveform.

The Fourier transform of the waveform in the stationary phase approximation for positive frequencies is given by

$$\tilde{h}(f) = \mathcal{A}f^{-7/6}\exp\left[i\sum_{\mu=1}^{4}\psi_\mu(f)\lambda^\mu - i\frac{\pi}{4}\right] \tag{5}$$

where $\mathcal{A}$ is a normalization constant and is fixed by specifying the SNR, and $\psi_\mu(f)$ are functions only of frequency given by

$$\psi_1 = 2\pi f, \quad \psi_2 = -1, \quad \psi_3 = \frac{6\pi f_a}{5}\left(\frac{f}{f_a}\right)^{-5/3}, \quad \psi_4 = 2\pi f_a\left(\frac{f}{f_a}\right)^{-1}. \tag{6}$$

Here λ^μ are related to the set of parameters introduced earlier: $\lambda^\mu = \{t_C, \Phi_C, \tau_0, \tau_1\}$, where

$$t_C = t_a + \tau_0 + \tau_1, \quad \Phi_C = \Phi - 2\pi f_a t_a + \frac{16\pi f_a\tau_0}{5} + 4\pi f_a\tau_1. \tag{7}$$

For $f < 0$, the Fourier transform can be obtained by the relation $\tilde{h}(-f) = \tilde{h}^*(f)$, obeyed by real functions $h(t)$.

Central to the computation of the covariance matrix of the parameters is the scalar product defined over the space of the signal waveforms. Given waveforms $h(t;\lambda)$ and

$h(t; \lambda')$ their scalar product is defined by,

$$\langle h, g \rangle = \int_{f_a}^{\infty} \frac{\tilde{h}(f; \lambda^{\mu}) \tilde{h}^{*}(f; \lambda'^{\mu})}{S_n(f)} \mathrm{d}f + \text{c.c.} \tag{8}$$

where $S_n(f)$ is the two-sided detector noise power spectral density. The scalar product defines a norm on the vector space. The SNR obtained for a signal $h(t; \lambda)$ when it is passed through the matched filter is the norm of the signal [4]

$$\rho = \langle h, h \rangle^{1/2}. \tag{9}$$

The covariance matrix $C_{\mu\nu}$ is the inverse of the Fisher information matrix $\Gamma_{\mu\nu}$ [13] given by,

$$\Gamma_{\mu\nu} = \left\langle \frac{\partial h}{\partial \lambda^{\mu}}, \frac{\partial h}{\partial \lambda^{\nu}} \right\rangle; \quad C_{\mu\nu} = [\Gamma^{-1}]_{\mu\nu}. \tag{10}$$

The diagonal elements of the covariance matrix provide us an estimate of the variances to be expected in the measured values of the parameters in a given experiment. In figure 1 we have shown as a solid line the behaviour of the computed 1σ uncertainties (square root of the variances) in the estimation of parameters t_a, τ_0, τ_1 and τ_C, as a function of the SNR. As is well-known the covariance matrix predicts that the errors fall off in inverse proportion to the SNR.

3. Simulations

In the actual detection problem the detector output $x(t)$ is filtered through a host of search templates corresponding to test parameters, ${}_t\lambda$ and the template that obtains the maximum correlation with the output gives us the measured values ${}_m\lambda$ of the signal. These measured values will in general differ from the actual signal parameters. With the aid of a large number of detectors one obtains an ensemble of measured values ${}_m\lambda$. Such an ensemble provides us with an estimate ${}_e\lambda$, the errors σ_λ and correlation coefficients $\Sigma_{\mu\nu}$.

$${}_e\lambda = \overline{{}_m\lambda}, \quad \sigma_\lambda^2 = \overline{({}_m\lambda - \overline{{}_m\lambda})^2}, \quad \Sigma^{\mu\nu} = \frac{\overline{{}_m\lambda^{\mu}\, {}_m\lambda^{\nu}}}{\sigma_\mu \sigma_\nu}, \tag{11}$$

where the overbar denotes the average over an ensemble. In reality we will have only a few detectors and hence a numerical simulation needs to be carried out to deduce the errors in the estimation of parameters. We have carried out such a numerical simulation by using in excess of 5000 realizations of detector noise. Thus, the results of our simulations are statistically significant and the errors in the estimation of various statistical quantities, such as the mean and the variance (cf. figure 1), are negligible. In figure 1 the dotted line corresponds to the behaviour of the errors in the estimation of various parameters computed at several values of the SNR. At low values of the SNR ($\rho \sim 10$) there is a significant departure of the observed errors from those predicted by the covariance matrix. However at an SNR of ~ 30 the two curves merge together indicating the validity of the covariance matrix at high enough SNRs. Since most of the events which the interferometric detectors will observe are expected to have an SNR

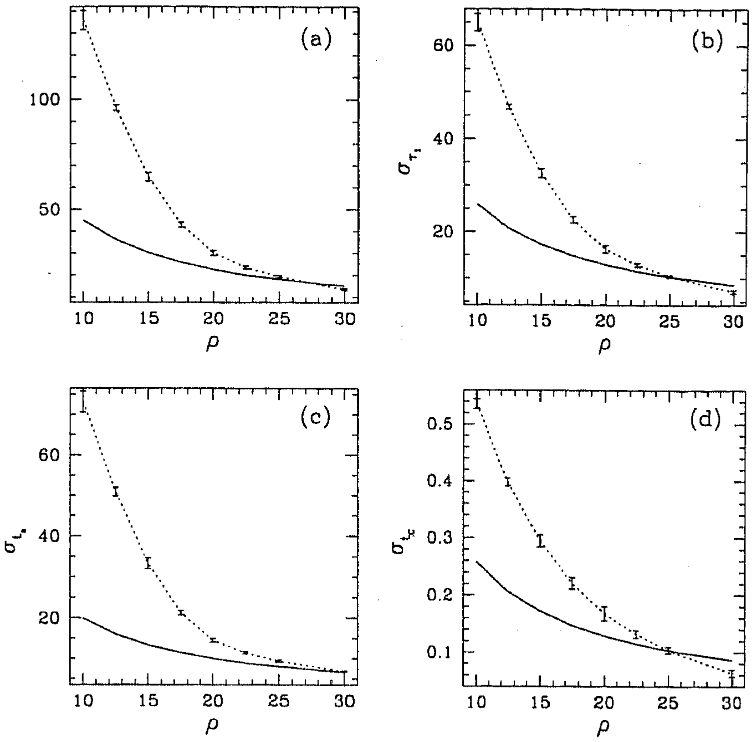

Figure 1. Dependence of the errors in the estimation of parameters of the post-Newtonian waveform i.e. $\{\sigma_{\tau_0}, \sigma_{\tau_1}, \sigma_{t_a}, \sigma_{t_c}\}$ as a function of SNR. The solid line represents the theoretically computed errors whereas the dotted line represents the errors obtained through Monte Carlo simulations. The simulations have been carried out for $10\,M_\odot - 1{\cdot}4\,M_\odot$ binary system. The errors in the parameters are expressed in ms.

less than 10, we conclude that the accuracy in the determination of the parameters is not as high as was thought to be. Detailed analysis suggests that this discrepancy is larger when higher post-Newtonian corrections are taken into account. Consequently, a more exhaustive analysis has been reported here or elsewhere [14] which is in order. At SNRs > 30 numerically estimated errors are somewhat lower than the covariance matrix, especially in the case of t_C, as opposed to what is expected from the Cramar-Rao bound. The reason for this discrepancy is that we do not have a high enough resolution (in particular, in the case of t_C) to determine the errors when the SNR is high. This is indicated by the increase in the error bars after the two curves cross over in figure 1. Moreover, the covariance matrix is computed using the stationary phase approximation to the Fourier transform of the signal while numerical estimation uses Fast Fourier Transforms.

With reference to figure 1 we point out that the instant of coalescence t_C can be determined to an accuracy much better than the time-of-arrival t_a. Typically σ_{t_C} is a factor of 50 less than σ_{t_a}. Consequently, with the aid of t_C we can fix the direction to the source at a much higher accuracy than with t_a. Since the parameters τ_0, τ_1 and t_a have negative covariances their sum, t_C, is more accurately determined. A possible physical interpretation of this result is that at the time-of-arrival the frequency is lower and

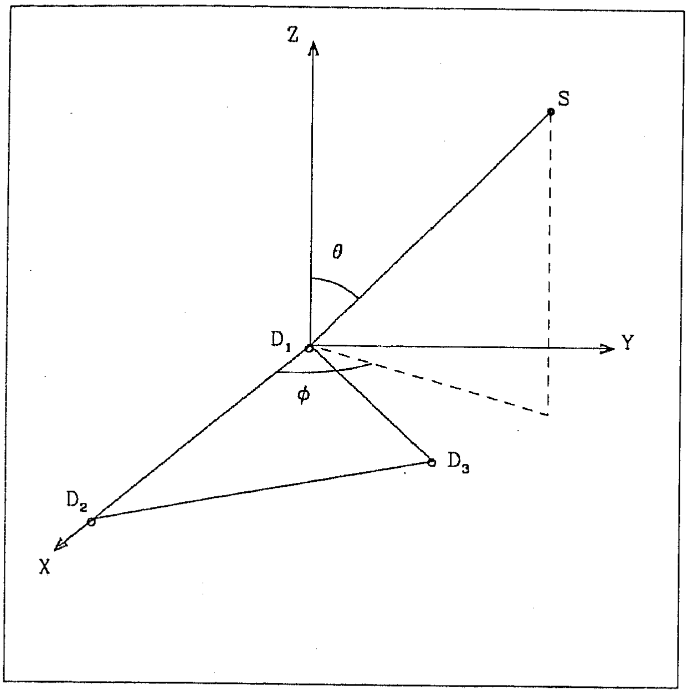

Figure 2. This figure illustrates the network of three identical detectors and the coordinate system employed. The detectors lie on the X–Y plane and θ and ϕ represent the angles determining the direction of the source (S).

hence allows for a larger variation in t_a without incurring a substantial error in the phase of the template relative to that of the signal. While close to coalescence the signal has a greater frequency and hence even a slight mismatch in t_C's of the signal and the template leads to a greater error in the phase. It is to be noted, however, that as of now we do not know the orbit of the binary accurately enough, to predict the exact instant of the coalescence. In fact the frequency cut off imposed by the onset of the plunge orbit will make it difficult to calculate t_C. In addition, for a realistic detector the noise increases with the frequency beyond about 150 Hz and the frequency which will be of more interest to us is the one where the power spectrum of the signal divided by that of the noise reaches a maximum. This, of course, assumes that the detectors, used in the measurement of the direction of the source, are all identical.

4. Determining source direction

Let t denote a convenient time parameter which, if measured at three detectors, gives the direction to the source. For our purpose we take this parameter to be either t_C or t_a. We assume a simple configuration of three identical detectors placed at the vertices of an equilateral triangle on the equatorial plane of the earth. We take the separation between any two detectors as L. Thus, the maximum delay in arrival times of the signal at the detectors is $\Delta = L/c$, where c is the speed of light. Figure 2 illustrates such a detector network and the coordinate system used. Given t_1, t_2 and t_3 as the measured values of t in the three detectors, respectively, we can deduce the direction to the source

(θ, ϕ) via the time delays $\gamma \equiv t_2 - t_1$ and $\delta \equiv t_3 - t_1$:

$$\phi = \tan^{-1}\left[\frac{2\delta - \gamma}{\sqrt{3}\gamma}\right], \tag{12}$$

and,

$$\theta = \sin^{-1}\left[\frac{2\gamma\sqrt{\gamma^2 + \delta^2 - \gamma\delta}}{\Delta(2\delta - \gamma)}\right]. \tag{13}$$

Our assumption of identical detectors means that the errors in t in different detectors are all the same. As the noises in different detectors are uncorrelated, the fluctuations in the measurements of t_1, t_2 and t_3 are uncorrelated; however, γ and δ have non-zero covariances. Thus,

$$\sigma_{t_1} = \sigma_{t_2} = \sigma_{t_3} = \sigma; \quad \sigma_\gamma = \sigma_\delta = \sqrt{2}\sigma. \tag{14}$$

The errors in the measurement of time parameter will induce errors, σ_ϕ and σ_θ, in the measurement of angles. Assuming the errors in time parameter to be small we can write,

$$\sigma_\phi = \frac{\sigma}{\Delta} g_\phi(\theta, \phi) \quad \text{and} \quad \sigma_\theta = \frac{\sigma}{\Delta} g_\theta(\theta, \phi), \tag{15}$$

where,

$$g_\phi(\theta, \phi) = \frac{1}{\sin\theta} \quad \text{and}$$

$$g_\theta(\theta, \phi) = \left(\frac{\cos^4\phi - \cos^2\phi - 1}{-1 + 3\cos^2\phi - 2\cos^4\phi - \cos^2\phi\cos^2\theta\sin^2\theta}\right)^{1/2} \tag{16}$$

The factors $g_\phi(\theta, \phi)$ and $g_\theta(\theta, \phi)$ are typically of order unity for most directions. For earth based detectors the value of Δ is ≈ 15 ms. This certainly rules out the use of t_a to determine the direction as the error in this parameter, even for high SNRs ~ 15, is much more than Δ. Though for the initial LIGO we can use t_C, the errors at an SNR of 10 will be around two degrees which is too large to make an optical identification of the source. In order to determine the direction to arcsecond resolution one needs the sensitivity of the advanced LIGO or the VIRGO.

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References

[1] A Abramovici *et al*, *Science* **256**, 325 (1992)
[2] C Bradaschia *et al*, *Nucl. Instrum. Methods*, **A289**, 518 (1990)
[3] K S Thorne, in *300 Years of gravitation*, edited by S W Hawking and W Israel (Cambridge Univ. Press, 1987)
[4] B F Schutz, in *The detection of gravitational radiation*, edited by D Blair (Cambridge Univ. Press, Cambridge, 1989) pp 406–427

[5] L S Finn and D F Chernoff, *Phys. Rev.* **D47**, 2198 (1993)
[6] C Cutler and E Flanagan, *Phys. Rev.* **D49**, 2658 (1994)
[7] E Poisson and C M Will, *Phys. Rev.* **D52**, 848 (1995)
[8] A Krolak, K D Kokkotas and G Schäfer, *Phys. Rev.* **D52**, 2089 (1995)
[9] P Jaranowski, K D Kokkotas, A Krolak and G Tsegas, report (unpublished)
[10] D Marković, *Phys. Rev.* **D48**, 4738 (1993) and references therein
[11] L Blanchet and B S Sathyaprakash, *Class. Quantum Gravit.*, **11**, 2807 (1994)
[12] L Blanchet and B S Sathyaprakash, *Phys. Rev. Lett.* **74**, 1067 (1995)
[13] C W Helstrom, *Statistical theory of signal detection*, (Pergamon, Cambridge, 1966) second edition
[14] R Balasubramanian, B S Sathyaprakash and S V Dhurandhar, *Gravitational waves from coalescing binaries: detection strategies and Monte Carlo estimation of parameters*, submitted for publication (1995)
[15] P Jaranowski and A Krolak, *Phys. Rev.* **D49** 1723 (1994)