Entanglement of Formation for Gaussian States

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The entanglement of formation (EOF) is computed for arbitrary two-mode Gaussian states. Apart from a conjecture, our analysis rests on two main ingredients. The first is a four-parameter canonical form we develop for the covariance matrix, one of these parameters acting as a measure of EOF, and the second is a generalisation of the EPR correlation, used in the work of Giedke *et al* [Phys. Rev. Lett. **91**, 107901 (2003)], to noncommuting variables. The conjecture itself is in respect of an extremal property of this generalized EPR correlation.

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Entanglement is an essential resource for many quantum information processing tasks, and hence it is important to be able to quantify this resource. A reasonable set of demands lead, in the case of bipartite pure states, to a simple and unique measure for this resource: it is the von Neumann entropy of either subsystem [1, 2, 3]. For mixed states, however, many different entanglement measures continue to be under consideration [4]. One of these measures with an attractive physical motivation is the entanglement of formation (EOF) [5]. The asymptotic version of EOF is the entanglement cost [5, 6]. EOF is defined as an infimum:

$$\mathrm{EOF}\,(\rho) \equiv \inf\,\{\,\sum_j p_j E(\psi_j)\;|\;\rho = \sum_j p_j |\psi_j\rangle\langle\psi_j|\,\}\,.$$

The infimum is to be taken over all realizations of the given mixed state ρ as convex sum of pure states, and $E(\psi_j) \equiv S(\mathrm{tr}_B[|\psi_j\rangle\langle\psi_j|])$ where $S(\cdot)$ is the von Neumann entropy. EOF has been computed in closed form for *arbitrary* two-qubit states [7], and for highly symmetric states like the isotropic states [8] and the Werner states [9].

Gaussian states, whose entanglement originates in nonclassicality of the squeezing type [10], have played a distinguished role in quantum information in respect of continuous variable systems [11]. Their use in teleportation [12, 13] and quantum cryptography [14] has been demonstrated. Questions related to their separability [15, 16, 17, 18] and distillability [19] have been resolved. More recently, analytic expression for their EOF has been obtained in the *symmetric* case [20]. This notable achievement seems to be the first computation of EOF for states of infinite rank. These authors exploit a certain *extremality* that the two-mode-squeezed vacuum enjoys in respect of the Einstein-Podolsky-Rosen (EPR) correlation [21] on the one hand and entanglement on the other. Further analysis of EOF in this case has been made [22] from the viewpoint of Bures distance

An interesting Gaussian-state-specific generalisation of EOF, the *Gaussian entanglement of formation*, has also been explored [23, 24]. But the EOF of asymmetric Gaussian states has remained an open problem [25].

In this Letter we compute, under a conjecture, the EOF for arbitrary two-mode Gaussian states. Our analysis rests on two principal ingredients. The first one is a four-parameter canonical form we develop for the covariance matrix; one of these parameters proves to be a measure of EOF. The second one is a family of generalised EPR correlations for *noncommuting* pairs of nonlocal variables; this family is indexed by a continuous parameter θ. And the conjecture is in respect of an extremal property of this generalised EPR correlation.

Canonical Form for Covariance Matrix: Given a two-mode Gaussian state, with the mode on Alice's side described by canonical quadrature variables x_A, p_A and that on Bob's side by x_B, p_B, we can assume without loss of generality that the first moments of all four variables vanish [16, 20]. Such a zero-mean Gaussian state is fully described by the covariance matrix [16, 20]

$$V_G = \frac{1}{2}\begin{bmatrix} \alpha\beta n & 0 & \beta k_x & 0 \\ 0 & \alpha^{-1}\beta^{-1} n & 0 & -\beta^{-1} k_p \\ \beta k_x & 0 & \alpha^{-1}\beta m & 0 \\ 0 & -\beta^{-1} k_p & 0 & \alpha\beta^{-1} m \end{bmatrix}, \quad (1)$$

where the phase space variables are assumed to be arranged in the order $(x_A, p_A, x_B, p_B) \equiv \xi$, and we have retained through the parameters α, $\beta > 0$ the freedom of independent local unitary (i.e., symplectic) scalings on the A and B sides. This freedom will be used shortly.

Note that V_G is left with no correlation between the 'spatial' variables x_A, x_B and the 'momentum' variables p_A, p_B. Thus it is sometimes convenient to view V_G as the direct sum of 2×2 matrices:

$$V_G = X_G \oplus P_G\,,$$
$$X_G = \frac{\beta}{2}\begin{bmatrix} \alpha n & k_x \\ k_x & \alpha^{-1} m \end{bmatrix},\;\; P_G = \frac{\beta^{-1}}{2}\begin{bmatrix} \alpha^{-1} n & -k_p \\ -k_p & \alpha m \end{bmatrix}.$$

Let $|\Psi_r\rangle$ denote the standard two-mode-squeezed vacuum state with squeeze parameter r. It takes the Schmidt form in the standard Fock basis:

$$|\Psi_r\rangle \;=\; \sum_{n=0}^{\infty} c_n |n\rangle_A \otimes |n\rangle_B \equiv \sum_{n=0}^{\infty} c_n |n, n\rangle\,,$$
$$c_n \;=\; \tanh^n r/\cosh r\,. \quad (2)$$

Denoting by E_r the entanglement of $|\Psi_r\rangle$, we have

$$E_r = \cosh^2 r \log_2(\cosh^2 r) - \sinh^2 r \log_2(\sinh^2 r)\,. \quad (3)$$

The covariance matrix of $|\Psi_r\rangle$ has the form

$$\begin{aligned} V_{\Psi_r} &= X_{\Psi_r} \oplus P_{\Psi_r}\,, \\ X_{\Psi_r} &= \frac{1}{2}\begin{bmatrix} C & S \\ S & C \end{bmatrix},\ P_{\Psi_r} = \frac{1}{2}\begin{bmatrix} C & -S \\ -S & C \end{bmatrix}, \\ C &\equiv \cosh 2r,\quad S \equiv \sinh 2r\,. \end{aligned} \quad (4)$$

Proposition 1: Given a two-mode covariance matrix V_G, the local scale parameters α, β can be so chosen that V_G gets recast in the form

$$V_0 = \frac{1}{2}\begin{bmatrix} C + u\,c^2 & 0 & S + u\,cs & 0 \\ 0 & C + v\,c^2 & 0 & -S - v\,cs \\ S + u\,cs & 0 & C + u\,s^2 & 0 \\ 0 & -S - v\,cs & 0 & C + v\,s^2 \end{bmatrix},$$
$$C \equiv \cosh 2r_0,\ S \equiv \sinh 2r_0;\quad c \equiv \cos\theta_0,\ s \equiv \sin\theta_0\,.$$

Note: We will call V_0 *the canonical form* of a two-mode covariance matrix; our results below will justify this elevated status. We assume without loss of generality $n \geq m$ or, equivalently, $0 < \theta_0 \leq \pi/4$. For a given V_G there will be two solutions for the above form. Canonical form will always refer to the one with the smaller squeeze parameter r_0, which is ensured by the restriction

$$\tan\theta_0 \geq \tanh r_0\,. \quad (5)$$

This condition proves central to our analysis. Its origin may be appreciated by inverse two-mode-squeezing the Gaussian state V_0 until it becomes just separable, and noting that there exists a range of further squeezing in which the *mixed* Gaussian state remains separable before becoming inseparable again. The parameters $u,\, v \geq 0$. *The essence of the canonical form is that V_0 differs from the covariance matrix of a two-mode-squeezed vacuum $|\Psi_{r_0}\rangle$ by a positive matrix which is a direct sum of two singular 2×2 matrices which are, modulo signature of the off-diagonal elements, multiples of one another.*

Proof: The canonical form demands, as a necessary condition, that α, β, and r be chosen to meet

$$\det(X_G - X_{\Psi_r}) = 0\,,\quad \det(P_G - P_{\Psi_r}) = 0\,. \quad (6)$$

These being two constraints on three parameters, one will expect to get a one-parameter family of solutions to these constraints. For each such solution we may denote the vector annihilated by the singular matrix $X_G - X_{\Psi_r}$ by $(\sin\theta,\, -\cos\theta)$, and that annihilated by $P_G - P_{\Psi_r}$ by $(\sin\theta',\, \cos\theta')$. The canonical form corresponds to that solution for which $\theta' = \theta$; it is this degenerate value that equals θ_0 of the canonical form.

That there exists such a degenerate value can be seen as follows. We may fix the scale parameter α through $\alpha = \sqrt{m/n}$, and then solve Eqs. (6) for β and r, the smaller r being the relevant one. We will find $\theta = \pi/4$ and $\theta' < \pi/4$ in this case. On the other hand if we take $\alpha = \sqrt{n/m}$ and then solve Eqs. (6), we will find $\theta' = \pi/4$ and $\theta < \pi/4$. It follows from continuity that there exists an intermediate value α_0 for the parameter α, in the range $\sqrt{m/n} < \alpha < \sqrt{n/m}$, for which $\theta' = \theta$ ($< \pi/4$ since $n > m$). And this yields the canonical form.

Viewed alternatively, the canonical form V_0 places the following two requirements on the scale factors α, β:

$$\begin{aligned} &\frac{\det X_G - 1/4}{\det P_G - 1/4} = \frac{\mathrm{tr}(\sigma_3 X_G)}{\mathrm{tr}(\sigma_3 P_G)}\,, \\ &\det(X_G - \sigma_3 P_G \sigma_3) = 0\,, \end{aligned} \quad (7)$$

where σ_3 is the diagonal Pauli matrix. These are simultaneous equations in α, β, and solving these equations yields, in terms of n, m, k_x, k_p, the values of α, β corresponding to the canonical form.

Two special cases may be noted. If $m = n$ we have $\alpha = 1$ (since $\sqrt{n/m} = \sqrt{m/n}$), and hence $\beta = \sqrt{(n-k_p)/(n-k_x)}$, so that the canonical squeeze parameter r_0 is given by $e^{-2r_0} = \sqrt{(n-k_x)(n-k_p)}$, reproducing the results of Ref. [20]. The parameter θ_0 always equals $\pi/4$ in this (symmetric) case. On the other hand, if $k_x = k_p = k$, the canonical form corresponds to $\alpha = \beta = 1$, and one obtains r_0 by simply solving

$$\det\begin{bmatrix} n - \cosh 2r_0 & k - \sinh 2r_0 \\ k - \sinh 2r_0 & m - \cosh 2r_0 \end{bmatrix} = 0\,, \quad (8)$$

which yields this closed-form expression for r_0:

$$\begin{aligned} \cosh(2\eta - 2r_0) &= \frac{nm - k^2 + 1}{\sqrt{(n+m)^2 - 4k^2)}}, \\ \mathrm{e}^{\pm 2\eta} &\equiv \frac{(n+m)\,\pm\,2k}{\sqrt{(m+n)^2\,-\,4k^2}}. \end{aligned}$$

Generalised EPR Correlation: To proceed further, we need to generalise the familiar EPR correlation [20]. Given any bipartite state $|\psi\rangle$, define

$$\begin{aligned} x_\theta &= \sin\theta\, x_A - \cos\theta\, x_B,\ p_\theta = \sin\theta\, p_A + \cos\theta\, p_B\,, \\ \Lambda_\theta(\psi) &= \langle\psi|(x_\theta)^2|\psi\rangle + \langle\psi|(p_\theta)^2|\psi\rangle\,. \end{aligned} \quad (9)$$

In defining $\Lambda_\theta(\psi)$ we have assumed $\langle\psi|x_\theta|\psi\rangle = 0 = \langle\psi|p_\theta|\psi\rangle$; if this is not the case then x_θ and p_θ in $\Lambda_\theta(\psi)$ should be replaced by $x_\theta - \langle\psi|x_\theta|\psi\rangle$ and $p_\theta - \langle\psi|p_\theta|\psi\rangle$ respectively. Clearly, the usual EPR correlation [20] corresponds to $\theta = \pi/4$. While $x_{\pi/4}$, $p_{\pi/4}$ commute, the generalised EPR (nonlocal) variables x_θ, p_θ *do not commute*, and hence the name generalised EPR correlation for $\Lambda_\theta(\Psi)$; indeed, we have $[x_\theta,\, p_\theta] = -i\cos 2\theta$. For the two-mode-squeezed vacuum $|\Psi_r\rangle$ the generalised EPR correlation reads

$$\Lambda_\theta(\Psi_r) = \cosh 2r - \sin 2\theta \sinh 2r\,. \quad (10)$$

Let us combine the quadrature variables of the oscillators of Alice and Bob into boson operators $a = (x_A + ip_A)/\sqrt{2}$ and $b = (x_B + ip_B)/\sqrt{2}$. Then, $\Lambda_\theta(\psi)$ has this expression quadratic in the boson variables:

$$\begin{aligned}\Lambda_\theta(\psi) &= \langle\psi|\hat{\Lambda}_\theta|\psi\rangle, \\ \hat{\Lambda}_\theta &= 1 + 2\sin^2\theta\, a^\dagger a + 2\cos^2\theta\, b^\dagger b \\ &\quad - 2\cos\theta\sin\theta(ab + a^\dagger b^\dagger)\,. \end{aligned} \qquad (11)$$

We may call $\hat{\Lambda}_\theta$ the *generalised EPR operator.*

The entanglement of $|\Psi_r\rangle$ monotonically increases with increasing value of the squeezing parameter r. In order that $\Lambda_\theta(\Psi_r)$ be useful as an entanglement measure of $|\Psi_r\rangle$ it should, for fixed value of θ, decrease with increasing r. The restriction $\tan\theta \geq \tanh r$, encountered earlier in Eq. (5) from a different perspective, simply ensures this. Through the monotonic relationship (3) between r and E_r, we will view this constraint as a restriction on the allowed range of values of θ, for a fixed value of entanglement.

Given a squeezed state $|\Psi_r\rangle$, let us denote by $|\Psi'_r\rangle$ the state obtained from $|\Psi_r\rangle$ by independent local canonical transformations [16] $S_A,\, S_B \in Sp(2, R)$, acting respectively on the oscillators of Alice and Bob.

Proposition 2: We have $\Lambda_\theta(\Psi'_r) \geq \Lambda_\theta(\Psi_r)$, $\forall\,\theta$ in the range $1 \geq \tan\theta \geq \tanh r$ and for all $S_A,\, S_B \in Sp(2, R)$.

Proof: Clearly, $\Lambda_\theta(\Psi'_r) = \frac{1}{2}\{\cosh 2r[\sin^2\theta\,\mathrm{tr}(S_A S_A^T) + \cos^2\theta\,\mathrm{tr}(S_B S_B^T)] - \sin 2\theta \sinh 2r\,\mathrm{tr}\,(\sigma_3 S_A \sigma_3 S_B^T)\}$. If $e^{\pm\gamma_A}$ are the singular values of S_A, and $e^{\pm\gamma_B}$ those of S_B, then $\mathrm{tr}(S_A S_A^T) = 2\cosh 2\gamma_A$, $\mathrm{tr}(S_B S_B^T) = 2\cosh 2\gamma_B$, and $\mathrm{tr}(\sigma_3 S_A \sigma_3 S_B^T) \leq 2\cosh(\gamma_A + \gamma_B)$. Thus the difference $\Delta(\gamma_A,\, \gamma_B) \equiv \Lambda_\theta(\Psi'_r) - \Lambda_\theta(\Psi_r)$ obeys $\Delta(\gamma_A,\, \gamma_B) \geq \cosh 2r[\sin^2\theta(\cosh 2\gamma_A - 1) + \cos^2\theta(\cosh 2\gamma_B - 1)] - \sin 2\theta \sinh 2r[\cosh(\gamma_A + \gamma_B) - 1]$. It is easily seen that $\Delta(\gamma_A,\, \gamma_B)$ is extremal at $\gamma_A = \gamma_B = 0$ corresponding to the standard squeezed state $|\Psi_r\rangle$. To show that this extremum is indeed minimum we note that the determinant of the Hessian matrix of the right hand side, evaluated at $\gamma_A = 0 = \gamma_B$, is proportional to $\sin 2\theta \cosh 2r - \sinh 2r$, and hence is positive if and only if $\tan\theta \geq \tanh r$.

Once again we see a role for the requirement $\tan\theta \geq \tanh r$. Let the equivalence $V_G \sim V_0$ denote the fact that the corresponding Gaussian states are connected by a local canonical transformation. The fact that $M \equiv V_0 - V_{\Psi_{r_0}} \geq 0$ implies $\Lambda_{\theta_0}(\rho_{V_0}) \geq \Lambda_{\theta_0}(\Psi_{r_0})$. In view of Proposition 2 this implies $\Lambda_{\theta_0}(\rho_{V_G}) \geq \Lambda_{\theta_0}(\rho_{V_0}) \geq \Lambda_{\theta_0}(\Psi_{r_0}) = \cosh 2r_0 - \sin 2\theta \sinh 2r_0$ for any Gaussian state V_G connected to V_0 by local canonical transformation. This assigns an alternative meaning to the canonical parameter r_0:

Proposition 3: Given a Gaussian state described by $V_G \sim V_0$, the canonical squeeze parameter r_0 is the smallest r for which the matrix inequality $V_G - V_{\Psi'_r} \geq 0$ is true.

It is well known that the two-mode-squeezed vacuum has several extremal properties of interest to entanglement [20, 26]. It seems that this state enjoys one more such distinction, this time in respect of our generalised EPR correlation.

Conjecture 1: Among all bipartite states of fixed entanglement numerically equalling E_r, and for every θ in the range $\tanh r \leq \tan\theta \leq 1$, the two-mode-squeezed vacuum $|\Psi_r\rangle$ yields the least value for the generalised EPR correlation $\Lambda_\theta(\cdot)$. In other words, no state $|\psi\rangle$ with entanglement $E(|\psi\rangle) \leq E_r$ can yield a generalised EPR correlation $\Lambda_\theta(\psi) < \Lambda_\theta(\Psi_r)$, for any θ in the range $\tan\theta \geq \tanh r$

The special case $\theta = \pi/4$ is the basis of the important work of Ref. [20]. Hence the present assertion can be viewed as a generalisation of their Proposition 1.

The original EPR correlation $\Lambda_{\pi/4}(\cdot)$ continuously decreases to zero with increasing entanglement. But this is not true of the generalised EPR correlation $\Lambda_\theta(\cdot)$.

Let us denote by r_θ the value of r determined by a given value of θ through the equation $\tan\theta = \tanh r$, and let θ_r denote the value of θ so determined by r. Then, for a given numerical E_r, the relevant range for θ in Conjecture 1 is $\theta_r \leq \theta \leq \pi/4$.

Proposition 4: The generalised EPR correlation $\Lambda_\theta(\cdot)$ obeys the basic inequality $\Lambda_\theta(\cdot) \geq \cos 2\theta$. The two-mode-squeezed vacuum saturates this inequality if and only if the squeeze parameter r solves $\tanh r = \tan\theta$.

Proof: It is clear that the relations $\tan\theta = \tanh r$, $\sin 2\theta = \tanh 2r$, and $\cos 2\theta = (\cosh 2r)^{-1}$ are equivalent to one another, and so also are the inequalities $\tan\theta \geq \tanh r$, $\sin 2\theta \geq \tanh 2r$, and $\cos 2\theta \leq (\cosh 2r)^{-1}$. Now consider the transformation $(a,\, b) \to U(r)(a,\, b)U(r)^\dagger$ where $U(r) = \exp\{r(a^\dagger b^\dagger - ab)\}$ is the unitary two-mode-squeeze operation:

$$a \to a\cosh r - b^\dagger \sinh r, \quad b \to b\cosh r - a^\dagger \sinh r.$$

This implies the following transformation for the anti-commutator $\{b, b^\dagger\} \equiv bb^\dagger + b^\dagger b$:

$$\begin{aligned}\{b, b^\dagger\} &\to (b^\dagger b - a^\dagger a) + \frac{1}{2}(\{a, a^\dagger\} + \{b, b^\dagger\})\cosh 2r \\ &\quad - (ab + a^\dagger b^\dagger)\sinh 2r \\ &= \cosh 2r\,\hat{\Lambda}_{\theta_r}, \quad \theta_r \equiv \arctan(\tanh r).\end{aligned}$$

Since $\{b, b^\dagger\} \geq 1$, so is also its unitary transform $\cosh 2r\,\hat{\Lambda}_{\theta_r}$. That is, $\hat{\Lambda}_{\theta_r} \geq (\cosh 2r)^{-1} = \cos 2\theta_r$.

Thus, saturation of the inequality $\Lambda_{\theta_r}(\psi') \geq \cos 2\theta_r$ is equivalent to the condition $\langle\psi|\{b, b^\dagger\}|\psi\rangle = 1$, where $|\psi'\rangle = U(r)|\psi\rangle$. A pure state which satisfies $\langle\psi|\{b, b^\dagger\}|\psi\rangle = 1$, is of the form $|\psi\rangle = |\phi\rangle_A \otimes |0\rangle_B$, where $|\phi\rangle_A$ is *any* vector in Alice's Hilbert space $\mathcal{H}_A$. It follows that states saturating the inequality $\Lambda_{\theta_r}(\rho) \geq \cos 2\theta_r$ constitute the set $\{\rho = U(r)\rho_A \otimes |0\rangle_{B\,B}\langle 0|U(r)^\dagger\}$, where ρ_A is any (pure or mixed) state of Alice's oscillator. Finally, Conjecture 1 claims that among all these states saturating this inequality the two-mode-squeezed vacuum $|\Psi_{r_\theta}\rangle$, corresponding to the choice $\rho_A = |0\rangle_{A\,A}\langle 0|$, has the least entanglement.

Entanglement of Formation: With the canonical form and the generalised EPR correlations in hand, we are now fully equipped to compute the EOF of an arbitrary two-mode Gaussian state.

Proposition 5: Given an inseparable zero-mean two-mode Gaussian state ρ_{V_0} with covariance matrix V_0 specified in the canonical form by u, v, θ_0 and r_0 with $u,\, v \geq 0$ and $0 < \tanh r_0 \leq \tan\theta_0 \leq 1$, its EOF equals E_{r_0}, the entanglement of the squeezed vacuum $|\Psi_{r_0}\rangle$.

Proof: The fact that $M \equiv V_0 - V_{\Psi_{r_0}} \geq 0$ guaranties that ρ_{V_0} can be realized as a convex sum of displaced versions $D(\xi)|\Psi_{r_0}\rangle$ of the squeezed vacuum state $|\Psi_{r_0}\rangle$, all of which have the same entanglement E_{r_0} as $|\Psi_{r_0}\rangle$:

$$\rho_{V_0} \sim \int \mathrm{d}^2\xi D(\xi)|\Psi_{r_0}\rangle\langle\Psi_{r_0}|D^\dagger(\xi)\exp(-\frac{1}{2}\xi^T M^{-1}\xi).$$

Here $D(\xi)$ is the unitary phase space displacement operator. The rank of M equals 2, and both M^{-1} and the two-dimensional integral refer to the restriction of the phase space variable ξ to the range of M.

Since a specific ensemble realization with average entanglement E_{r_0} is exhibited, $\mathrm{EOF}(\rho_{V_0}) \leq E_{r_0}$. On the other hand, evaluation of the generalised EPR correlation $\Lambda_\theta(\rho_{V_0}) = \mathrm{tr}\,(\hat{\Lambda}_\theta\rho_{V_0})$, for the particular value of θ occurring in V_0 shows that $\Lambda_{\theta_0}(\rho_{V_0}) = \cosh 2r_0 - \sin 2\theta_0 \sinh 2r_0$. And by Conjecture 1, this implies $\mathrm{EOF}(\rho_{V_0}) \geq E_{r_0}$. We have thus proved $\mathrm{EOF}(\rho_{V_0}) = E_{r_0}$.

An attractive feature of the canonical form of the covariance matrix is that the two-mode-squeezing $U(r)$ acts on it in a covariant or form-preserving manner.

Proposition 6: Under the two-mode-squeezing transformation $U(r)$ we have

$$\begin{aligned} V_0(r_0,\theta_0,u,v) &\rightarrow V_0(r_0',\theta_0',u',v')\,; \\ r_0' = r_0 + r, \qquad \sin 2\theta_0' &= \frac{\sinh 2r + \cosh 2r \sin 2\theta_0}{\cosh 2r + \sin 2\theta_0 \sinh 2r}, \\ (u',\, v') &= (u,\, v) \times (\cosh 2r + \sin 2\theta_0 \sinh 2r). \end{aligned}$$

This is easily verified by direct computation. While the canonical squeeze parameter r_0 simply gets translated by r, the parameters u and v get scaled by a *common factor*. If we define r_{θ_0}, $r_{\theta_0'}$ through $\tan\theta_0 \equiv \tanh r_{\theta_0}$ and $\tan\theta_0' \equiv \tanh r_{\theta_0'}$, the transformation law for θ_0 takes the form of translation: $r_{\theta_0'} = r_{\theta_0} + r$.

As a consequence of this covariance, the convex decomposition which minimizes the average entanglement goes covariantly to such a decomposition under two-mode-squeezing: the minimal decomposition commutes with squeezing. This implies, in particular, the following simple behaviour of EOF under squeezing: $E_{r_0} \rightarrow E_{r_0+r}$.

Finally, the just separable Gaussian states on the separable-inseparable boundary, correspond to the canonical form with $r_0 = 0$ [16]. As was to be expected, the condition (5) places no restriction on θ_0 in this case.

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