Pramāṇa – J. Phys., Vol. 27, No. 4, October 1986, pp. 485–495. © Printed in India.

Charged particle orbits in the field of two charges placed symmetrically near a Schwarzschild black hole

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MS received 24 October 1985; revised 2 June 1986

Abstract. From the Copson and Linet solution for the electrostatic field due to a point charge near a Schwarzschild black hole, we have deduced the field due to two equal charges placed symmetrically (diametrically opposite) about the hole. It turns out that the motion of a test-charged particle is completely solvable only in the equatorial plane, because the θ-equation does not yield the first integral for $\theta \neq \pi/2$. We have however considered circular orbits about the axis for $\theta = \text{constant} \neq \pi/2$ by requiring both θ and r to remain fixed all through the motion. For $\theta \neq \pi/2$ orbits, in contrast to the similar classical situation, there occur forbidden θ-ranges. This seems to be a relativistic effect.

Keywords. Black hole; electrostatic field; Penrose process; charged-particle trajectories.

PACS No. 04·20

1. Introduction

Of late, high energy astrophysicists have shown considerable interest in electromagnetic fields superposed on the curved space-time background of the black holes. It is generally believed that electromagnetic extraction of hole's energy may be the most likely mechanism for powering the active galactic nuclei, quasars, and x-ray binaries (see the two excellent reviews by Rees 1984 and Begelman *et al* 1984). It is therefore of interest to study charged particle orbits for various situations involving electromagnetic fields on black holes. The most favoured setting may perhaps be a rotating black hole in magnetic field. In a forthcoming review, Dadhich (1986) discusses the electromagnetic extraction of energy from a rotating black hole in electromagnetic field by the Penrose process. It turns out that the electromagnetic interaction tremendously enhances the efficiency of the energy extraction, which may win over the other competing processes. (Wagh *et al* 1986; Parthasarathy *et al* 1986).

In this investigation we consider rather a simple setting of two equal point charges placed symmetrically (diametrically opposite at equal distances) near a Schwarzschild black hole. The electrostatic field due to a single charge near the hole was considered by Cohen and Wald (1971) and by Hanni and Ruffini (1973). However, this field has been given in a closed algebraic form by Copson (1928), with a correction by Linet (1976);

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Misra (1977) derived the same independently. From this solution we deduce the field due to two charges having coordinates $r = b$, $\theta = 0$ and $r = b$, $\theta = \pi$. The Copson solution together with the Linet's correction is equivalent to the Cohen-Wald solution. The former has the advantage of being in the closed algebraic form over the latter which is given by a series.

It should however be noted that astronomical observations do not favour significant quantity of charge on or around the hole, while there may perhaps occur a transient phase involving some charge distribution. We shall therefore consider two symmetrically placed test (small) charges near the hole and our results would be of relevance for such a transient phase.

For motion in the equatorial plane, we could obtain all the four first integrals of motion and it could be discussed qualitatively in the usual manner by writing an effective potential, while for motion off the equatorial plane the θ-equation does not readily yield a first integral. In this case we have considered circular orbits about the axis by imposing the conditions $\dot{\theta} = \ddot{\theta} = 0$ and $\dot{r} = \ddot{r} = 0$ simultaneously. A similar study was carried out by Sonar *et al* (1985) for a single charge placed near a Schwarzschild black hole. In their case there could occur no circular orbits in the equatorial plane as it was not the plane of symmetry. By putting another charge symmetrically, we have restored the reflection symmetry to the equatorial plane and hence allowing for the orbits.

In § 2 we obtain from the Copson-Linet (which is equivalent to the Cohen-Wald) solution, the field due to two charges in the Schwarzschild spacetime. Equations of motion for charged particles are written down in § 3 while the motion in the equatorial plane is discussed in § 4, and circular orbits about the axis of symmetry (for $\theta \neq \pi/2$) are considered in § 5.

2. Solution for two charges

Here we obtain the expression for electrostatic field due to two point charges at rest near a Schwarzschild black hole. Cohen and Wald (1971) and Hanni and Ruffini (1973) solved the Maxwell's equations in the background Schwarzschild spacetime and obtained a solution in the series form for a point charge held at rest near the black hole. This question was first considered by Copson (1928) and he had obtained the solution in a closed algebraic form. Linet (1976) showed that the Copson solution with the correction term is equivalent to the Cohen-Wald solution (Misra 1977). Using the Copson-Linet solution, we obtain the field due to two charges symmetrically placed at a distance $r = b$ (at $\theta = 0$ and $\theta = \pi$) from the black hole.

The background geometry is given by the Schwarzschild solution,

$$ds^2 = -\left(1 - \frac{2m}{r}\right)dt^2 + \left(1 - \frac{2m}{r}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta \, d\phi^2), \tag{1}$$

where m is the mass of the black hole. Here and in the following, geometric units are used; $G = c = 1$.

The electrostatic field is assumed to be a perturbative test field, i.e., it does not contribute appreciably so as to alter the background Schwarzschild geometry. That means, the electrostatic energy of the point charge is negligible as compared to the mass of the black hole.

The Maxwell's equations in curved space are written as

$$F^{ik}_{;i} = \frac{1}{\sqrt{-g}} \frac{\partial}{\partial x^i} (\sqrt{-g}\, F^{ik}) = 4\pi j^k \tag{2}$$

and

$$*F^{ik}_{;i} = 0, \tag{3}$$

where $F_{ik} = A_{k;i} - A_{i;k} = A_{k,i} - A_{i,k}$ is the electromagnetic field tensor ($*F^{ik}$ is the dual of F^{ik}), A_i is the 4-potential, j^k is the 4-current and a semicolon denotes covariant derivative while a comma denotes ordinary derivative.

Since the charge is at rest on the axis, spacelike components of the 4-current vector vanish and so will be the space-like components of the 4-potential. This means, the field, as expected, is purely electrostatic, and hence we need only to solve equation (2). Further the system is static and axially symmetric, A_t will be a function of r and θ only.

In the spacetime (1), the Maxwell's equations (2) will read as

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial A_t}{\partial r}\right) + \frac{1}{\left(1 - \frac{2m}{r}\right) r^2 \sin\theta} \frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial A_t}{\partial \theta}\right) = -4\pi j^t. \tag{4}$$

2.1 *Solution for a charge at $r = b$, $\theta = 0$*

For a point charge e at rest at $r = b$, $\theta = 0$, we have, $j^t = e\delta(r-b)\delta(\cos\theta - 1)$, and hence solution of (4) could be written in the closed algebraic form by using the Copson-Linet solution as

$$A_{1t}(r = b,\, \theta = 0)$$

$$= \frac{e[(r-m)(b-m) - m^2\cos\theta]}{br[(r-m)^2 + (b-m)^2 - m^2 - 2(r-m)(b-m)\cos\theta + m^2\cos^2\theta]^{1/2}} + \frac{em}{br}. \tag{5}$$

For solution due to a charge at $r = b, \theta = \pi$, one should just replace $\cos\theta$ by $-\cos\theta$ in the above solution.

2.2 *Solution for two equal charges at $r = b$, $\theta = 0$ and $r = b$, $\theta = \pi$*

The curved spacetime Maxwell's equations (perturbative) are linear and hence we can superpose the two solutions and obtain

$$A_t = A_{1t}(r = b,\, \theta = 0) + A_{2t}(r = b,\, \theta = \pi)$$

$$= \frac{e[(r-m)(b-m) - m^2\cos\theta]}{br[(r-m)^2 + (b-m)^2 - m^2 - 2(r-m)(b-m)\cos\theta + m^2\cos^2\theta]^{1/2}}$$

$$+ \frac{e[(r-m)(b-m) + m^2\cos\theta]}{br[(r-m)^2 + (b-m)^2 - m^2 + 2(r-m)(b-m)\cos\theta + m^2\cos^2\theta]^{1/2}}$$

$$+ \frac{2em}{br}. \tag{6}$$

As in the flat spacetime, potential as well as field diverge as the location of a point charge is approached.

In the orthonormal frame,

$$w^0 = \left(1 - \frac{2m}{r}\right)^{1/2} \mathrm{d}t, \qquad w^1 = \left(1 - \frac{2m}{r}\right)^{-1/2} \mathrm{d}r,$$

$$w^2 = r\,\mathrm{d}\theta, \qquad w^3 = r \sin\theta\, \mathrm{d}\phi,$$

the only non-vanishing components of the field tensor are

$$\tilde{F}_{01} = -\tilde{F}_{10} = -\frac{\partial A_t}{\partial r} \text{ and } \tilde{F}_{02} = -\tilde{F}_{20} = -\frac{1}{r}\left(1 - \frac{2m}{r}\right)^{-1/2} \frac{\partial A_t}{\partial \theta}. \tag{7}$$

It is clear from (6) and (7) that as $r \to 2m$, $\tilde{F}_{20} \to 0$, while $\tilde{F}_{10}$ remains finite. This shows that the field is radial in the close vicinity, $r \to 2m$, of the hole.

3. Equations of motion

In this section we derive the equations of motion of a charged test particle in the Schwarzschild geometry with the superposed electrostatic field discussed in the previous section. Now, the motion of a charged test particle is not only governed by the background gravitational field of the black hole, but also by the Lorentz force acting on the particle due to the perturbative electrostatic field.

The equations of motion of a charged test particle of rest mass μ and charge q in an electromagnetic field are given by,

$$\frac{\mathrm{d}u^i}{\mathrm{d}s} + \Gamma^i_{jk} u^j u^k = \frac{q}{\mu} F^i_{\ p} u^p, \tag{8}$$

where u^i is the 4-velocity of the particle. The right side of (8) represents the Lorentz force indicating the non-geodesic character of motion. The second term on the left side is due to the gravitational interaction derived from the metric.

These equations of motion are obtained from the Lagrangian:

$$\mathscr{L} = \tfrac{1}{2} g_{ij} \dot{x}^i \dot{x}^j - \frac{q}{\mu} A_i \dot{x}^i, \tag{9}$$

where a dot denotes the ordinary differentiation with respect to the proper time s. Using (1) in (9), we obtain,

$$\mathscr{L} = \frac{1}{2}\left[-\left(1 - \frac{2m}{r}\right)\dot{t}^2 + \left(1 - \frac{2m}{r}\right)^{-1} \dot{r}^2 + r^2(\dot{\theta}^2 + \sin^2\theta\,\dot{\phi}^2)\right] - \frac{q}{\mu} A_t \dot{t}. \tag{10}$$

Since the electrostatic field is axisymmetric and the background geometry is spherically symmetric, the Lagrangian is independent of the azimuthal coordinate ϕ. Further, since the field considered is static, the Lagrangian is also independent of time coordinate t.

Consequently, we get two constants of motion L and E as,

$$r^2 \sin^2\theta\, \dot{\phi} = L, \tag{11}$$

$$\left(1-\frac{2m}{r}\right)\dot{t} = E - \frac{q}{\mu}A_t, \tag{12}$$

where L is the angular momentum and E is the energy per unit rest mass of the particle as measured by an observer at rest at infinity.

The equations of motion corresponding to r and θ coordinates are given by,

$$\ddot{r} - \frac{m}{r^2}\left(1-\frac{2m}{r}\right)^{-1}\dot{r}^2 = r\left(1-\frac{2m}{r}\right)(\dot{\theta}^2 + \sin^2\theta\,\dot{\phi}^2) - \frac{m}{r^2}\left(1-\frac{2m}{r}\right)\dot{t}^2 - \frac{q}{\mu}\left(1-\frac{2m}{r}\right)A_{t,r}\dot{t}, \tag{13}$$

$$\ddot{\theta} + \frac{2}{r}\dot{r}\dot{\theta} = \sin\theta\cos\theta\,\dot{\phi}^2 - \frac{q}{\mu}\frac{1}{r^2}A_{t,\theta}\dot{t}. \tag{14}$$

4. Motion in the equatorial plane

We shall now consider motion in the equatorial plane, $\theta = \pi/2$. By the symmetry of the system, it is clear that a test particle can have motion confined to the equatorial plane.

$$A_{t,\theta} = (2m - r)(b - 2m)\sin\theta\left(\frac{1}{A^{3/2}} - \frac{1}{B^{3/2}}\right), \tag{15}$$

where $A = (r-m)^2 + (b-m)^2 - m^2 - 2(r-m)(b-m)\cos\theta + m^2\cos^2\theta,$

and $B = (r-m)^2 + (b-m)^2 - m^2 + 2(r-m)(b-m)\cos\theta + m^2\cos^2\theta.$

For $\theta = \pi/2$, $A_{t,\theta} = 0$ and hence a particle initiating its motion in the equatorial plane, remains confined to it. This is in contrast to the system consisting of a single charge at rest near the black hole (Sonar *et al* 1985). This is because of restoration of the reflection symmetry to the equatorial plane by putting another charge at $r = b$, $\theta = \pi$.

Since the 4-velocity vector has unit norm and is timelike, from (1) we get,

$$-1 = -\left(1-\frac{2m}{r}\right)\dot{t}^2 + \left(1-\frac{2m}{r}\right)^{-1}\dot{r}^2 + r^2(\dot{\theta}^2 + \sin^2\theta\,\dot{\phi}^2). \tag{16}$$

Substituting for $\dot{\phi}$ and $\dot{t}$ from (11) and (12) respectively, (16) yields, for $\theta = \pi/2$,

$$-\left(1-\frac{2m}{r}\right) = -\left(E - \frac{q}{\mu}A_t\right)^2 + \dot{r}^2 + \frac{L^2}{r^2}\left(1-\frac{2m}{r}\right). \tag{17}$$

The effective potential V is then defined by taking $\dot{r} = 0$, and is given by,

$$V = \frac{q}{\mu}A_t + \left[\left(1-\frac{2m}{r}\right)\left(1+\frac{L^2}{r^2}\right)\right]^{1/2}. \tag{18}$$

Here we have chosen the positive sign for the radical, for $\dot{t} > 0$ for a future moving particle. For convenience, we now introduce dimensionless quantities,

$$\bar{r} = r/m, \quad \bar{b} = b/m, \quad \bar{s} = s/m, \quad \bar{L} = L/m, \quad \bar{t} = t/m,$$
$$A_t = (e/m)\bar{A}_t \quad \text{and} \quad \lambda = eq/m\mu,$$

and shall drop the overhead bars in subsequent discussion. Equation (18) will now read as,

$$V = \lambda A_t + \left[\left(1 - \frac{2}{r}\right)\left(1 + \frac{L^2}{r^2}\right)\right]^{1/2}, \tag{19}$$

where $$A_t = \frac{2}{br}\left[\frac{(r-1)(b-1)}{\{(r-1)^2 + (b-1)^2 - 1\}^{1/2}} + 1\right].$$

It is clear that V can be negative for some $r > 2$ for $\lambda < 0$. The extent of the negative energy state region depends on the value of λ. This is evident from figures 1 and 2, where we have given some typical plots of V against r, which exhibit its dependence on the parameters λ and L. Figure 1 shows V for a fixed $\lambda = -5$, and for various values of $L = 10$, 50 and 100, while figure 2 shows V for a fixed $L = 50$, and for various values of $\lambda = -50, -10, 1, 10$ and 50. We observe that for a large negative λ, V is negative for large r and attains large negative values. Hence the Penrose process of energy extraction

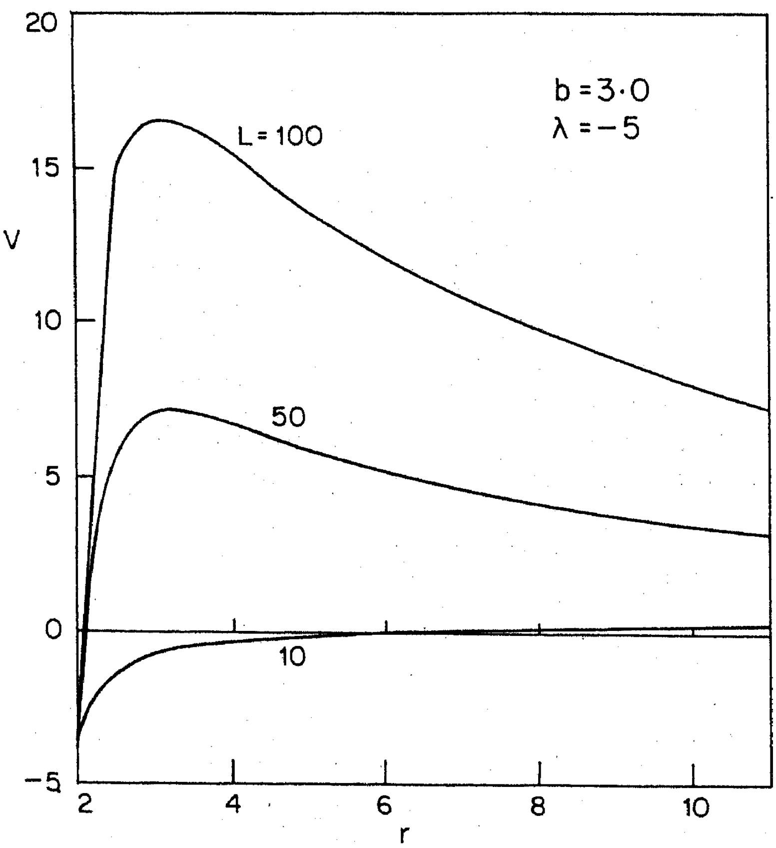

Figure 1. The curves show the effective potential V vs r plotted for motion in the equatorial plane for fixed $b = 3{\cdot}0$, $\lambda = -5{\cdot}0$ and for varying $L = 10$, 50 and 100.

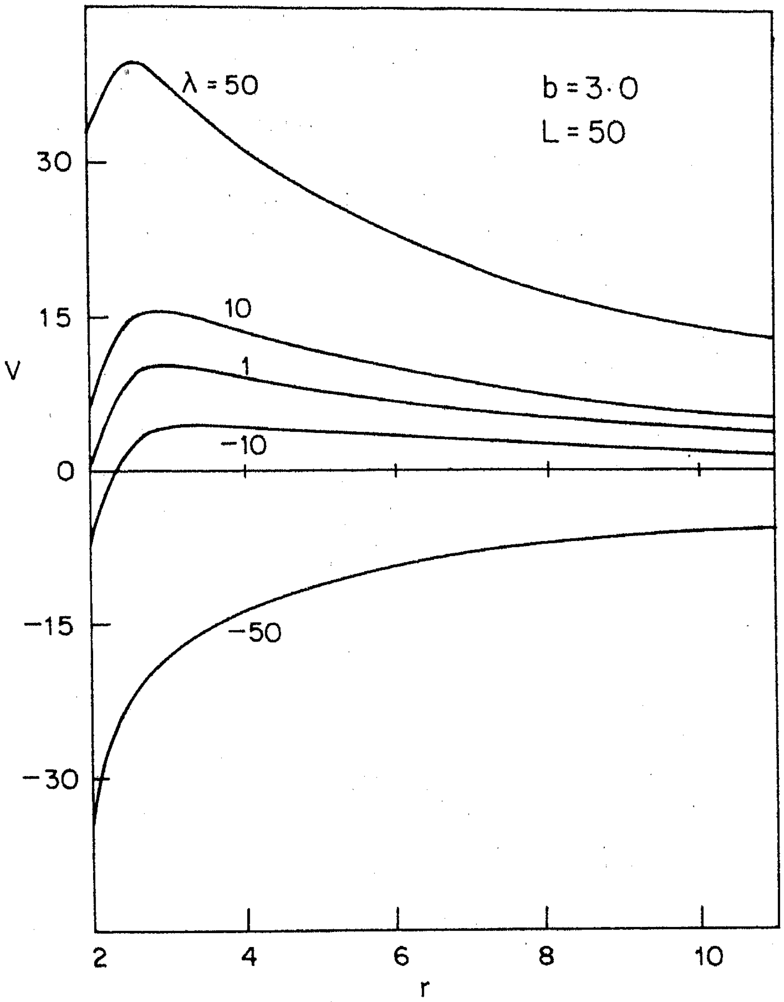

Figure 2. The curves show V for fixed $b = 3{\cdot}0$, $L = 50$ and for varying $\lambda = -50$, -10, 1, 10, 50.

can be set up to extract electrostatic energy (Denardo and Ruffini 1973; Dhurandhar and Dadhich 1984a, b; Penrose 1969). However, it should be remembered that electrostatic energy which alone can be extracted by this process is quite insignificant in view of the perturbative character of the field.

For $\lambda = 0$, V reduces to that of the Schwarzschild case. In view of the electrostatic interaction, one would expect that the existence threshold for circular orbits gets pulled closer to the black hole, i.e., threshold radius < 3—the Schwarzschild threshold. It could be brought arbitrarily close to the horizon by increasing λ. V_{max} in figures 1 and 2 indicates the existence of unstable circular orbits for $r < 3$. There could, however, exist stable circular orbits for $E = V_{min}$.

5. Circular orbits about the axis off the equatorial plane

As the system of black hole plus two charges is axially symmetric, there will not in general occur plane orbits off the equatorial plane. In contrast to the spherically symmetric case (where θ-equation is thrown out of consideration by putting $\theta = \pi/2$, as any plane can be taken as the equatorial plane), (14) does not yield a constant of motion for $\theta = \pi/2$. Hence the motion is not integrable in general.

We shall here examine the special case: under what conditions can circular orbits occur for a fixed $\theta \neq \pi/2$? That means, we should have $\dot{\theta} = \ddot{\theta} = 0$ and $\dot{r} = \ddot{r} = 0$ simultaneously. The additional conditions $\dot{\theta} = \ddot{\theta} = 0$ do not allow free choice of L but instead they determine it for a fixed r. Further, there will also be a consistency relation to be satisfied as both $\dot{r} = 0$ and $\ddot{r} = 0$ will give expressions for energy E.

Putting $\dot{\theta} = \ddot{\theta} = 0$ in (14), we get,

$$\frac{L^2}{r^2} = \frac{\lambda \sin^3\theta}{\cos\theta}\left(1-\frac{2}{r}\right)^{-1}(E-\lambda A_t)A_{t,\theta}. \tag{20}$$

Also setting $\dot{r} = 0$ in the line element (1), we obtain,

$$1 = \left(1-\frac{2}{r}\right)^{-1}(E-\lambda A_t)^2 - \frac{L^2}{r^2\sin^2\theta}. \tag{21}$$

We solve the above equations for E,

$$E = \lambda A_t + \frac{1}{2}\lambda \tan\theta\, A_{t,\theta} + \frac{1}{2}\left[\lambda^2\tan^2\theta\, A_{t,\theta}^2 + 4\left(1-\frac{2}{r}\right)\right]^{1/2}. \tag{22}$$

The condition $\ddot{r} = 0$ in (13) yields another expression for E as,

$$E = \lambda A_t + \lambda r^2\left(1-\frac{2}{r}\right)\left(\frac{\tan\theta}{r}A_{t,\theta} - A_{t,r}\right). \tag{23}$$

Hence from (22) and (23), the consistency demands,

$$\lambda\tan\theta\, A_{t,\theta}\left[2r\left(1-\frac{2}{r}\right)-1\right] - 2\lambda r^2\left(1-\frac{2}{r}\right)A_{t,r} - \left[\lambda^2\tan^2\theta\, A_{t,\theta}^2 + 4\left(1-\frac{2}{r}\right)\right]^{1/2} = 0. \tag{24}$$

Denoting the left side expression by S, we write,

$$S(r,\theta,b,\lambda) = \lambda(2r-5)\tan\theta\, A_{t,\theta} - 2\lambda r(r-2)A_{t,r} - \left[\lambda^2\tan^2\theta\, A_{t,\theta}^2 + 4\left(1-\frac{2}{r}\right)\right]^{1/2} = 0. \tag{25}$$

We observe that for given values of r, b and λ, a discrete set of values of θ can only satisfy this equation. The roots of $S = 0$ give the radii of circular orbits for given values of θ, b and λ. We are no longer free to choose L and E but they are determined by (20) and (22) or (23) and in addition (25) should be satisfied. This is because we are unable to obtain the first integral of the θ-equation for $\theta \neq \pi/2$ and hence there is no proper effective potential defined. In general, orbits off the equatorial plane are not confined to a plane.

From (25) we obtain,

$$\lambda^2 = \frac{4\left(1-\frac{2}{r}\right)}{[(2r-5)\tan\theta\, A_{t,\theta} - 2r(r-2)\, A_{t,r}]^2 - \tan^2\theta\, A_{t,\theta}^2}. \tag{26}$$

It turns out that there occur θ-ranges for which $\lambda^2 < 0$, implying the forbidden regions. That is, circular orbits cannot occur for all θ-values. The denominator of λ^2 can be factorized as,

$$[\sin\theta\, A_{t,\theta} - r\cos\theta\, A_{t,r}][(r-3)\sin\theta\, A_{t,\theta} - r(r-2)\cos\theta\, A_{t,r}],$$

either of them getting a negative gives a forbidden region. The boundary of a forbidden region is given by,

$$\sin\theta\, A_{t,\theta} - r\cos\theta\, A_{t,r} = 0, \tag{27}$$

or

$$(r-3)\sin\theta\, A_{t,\theta} - r(r-2)\cos\theta\, A_{t,r} = 0. \tag{28}$$

In figure 3, we have shown, for $b = 3$ the forbidden θ-ranges for circular orbits at $r = 2{\cdot}2$, $2{\cdot}4$, $2{\cdot}6$, $2{\cdot}8$ and $2{\cdot}9$. The angular range depends on the location of the circular orbit. For various values of b, the forbidden angular regions are given in table 1. The forbidden ranges always maintain the reflection symmetry about the equatorial plane and occur only for $r < b$. Numerical computations show that there is no forbidden region for $r > b$.

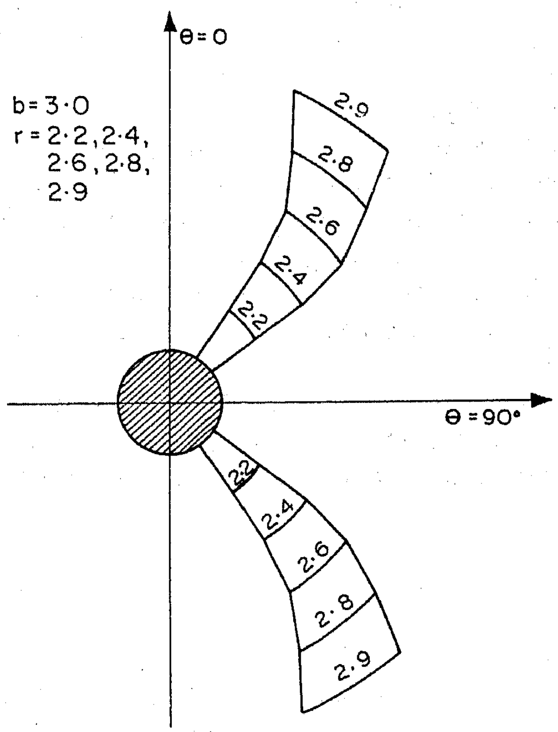

Figure 3. The forbidden θ-ranges are shown for $b = 3{\cdot}0$ and $r = 2{\cdot}2$, $2{\cdot}4$, $2{\cdot}6$, $2{\cdot}8$, $2{\cdot}9$ (see table 1).

Table 1. Forbidden θ-ranges for b and r values.

b	r = 2·2	2·4	2·6	2·8	2·9
3·0	(33·9°, 54·5°) (125·5, 146·1)	(33·9°, 54·5°) (125·5, 146·1)	(32·1°, 52·1°) (127·9, 147·9)	(27·0°, 47·1°) (132·9, 153·0)	(22·4°, 42·5°) (137·5, 157·6)
3·4	(36·2, 56·8) (123·2, 143·8)	(37·8, 59·6) (120·4, 142·2)	(37·8, 60·3) (119·7, 142·2)	(37·2, 59·1) (120·9, 142·8)	(36·1, 58·0) (122·0, 143·9)
3·8	(37·2, 58·5) (121·5, 142·8)	(39·5, 62·5) (117·5, 140·5)	(40·8, 64·8) (115·2, 139·2)	(41·3, 66·6) (113·4, 138·7)	(41·3, 66·6) (113·4, 138·7)

The occurrence of forbidden region seems to be the relativistic effect. It could be easily checked that for a similar classical system involving a mass point and two point charges, there will not occur such forbidden regions. A numerical check shows that as b increases the forbidden angular region shrinks and the range of forbidden radius also increases (for $b \geqslant 8$, $r \geqslant 5$). This effect may presumably be due to (i) the curvature of spacetime modifying the electrostatic field and (ii) in the neighbourhood of a black hole particle orbits, as is well known, have a non-Keplerian character.

6. Conclusion

In contrast to a single charge and black hole system (Sonar *et al* 1985), there do occur particle orbits confined to the equatorial plane in our case. This is because by putting another charge symmetrically, we have restored the reflection symmetry in the charge distribution relative to the equatorial plane.

The most interesting result is the occurrence of forbidden regions for circular orbits about the axis off the equatorial plane. This appears to be caused by the curvature of spacetime, for, a similar classical system does not exhibit this phenomena. One wonders whether this effect could ever provide a test for general relativity. For, astrophysical observations do not unfortunately favour the presence of static charges in the vicinity of black holes. The present investigation therefore has a very limited relevance (only for an intermediary stage) for a realistic situation.

We have taken charges to be static near the black hole and the question: what keeps them so? is not incorporated in our study. It would perhaps be too difficult to incorporate, however, so long as the charges remain static and as we insist by whatever means our results will be valid.

Acknowledgements

One of us (VC) thanks Dr M E Cherian, President, Spicer Memorial College, Pune for the financial assistance under the college sponsorship scheme. We also thank the referees for their comments and suggestions that have led to improvement in the paper. This investigation constitutes the dissertation submitted by VC, towards partial fulfilment for the M.Phil. degree, to the University of Poona.

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