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Energetics of the Kerr-Newman Black Hole by the Penrose Process

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Abstract. We have studied in detail the energetics of Kerr–Newman black hole by the Penrose process using charged particles. It turns out that the presence of electromagnetic field offers very favourable conditions for energy extraction by allowing for a region with enlarged negative energy states much beyond $r = 2M$, and higher negative values for energy. However, when uncharged particles are involved, the efficiency of the process (defined as the gain in energy/input energy) gets reduced by the presence of charge on the black hole in comparison with the maximum efficiency limit of 20.7 per cent for the Kerr black hole. This fact is overwhelmingly compensated when charged particles are involved as there exists virtually no upper bound on the efficiency. A specific example of over 100 per cent efficiency is given.

Key words: black hole energetics—Kerr-Newman black hole—Penrose process—energy extraction

1. Introduction

The problem of powering active galactic nuclei, X-raybinaries and quasars is one of the most important problems today in high energy astrophysics. Several mechanisms have been proposed by various authors (Abramowicz, Calvani & Nobili 1983; Rees *et al.,* 1982; Koztowski, Jaroszynski & Abramowicz 1978; Shakura & Sunyaev 1973; for an excellent review see Pringle 1981). Rees *et al.* (1982) argue that the electromagnetic extraction of black hole's rotational energy can be achieved by appropriately putting charged particles in negative energy orbits. Blandford & Znajek (1977) have also proposed an interesting mechanism by considering the electron-positron pair production in the vicinity of a rotating black hole sitting in a strong magnetic field. It is, therefore, important to study the energetics of a black hole in electromagnetic field.

An ingenious and novel suggestion was proposed by Penrose (1969) for the extraction of energy from a rotating black hole. It is termed as the Penrose process and is based on the existence of negative energy orbits in the ergosphere, the region bounded by the horizon and the static surface (Vishveshwara 1968). Though there does not exist an ergosphere for the Reissner-Nordstrϕm black hole, there do exist negative energy states for charged particles (Denardo & Ruffini 1973), which means that the electromagnetic energy can also be extracted by the Penrose process.

Though Penrose (1969) did not consider astrophysical applications of the process, Wheeler (1971) and others proposed that it could provide a viable mechanism for high energy jets emanating from active galactic nuclei. The mechanism envisaged a star-like body which on grazing a supermassive black hole breaks up into fragments due to enormous tidal forces (Mashhoon 1973; Fishbone 1973). Some fragments may have negative energy orbits and they fall into the black hole resulting in reduction of its rotational energy while the others come out with very high velocities to form a jet. However, this process fell out of favour for its astrophysical applications owing to limits on the relative velocity between the fragments (Bardeen, Press & Teukolsky 1972; Wald 1974): No significant gain in energy results for an astrophysically reasonable orbit of an incident star unless the splitup itself is relativistic, *i.e.* relative velocity between the fragments $\geqslant 1/2$. Very recently, Wagh, Dhurandhar & Dadhich (1985) have shown that these limits can be removed with the introduction of an electromagnetic field around the black hole. The electromagnetic binding energy offers an additional parameter which is responsible for removal of the limits. Thus the Penrose process is revived as a mechanism for high energy sources.

In this paper we wish to study the negative energy states for charged particles in the Kerr-Newman spacetime with a view to extracting energy by the Penrose process. A comparative analysis of negative energy states for charged particles in the Kerr-Newman field and for a Kerr black hole in a dipole magnetic field is done by Prasanna (1983). We study the negative energy states in a greater detail, and set up a Penrose process for energy extraction and also examine its efficiency in this case. It is known (Chandrasekhar 1983) that the maximum efficiency of this process is 20.7 per cent in the case of a Kerr black hole. The presence of charge on the Kerr-Newman black hole decreases the efficiency further when uncharged particles participate in the process while the efficiency is enormously enhanced (as high as over 100 per cent, in fact there is no limit!) when charged particles are involved.

Astrophysically massive bodies are not known to have significant charge on them $[Q/(\sqrt{G}M) \ll 1]$. That means the charge Q on the black hole should be taken as very small. But a small, nonzero Q can have appreciable effect on the test charge orbits due to the Lorentzian force. It is the Coulombic binding energy that contributes significantly to the energy of the test particle. It is not unjustified, therefore, to study the Penrose process with this assumption.

In Section 2, we establish the equations of motion and the effective potential for charged particles in the Kerr-Newman field while in Section 3 the negative energy states are examined. Section 4 deals with the setting up of an energy extraction process and finally in Section 5 we investigate the efficiency of the process.

2. The Kerr-Newman field

The Kerr-Newman spacetime in the BoyerLindquist coordinates is described by the metric

$$ds^2 = -(\Delta/\rho^2)(dt - a\sin\theta d\phi)^2 + (\rho^2/\Delta)dr^2 + \rho^2 d\theta^2 + (\sin^2\theta/\rho^2)\left[(r^2+a^2)d\phi - a dt\right]^2 \tag{2.1}$$

where

$$\Delta = r^2 + a^2 - 2mr + Q^2$$

$$\rho^2 = r^2 + a^2\cos^2\theta.$$

Here m is the mass, a is the angular momentum per unit mass and Q is the charge on the black hole. We have used the geometrised units (c = 1, G = 1). The event horizon is given by the larger root r_+ of $\Delta = 0$, $r_+ = M + (M^2 - a^2 - Q^2)^{1/2}$.

In this spacetime there exists an electromagnetic field due to the presence of charge Q. This field is obtained from the vector potential A_i,

$$A_i = (-Qr/\rho^2, 0, 0, aQr\sin^2\theta/\rho^2). \quad (2.2)$$

That means the rotation of the black hole also gives rise to a magnetic dipole potential in addition to the usual electrostatic potential.

2.1 *The Equations of Motion*

Let a test particle of rest mass μ and electric charge e move in the exterior field of the black hole. Its motion will be governed by the gravitational field of a charged rotating black hole as well as by the Lorentz force due to electromagnetic interaction. The equations of motion of the particle can be derived either from the Lagrangian $\mathscr{L}$

$$\mathscr{L} = \frac{\mu}{2} g_{ij}\dot{x}^i\dot{x}^j + eA_i\dot{x}^i \quad (2.3)$$

or from the Hamiltonian H,

$$H = \tfrac{1}{2}g^{ij}p_i p_j \quad (2.4)$$

where a dot denotes derivative with respect to the affine parameter τ/μ (τ being the proper time) and p_i is 4-momentum of the particle. Since the metric and the electromagnetic field are time independent and axially symmetric, the energy and the φcomponent of the angular momentum will be conserved yielding two constants of motion. Carter (1968) showed that the HamiltonJacobi equation is separable in this system giving the constant related to the θ-motion of the particle. It is known as the Carter constant (Misner, Thorne & Wheeler 1973, hereinafter MTW). Hence all the four first integrals are obtained as the rest mass of the particle is also a constant of motion which gives the remaining integral.

From Equation (2.3) we have

$$\frac{\partial\mathscr{L}}{\partial\dot{t}} = p_t + eA_t = -\mu E \quad (2.5)$$

$$\frac{\partial\mathscr{L}}{\partial\dot{\phi}} = p_\phi + eA_\phi = \mu L \quad (2.6)$$

where E and L are the energy and the ϕ-component of the angular momentum per unit rest mass of the particle as measured by an observer at infinity.

The rest mass μ of the particle gives another first integral

$$-\mu^2 = g^{ij}p_i p_j. \quad (2.7)$$

Now, on substituting Equations (2.5) and (2.6) in (2.7) we obtain

$$g_{\phi\phi}(E + eA_t)^2 + 2g_{t\phi}(E + eA_t)(L - eA_\phi) + g_{tt}(L - eA_\phi)^2 + \psi\,(g^{rr}p_r^2 + g^{\theta\theta}p_\theta^2 + \mu^2) = 0, \quad (2.8)$$

Which gives

$$E = -eA_t + \omega(L - eA_\phi) + (\sqrt{-\psi}/g_{\phi\phi})[(L - eA_\phi)^2 + g_{\phi\phi}(g^{rr}p_r^2 + g^{\theta\theta}p_\theta^2 + \mu^2)]^{1/2}, \tag{2.9}$$

where

$$\psi = g_{tt}g_{\phi\phi} - g_{t\phi}^2 < 0 \quad \text{for} \quad r > r_+ \tag{2.10}$$

$$\omega = -g_{t\phi}/g_{\phi\phi} > 0.$$

The event horizon r_+ is given by the larger root of $\psi = 0$. It can be easily verified that $\psi = 0 \Leftrightarrow \Delta = 0$.

For convenience we introduce the dimensionless quantities

$$\bar{r} = r/m, \quad \bar{a} = a/m, \quad l = L/m, \quad \bar{Q} = Q/m$$

$$\bar{A}_t = A_t/m, \quad \bar{A}_\phi = A_\phi/m, \quad \bar{e} = e/\mu$$

and drop the bars on these symbols in further discussion.

2.2 *The Effective Potential*

By the symmetry of the metric and the electromagnetic field, it follows that the particle commencing its motion with $p_\theta = 0$ in the equatorial plane will stay in the plane for all time, *i.e.* $p_\theta = 0$ all through the motion. This can also be verified by considering the equations of motion

$$\ddot{x}^i + \Gamma^i_{kl}\dot{x}^k\dot{x}^l = eF^i_k\dot{x}^k \tag{2.11}$$

for the θ-coordinate. The Lorentz force term on the right gives a force directed in the $\theta = \pi/2$ plane for A_i given in Equation (2.2) and $F_{ik} = A_{k,i} - A_{i,k}$. Henceforth we shall consider motion in the equatorial plane and set $p_\theta = 0$. As our main aim in this investigation is to analyse negative energy states, the restriction of motion in the equatorial plane will not matter much.

The effective potential for radial motion could be obtained by putting $p_r = p_\theta = 0$ in Equation (2.9). We write

$$V = -eA_t + \omega(L - eA_\phi) + (\sqrt{-\psi}/g_{\phi\phi})[(L - eA_\phi)^2 + g_{\phi\phi}]^{1/2},$$

$$\omega = -g_{t\phi}/g_{\phi\phi} > 0. \tag{2.12}$$

The positive sign for the radical is chosen to ensure that the 4-momentum of the particle is future directed. The quantity ω represents the angular velocity of a locally nonrotating observer (LNRO) at a given r and θ. That is, a particle with $L = 0$ will have $d\phi/dt = \omega \neq 0$.

Equations (2.8) and (2.12) can be rewritten as

$$\alpha E^2 - 2\beta E + \gamma = 0, \tag{2.13}$$

and

$$V = \frac{\beta + (\beta^2 - \alpha\gamma)^{1/2}}{\alpha}, \tag{2.14}$$

where

$$\alpha = (r^2 + a^2) - \Delta a^2,$$

$$\beta = (r^2 + a^2)(al + eQr) - \Delta al,$$
$$\gamma = (al + eQr)^2 - (r^2 + l^2). \qquad (2.15)$$

The effective potential at the horizon reads as

$$V(r_+) = [-eA_t + \omega(l - eA_\phi)](r_+),$$
$$V(r_+) = \frac{eQ}{r_+} + \frac{a}{r_+^2 + a^2}\left(l - \frac{eQa}{r_+}\right), \qquad (2.16)$$

where

$$\omega(r_+) = \frac{a}{r_+^2 + a^2}.$$

$V(r_+)$ can become negative if $-eA_t < 0$ and $(l - eA_\phi) < 0$. It should be noted that it is the sign of $(l - eA_\phi)$ which is relevant for V getting negative (Dadhich 1983). The particle rotates slower than the LNRO if $l - eA_\phi < 0$. This can be seen from the following.

The angular velocity $\Omega = d\phi/dt$ of a particle can be obtained by using Equations (2.5) and (2.6),

$$\Omega - \omega = (-\psi/g_{\phi\phi}^2)(l - eA_\phi)[E + eA_t - \omega(l - eA_\phi)]^{-1},$$

which, in view of Equations (2.9) and (2.10), directly relates the sign of $(l - eA_\phi)$ to $\Omega - \omega$. As argued by Dadhich (1985), $\Omega \;\; \omega \gtrless 0$ defines co/counter-rotation relative to an LNRO. It is the LNRO frame that is physically meaningful in these considerations.

3. The negative energy states

In this section we shall discuss the behaviour of the effective potential in relation to the occurrence of negative energy states (NES).

The NES could occur due to the electromagnetic interaction (as in the ReissnerNordstrøm case) as well as due to the counter-rotating orbits (as in the Kerr case). The Kerr-Newman solution represents the gravitational field of a charged and rotating black hole. The rotation of a black hole also gives rise to the magnetic dipole field in addition to the usual electrostatic field. The presence of electromagnetic field will favour the occurrence of NES (Dhurandhar & Dadhich 1984a, b) (i) by allowing larger negative values for energy, and (ii) by increasing the region of occurrence of NES. It is also known to cause in certain situations the splitting of NES region into two disjoint patches (Dhurandhar & Dadhich 1984a, b). However, in the Kerr-Newman field it turns out that NES may occur only in one patch extending upto the horizon (Prasanna 1983) as in the Kerr case. In the following we shall investigate NES with reference to counterrotation and electromagnetic interaction.

3.1 *The Effective Potential Curves*

Let us first look at some typical plots of the effective potential which exhibit its dependence on the parameters l and $\lambda = eQ$. Fig. 1 (a) shows the effective potential V for fixed $\lambda = -5$ and for various values of $l = -100, -50, -10, 0, 5$. It depicts (i) V is large negative for large negative l, (ii) NES region extends beyond the ergosphere

r = 2, and (iii) as l becomes less negative, V becomes less negative but NES region enlarges. It is interesting to see that $V < 0$ even for $l = 0$ and $l = 5$. This is in contrast to the Kerr case, and is purely due to the electromagnetic interaction.

In Fig. l(b) V is plotted for fixed $l = -10$ and for various values of $\lambda = -10, -5, -2, 0, 5$. It shows that larger negative λ implies larger negative V as well as enlarged NES region. Hereagain we have the occurrence of NES for $\lambda = 0$ and $\lambda = 5$ which is in contrast with the ReissnerNordstrϕm case (Denardo & Ruffini 1973). The contribution due to counter-rotation ($\Omega - \omega$) dominates over the electrostatic term. These plots are in agreement with Prasanna's results (1983).

3.2 *The Single-Band NES Structure*

The V curves in Figs 1 and 2 exhibit the singleband NES structure as also noted by Prasanna (1983). We establish this character analytically.

From Equations (2.13) and (2.14), $V = 0$ requires $\gamma = 0$ and $\beta < 0$. From Equation (2.15) $\gamma = 0$ gives

$$(al + eQr)^2 - \Delta(r^2 + l^2) = 0 \tag{3.1}$$

We now show that there is only one root for the above equation for $r > r_+ = 1 + (1 -$

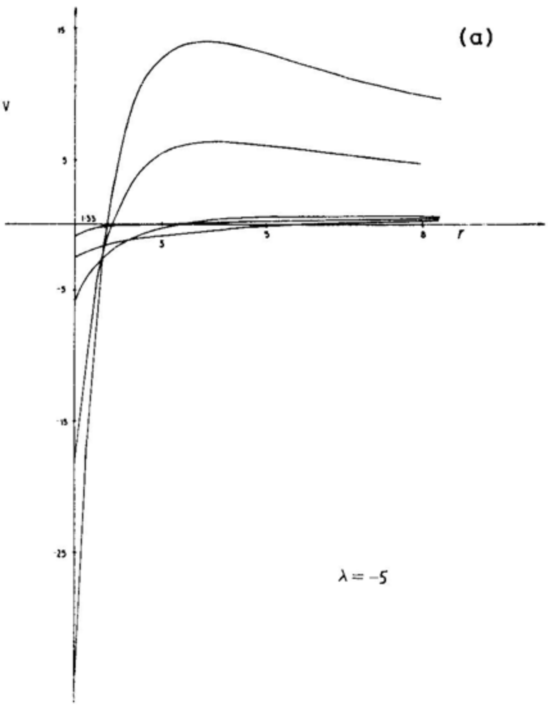

Figure 1. The effective potential V is plotted for a = 0.8 and Q =0.5.The vertical axis is drawn at the horizon (r_+ = 1.33). (a) l takes the values −100, −50, −10, 0, 5; (b) λ ranges through −10, − 5, − 2, 0, 5. The curve corresponding to a particular value of l and a particular value of λ can be picked up from the property that $V(r_+)$ is a monotonically increasing function of l and λ.

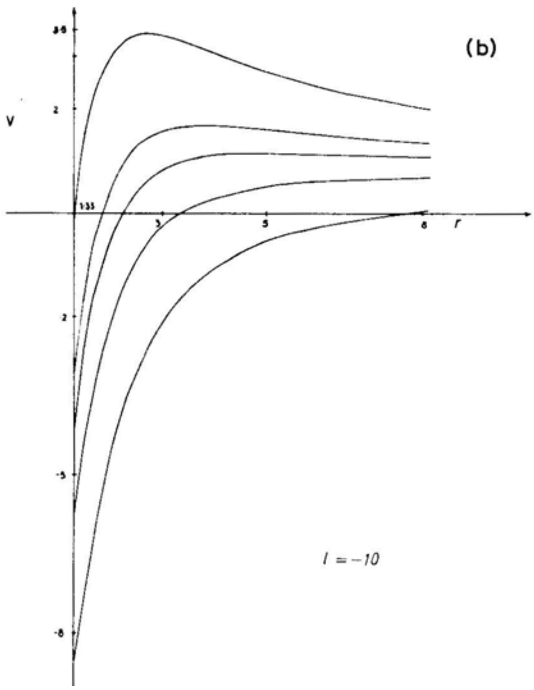

Figure 1. Continued.

$a^2 - Q^2)^{1/2}$. Write $R = r - r_+$. The above equation then reads as

$$R^4 + AR^3 + BR^2 + CR + D = 0, \tag{3.2}$$

where

$$A = 2(1 + 2\sqrt{1 - a^2 - Q^2})$$

$$B = 5r_+^2 - 4r_+ + l^2 - \lambda^2$$

$$C = 2[r_+^3 - r_+^2 + r_+(l^2 - \lambda^2) - l^2 - al\lambda]$$

$$D = -(al + eQr_+)^2.$$

To establish the result we apply Descartes' rule of signs. As $A > 0$ and $D < 0$, the above equation can have more than one positive root only when $B < 0$ and $C > 0$. We now show that this is not possible.

Let $B < 0$, which implies

$$\lambda^2 > 5r_+^2 - 4r_+ + l^2, \tag{3.3}$$

which makes

$$C < -2\,(4r_+^3 - 3r_+^2 + l^2 + al\lambda).$$

If $l\lambda > 0$, then $C < 0$. However, for $l\lambda < 0$, $C < 0$ will require

$$4r_+^3 - 3r_+^2 + l^2 > |al\lambda|.$$

Squaring both sides of the above inequality and using (3.3) we deduce $C < 0$ for this case too. This proves the result. Thus $\gamma = 0$ has only one root $r > r_+$. As $r \to \infty$, $V \to$ l, and hence the NES band will occur only when $V < 0$ at the horizon.

The single-band nature of NES prescribes a linear relationship between l and λ, which could be inferred from $V(r) < 0$. From Equation (2.16) this will imply,

$$l < -\lambda r_+ /a.$$

3.3 *The Extent of the NES Band*

To find the extent of the NES band we consider the quartic Equation (3.1) in various limits as the exact solution is not easily obtainable. We take $|l| \gg 1$ and $|\lambda| \gg 1$ in $V = 0$ for larger r. Then the quartic reduces to a cubic

$$r^3 - 2r^2 + (a^2 + Q^2 + l^2 - \lambda^2)r - 2l(a\lambda + l) = 0 \tag{3.4}$$

by dropping Q^2 terms as $Q^2 \gg 1$.

Case (*i*): Let $|\lambda| \sim |l|$, $l(a\lambda + l) > 0$. For large r, terms in r^2 and r can be neglected implying

$$r \simeq [2l(a\lambda + l)]^{1/3}$$

Case (ii): $|l - \lambda| \gg 1$, then Equation (3.4) reduces to

$$r^2 \simeq (l^2 - \lambda^2 + a^2 + Q^2)$$

by neglecting r^2 and the constant terms. Since, a, $Q < 1$ and if $l^2 - \lambda^2 > 0$ then $r \sim \sqrt{l^2 - \lambda^2}$

In the general case we need to resort to numerical computations. Table 1 below gives the extent of NES. It gives the root of $V = 0$ for various values of / and λ for fixed $a(= 0.8)$ and Q (= 0.5). The horizon in this case is at $r_+ = 1.3317$.

It is apparent from the table that for a fixed $\lambda < 0$, the value of $r = r_0$, say, where V gets negative, increases as l increases until l becomes positive and dominant, then it drops off below r_+. On the other hand, for $\lambda > 0$, r_0 decreases as / increases and there obviously exists no r_0 for $l > 0$. For fixed $l < 0$, it decreases as X becomes less negative but it slightly increases for $|\lambda|$ small and then steadily decreases as X increases further in the positive range. For $l > 0$, only large negative values of λ give $r_0 > r_+$. The large negative λ favours large values for r_0, as is borne out by the special'cases discussed above.

Table 1. Roots of $V = 0$ for $a = 0.8$, Q = 0.5 and various values of l and λ.

λ \ l	−100	−50	−10	0	10	50
−50	3.6384	> 10.0	> 10.0	> 10.0	> 10.0	> 10.0
−10	2.0584	2.3020	7.7847	> 10.0	3.9459	—
−1	1.8489	1.8318	1.7037	—	—	—
0	1.8659	1.8655	1.8532	—	—	—
1	1.8489	1.8318	1.7037	—	—	—
5	1.7843	1.7107	1.3564	—	—	—
50	1.3538	—	—	—	—	—

3.4 *The Factors Causing NES*

From Equation (2.12) it is seen that V can be negative only when $\lambda = eQ < 0$ (*i.e.* $eA_t < 0$) and /or $(l - eA_\phi) < 0$. Here we wish to compare the contributions of these factors in rendering $V < 0$. There are the following six possible cases.

$$\begin{array}{llll}
(1) & -eA_t < 0 & -eA_\phi > 0 & l > 0, \\
(2) & -eA_t < 0 & -eA_\phi > 0 & l < 0, \\
(3) & -eA_t < 0 & -eA_\phi < 0 & l > 0, \\
(4) & -eA_t < 0 & -eA_\phi < 0 & l < 0, \\
(5) & -eA_t > 0 & (l - eA_\phi) < 0 & l > 0, \\
(6) & -eA_t > 0 & (l - eA_\phi) < 0 & l < 0.
\end{array}$$

One can immediately see that case (3) is not possible because the conditions put on the parameters are inconsistent in view of Equation (2.2).That is, $\lambda < 0$and $l > 0$ do not permit counter-rotating orbit $(\Omega - \omega < 0)$.

The second law of the black hole physics rules out case 5. It implies (MTW),

$$\delta m \geqslant (a\delta J + Q\delta Q r_+)/2r_+,$$

where $\delta m = \mu E$, $\delta J = \mu m l$, $\delta Q = e\mu$.

Clearly $e>0$, and $l > 0$ does not allow $\delta m < 0$, thus ruling out NES. That is, the magnetic field alone cannot make $V < 0$.

The rest of the four cases allow for the NES. In the first case, the electrostatic energy is responsible for the NES while in case 2 it is the electrostatic and rotation, in case 6 the rotation and magnetic field, whereas in case 4 all the three factors join hands.

We shall consider the cases 1, 2 and 6 for $Q \to 0$ but $\lambda = eQ$ finite.

From Equation (2.16), $V(r_+) < 0$ gives

$$\frac{\lambda}{r_+} + \omega\left(l - \frac{a\lambda}{r_+}\right) < 0,$$

Where $\omega(r_+) \simeq \frac{a}{2r_+}$ by neglecting Q^2. Then

$$l < -\lambda\left(\frac{2}{a} - \frac{a}{r_+}\right). \tag{3.5}$$

In case 1, the inequality (3.5) gives

$$\frac{l}{-\lambda} < \frac{2}{a} - \frac{a}{r_+}$$

which, in the extreme case $a \to 1$, implies $1 < |\lambda|$. In case 2, it will always be satisfied, while in case 6 it gives

$$\frac{l}{\lambda} < \frac{a}{r_+} - \frac{2}{a},$$

which will imply for $a \to 1$, $|l| > \lambda$.

4. The energy extraction

In this section we consider the process of energy extraction from the black hole. In this process proposed by Penrose (1969), it is envisaged that a particle falling onto a black hole splits up into two fragments at some $r > r_+$ where $V < 0$. Then, if one of the fragments has negative energy (relative to infinity), it will be absorbed by the black hole while the other fragment will come out, by conservation of energy, with the energy greater than the parent particle. This will result in extraction of energy from the black hole. In the case of the Kerr-Newman black hole, the extracted energy may be provided by the rotational and/or the electromagnetic energy (Christodoulou 1970). In the following we shall first consider the conservation equations for the 4-momenta of the participating particles, and then give a recipe for energy extraction.

At the point of split, we assume that the 4-momentum is conserved, *i.e.*,

$$P_1 = P_2 + P_3 \tag{4.1}$$

where p_i ($i = 1, 2, 3$) denotes the 4-momentum of the ith particle. The above relation stands for the following three relations.

$$E_1 = \mu_2 E_2 + \mu_3 E_3 \tag{4.2}$$

$$l_1 = \mu_2 l_2 + \mu_3 l_3 \tag{4.3}$$

$$\dot{r}_1 = \mu_2 \dot{r}_2 + \mu_3 \dot{r}_3 \tag{4.4}$$

where we have set $\mu_1 = 1$. The other conservation relation follows from the conservation of charge,

$$\lambda_1 = \mu_2 \lambda_2 + \mu_3 \lambda_3. \tag{4.5}$$

The quantities μ_i ,l_i; λ_i, E_i r_i refer to the ith particle. These relations contain in all eleven parameters, of which 7 can be chosen freely. The choice of these parameters will be constrained by the requirements that particle 1 should reach the point of split where $V < 0$ for some suitable I, λ values such that particle 2 can have $E_2 < 0$ and particle 3 has an escape orbit. To ensure uninterrupted progress of particle 1 down to the horizon, we set $l_1 = 0 = \lambda_1$. The l and λ parameters for particle 2 should be so chosen that $E_2 < 0$. We further chose $\dot{r}_2 = 0$ which will imply $E_2 = V$ at the point of split. Such a choice is favourable for high efficiency of the process. (For further discussion refer to Dhurandhar & Dadhich 1984b.)

For these calculations we assume $Q \ll 1$. This assumption is realistic as can be seen from the following relation

$$Q\,(\text{metres}) = (G / \varepsilon_0 c^A)^{1/2}\, Q\,(Coulombs).$$

Though Q could be small, $eQ = \lambda$ can produce the Lorentz force on a particle of the charge/mass ratio of an electron, comparable to the corresponding gravitational force. So we neglect Q in the metric but retain λ in the equations of motion.

We shall now adopt the scheme for calculations due to Parthasarathy *et al.* (1985). From Equation (2.8) we can readily write the equations for radial motion of the particle,

$$\dot{r}^2 = \frac{1}{r^3}\left[R\bar{E}^2 - 4a\bar{E}\bar{L} - (r-2)\bar{L}^2 - r\Delta\right], \tag{4.6}$$

where

$$R = r(r^2 + a^2) + 2a^2,$$
$$\overline{E} = E + eA_t,$$
$$\overline{L} = l - eA_\phi.$$

Since we have taken $\dot{r}_2 = 0$, which means

$$\dot{r}_1 = \mu_3 \dot{r}_3, \tag{4.7}$$

by writing Equation (4.6) for particles 1 and 3 and using Equations (4.2), (4.3), (4.5) and (4.7) we obtain E_1 as follows

$$E_1 = \frac{[R\overline{E}_2^2 - 4a\overline{E}_2\overline{L}_2^2 - (r-2)\overline{L}_2^2]\mu_2^2 + r\Delta(1-\mu_3^2)}{2\mu_2(R\overline{E}_2 - 2a\overline{L}_2)}. \tag{4.8}$$

For the parent particle to be thrown from infinity $E_1 \geqslant 1$, and Equation (4.8) reduces to the inequality

$$\mu_2^2[R\overline{E}_2^2 - 4a\overline{E}_2\overline{L}_2 - (r-2)\overline{L}_2^2] + r\Delta(1-\mu_3^2) - 2\mu_2(R\overline{E}_2 - 2a\overline{L}_2) \geqslant 0. \tag{4.9}$$

The above inequality can be analysed in $\mu_2 - \mu_3$ plane. The equality sign gives the boundary of the region for the permissible values of μ_2 and μ_3. For the numerical values that we consider, this boundary is a hyperbola given by

$$\mu_2 = \frac{(R\overline{E}_2 - 2a\overline{L}_2) \pm [(R\overline{E}_2 - 2a\overline{L}_2)^2 - r\Delta(1-\mu_3^2)\{R\overline{E}_2^2 - 4a\overline{E}_2\overline{L}_2 - (r-2)\overline{L}_2^2\}]^{1/2}}{R\overline{E}_2^2 - 4a\overline{E}_2\overline{L}_2 - (r-2)\overline{L}_2^2}, \tag{4.10}$$

the relevant branch of which will be decided by the following considerations.

Squaring Equation (4.1), and using $p_2 \cdot p_3 < 0$ (future-pointing timelike vectors) we have,

$$\mu_2^2 + \mu_3^2 < 1. \tag{4.11}$$

This isaregion inside a unitcirclein the $\mu_2 - \mu_3$ plane.The inequality (4.9)requires μ_2 to be greater than the larger root or less than the smaller root given in Equation (4.10). It is the smaller root (*i.e.* with the negative sign for the radical) that gives the nonvoid intersection with the unit circle (4.11). However, μ_2 and μ_3 should be greater than zero. Fig. 2 shows the boundary of the permissible region.

The above prescription ensures that particle 1 from infinity reaches the desired splitting point, and particle 2 has negative energy. By Equation (4.2), particle 3 has greater energy than the incident particle. It now remains to ensure that particle 3 escapes to infinity. This further restricts the allowed region for μ_2 and μ_3. For particle 3 to escape to infinity two conditions must be satisfied. The particle must bounce outside the horizon and then it should continue its motion uninterrupted. That is,

(i) $E_3 < V_3 \qquad r_0 > r > r_+$

(ii) $E_3 > V_3 \qquad r > r_0$

where r_0 is the point of split. Numerical computations to this effect show that for $0 \leqslant \mu_3 < 1$, and for small values of μ_2, the particle does not escape, while for μ_2 close to the hyperbolic boundary the particle always escapes. Therefore, for a critical value of μ_2, say μ_{2c}, we have the particle escaping to infinity for $\mu_2 > \mu_{2c}$. So the allowed region

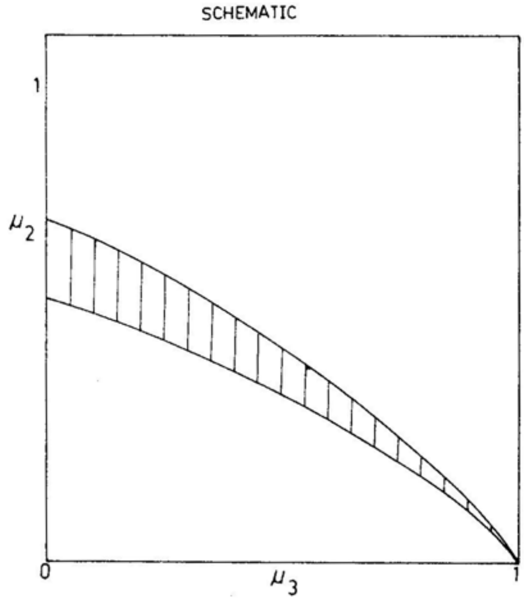

Figure 2. Schematic diagrams for μ_2 (max) and μ_2 (crit) are drawn. Here the numbers involved are too inconvenient to permit a figure to scale. The shaded region lying between μ_2 (max) and μ_2 (crit) is the allowed region.

now shrinks between μ_{2c} and the hyperbola. This is shown in Fig. 2 by the shaded region.

5. Efficiency of the process

The most important question in the black hole energetics is the efficiency of the energy extraction process. It is therefore very pertinent to examine how efficient the Penrose process is. The maximum efficiency of the process in extracting rotational energy of the black hole (Chandrasekhar 1983) turns out to be approximately 20.7 per cent. We shall rederive this result independently following the detailed analysis done by Parthasarathy *et al.* (1985) and show that the presence of charge on the black hole reduces the efficiency of the process. However, it further turns out that there exists no upper limit on the efficiency when one considers the process with electromagnetic interaction. Our numerical results show that there do occur events with more than 100 per cent efficiency.

5.1 *Efficiency in the Absence of Electromagnetic Interactions*

The maximum efficiency is obtained if we take the radial components of the velocities to be zero, the point of split being as close as possible to the horizon (MTW). We first derive the expression for efficiency at some $r > r_+$ and then take the limit as $r \to r_+$.

Let $U_i (i=1,2,3)$ denote the 4-velocity of the ith particle at the point of split,

$$U_1 = f_1(1, 0, 0, \Omega_1) \tag{5.1}$$

where

$$f_1 = -(g_{tt} + g_{t\phi}\Omega_1)^{-1}, \tag{5.2}$$

$$\Omega_1 = \frac{-g_{t\phi}(1+g_{tt}) + (-\psi(1+g_{tt}))^{1/2}}{g_{t\phi}^2 + g_{\phi\phi}}, \tag{5.3}$$

Ω_1 is the angular velocity of particle 1 with respect to the asymptotic Lorentz frame, and we have taken $E_1 = 1.f_1$ is obtained by considering unit length of the 4-velocity vector U_1.At the point of split, the light cone imposes restrictions on the angular velocity Ω of a future moving timelike particle that $\Omega_- < \Omega < \Omega_+$ where

$$\Omega_\pm = (-g_{t\phi} \pm \sqrt{-\psi})/g_{\phi\phi}. \tag{5.4}$$

The best result will be obtained by choosing the angular velocity of the second particle to be $\Omega_2 \to \Omega_-$ and that of the third to be $\Omega_3 \to \Omega_+$. In the limit,

$$\mu_2 U_2 = k_2(1, 0, 0, \Omega_-), \tag{5.5}$$

$$\mu_3 U_3 = k_3(1, 0, 0, \Omega_+). \tag{5.6}$$

The conservation of 4-momentum can be rewritten as

$$U_1 = \mu_2 U_2 + \mu_3 U_3. \tag{5.7}$$

By algebraically manipulating the above equations we obtain

$$\mu_3 E_3 = \left(\frac{\Omega_1 - \Omega_-}{\Omega_+ - \Omega_-}\right)\left(\frac{g_{tt} + g_{t\phi}\Omega_+}{g_{tt} + g_{t\phi}\Omega_1}\right). \tag{5.8}$$

The efficiency η is defined as

$$\eta = \frac{\text{gain in energy}}{\text{input energy}},$$

$$\begin{aligned} \eta &= \frac{\mu_3 E_3 - E_1}{E_1} \\ &= \mu_3 E_3 - 1 \quad \text{for} \quad E_1 = 1. \end{aligned} \tag{5.9}$$

Now we take the limit as split point tends to r_+. Then

$$\mu_3 \mathrm{E}_3 = [(1+g_{tt})^{1/2} + 1]/2. \tag{5.10}$$

For the extreme Kerr-Newman black hole $(a^2 + Q^2 = 1)$, the relevant g_{ij} at the horizon are given as

$$\begin{aligned} g_{tt} &= +(1-Q^2), \\ g_{t\phi} &= -(2-Q^2)(1-Q^2)^{1/2}, \\ g_{\phi\phi} &= (2-Q^2)^2. \end{aligned} \tag{5.11}$$

Putting in these values in Equation (5.10) we obtain

$$\mu_3 E_3 = [1 + (2-Q^2)^{1/2}]/2 \tag{5.12}$$

which will imply

$$\eta = [(2 - Q^2)^{1/2} - 1]/2. \tag{5.13}$$

For $Q = 0$

$$\eta = \frac{\sqrt{2} - 1}{2} = 0.207$$

which is in agreement with the known result. Thus the presence of charge on the black hole decreases the maximum efficiency of the Penrose process in the absence of electromagnetic interaction (participating particles being uncharged).

5.2 *Efficiency in the Presence of Electromagnetic Interactions*

When we consider the participating particles being charged, the t-component of the conservation Equation (5.7) will read as

$$E_1 + e_1 A_t = \mu_2 (E_2 + e_2 A_t) + \mu_3 (E_3 + e_3 A_t). \tag{5.14}$$

Here, the charges on particles can be chosen arbitrarily large and hence this will not give any upper limit on the efficiency (Parthasarathy *et al.*, 1985). In fact the term $eA_t = - eQ/r$ can assume arbitrarily large values for large e. This is borne out by the numerical example considered below.

Let us assume $a = 0.8$, $Q = 0.5$. The particle 1 comes from infinity, and has parameters $\mu_1 = 1$, $E_1 = 1$, $l_1 = 0$, $e_1 = 0$. The split is taken to occur at $r = 4.0$. For $l_2 = -10$ and $e_2 = -50$ we give in Table 2 the maximum efficiency for various values of μ_3. For η (max), $\mu_2 = \mu_2$ (max) given by the hyperbolic boundary, and μ_{2c} defines the lower boundary of the permissible region (see Fig. 2). The first row of the table gives an instance when efficiency is 104 per cent.

6. Conclusion

The presence of electromagnetic fields around a black hole (inherent in the metric as in the present case, or externally superposed) influences the behaviour of negative energy states for charged particles in the following two ways (Dhurandhar & Dadhich 1984a, b).

(a) The NES region is enlarged beyond the ergosphere $r = 2M$.
(b) E can have larger negative values.

Table 2. The maximum efficiency of the Penrose process for various values of μ_3, when electromagnetic interactions are included.

μ_3	μ_2 (max)	μ_{2c}	η (max)
0.001	0.2001	0.1199	1.045
0.191	0.1932	0.1930	1.009
0.381	0.1724	0.1723	0.901
0.581	0.1374	0.1373	0.718
0.761	0.0871	0.0870	0.455
0.951	0.0202	0.02019	0.105

Both these factors contribute positively to the energy extraction process. The former brings in NES at comfortable r-values, thereby increasing the probability of larger number of events yielding energy extraction, while the latter tends to increase the energy gain per event resulting in greater efficiency. For the Kerr-Newman black hole, large negative charge on the test particle (*i.e.* large $\lambda < 0$) causes (a), while both λ and l large and negative give rise to (b) (see Fig. 1).

It has been shown that the extraction of energy from the Kerr-Newman black hole is more efficient—in fact, there exists no upper bound on the efficiency when charged particles participate in the process (Table 2 shows an event of over 100 per cent efficiency)—in contrast to when uncharged particles are involved. In the latter case, the charge on the black hole reduces the maximum efficiency which is 20.7 per cent for the Kerr black hole. The electromagnetic extraction of black hole's energy is highly efficient.

As massive bodies cannot have significant charge on them, in our efficiency calculations we have taken $Q/M \ll 1$. We have hence neglected it in the metric but have retained its interaction with the test particle in the equations of motion. If a black hole acquires slight charge, our results would apply and will be indicative of the general behaviour of NES and energy extraction process.

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References

Abramowicz, M. A., Calvani, M., Nobili, L. 1983, *Nature,* **302**, 597.
Bardeen, J. M., Press, W. H., Teukolsky, S. A. 1972, *Astrophys. J.,* **178**, 347.
Blandford, R. D., Znajek, R. L. 1977, *Mon. Not. R. astr. Soc,* 1**79**, 433.
Carter, B. 1968, *Phys. Rev.,* **174**, 1559.
Chandrasekhar, S. 1983,77ie *MathematicalTheory of Black Holes,* Oxford University Press, New York.
Christodoulou, D. 1970, *Phys. Rev. Lett.,* **25**, 1596.
Dadhich, N. 1983, *Phys. Lett.,* **98A**, 103.
Dadhich, N. 1985, in *A Random Walk in Relativity and Cosmology,* Eds N. Dadhich, J. Krishna Rao, C. V. Vishveshwara & J. V. Narlikar, Wiley, Eastern, p. 72.
Denardo, G., Ruffini, R. 1973, *Phys. Lett.,* **45B**, 259.
Dhurandhar, S. V., Dadhich, N. 1984a, *Phys. Rev.,* **D29**, 2712.
Dhurandhar, S. V., Dadhich, N. 1984b, *Phys. Rev.,* **D30**, 1625.
Fishbone, L. G. 1973, *Astrophys.* J., **185**, 43.
Kozfowski, M., Jaroszynski, M., Abramowicz, M, A. 1978, *Astr. Astrophys.,* **63**, 209.
Mashhoon, B. 1973, *Astrophys. J.,* **181**, L65.
Misner, C. W., Thorne, K. S., Wheeler, J. A. 1973, *Gravitation,* W. H. Freeman, San Francisco.
Parthasarathy, S., Wagh, S. M., Dhurandhar, S. V., Dadhich, N. 1985, Astrophys. J., (submitted).
Penrose, R. 1969, *Rev. Nuovo Cimento,* 1 (Special Number), 252.
Prasanna, A. R. 1983, *Astr. Astrophys.,* **126**, 111.

Pringle J E. 1981, *A. Rev. Astr. Astrophys.,* 1**9**, 137.
Reest M. I, Begelman, M. C, Blandford, R. D, Phinney, E. S. 1982, *Nature,* **295**, 17.
Shakura, N. I., Sunyaev, R. A. 1973, *Astr. Astrophys.,* **24**, 337.
Vishveshwara, C. 1968, *J. Math. Phys.,* **9**, 1319.
Wagh, S. M., Dhurandhar, S. V., Dadhich, N. 1985, *Astrophys. J.*, **290**, 12.
Wald, R.. M. 1974, *Astrophys. J.,* **191**, 231.
Wheeler, J. A. 1971, in *Nuclei of Galaxies,* Ed. D. J. K. O'Connell, NorthHolland, Amsterdam, p. 539.