NPPT Bound Entanglement Exists

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Every $d\times d$ bipartite system is shown to have a large family of undistillable states with nonpositive partial transpose (NPPT). This family subsumes the family of conjectured NPPT bound entangled Werner states. In particular, all one-copy undistillable NPPT Werner states are shown to be bound entangled.

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Introduction: Maximally entangled bipartite states form an indispensable resource in several quantum information processing situations [1, 2, 3, 4]. In reality, however, readily available states may be mixed and less than maximally entangled due to a variety of reasons including influence of the environment. Thus distillation, the process by which some copies of (nearly) maximally entangled pure states are extracted from several copies of partially entangled mixed states using local quantum operations and classical communication (LOCC), is of singular importance.

Rapid progress has been achieved in this regard, beginning with the works of Popescu [5], Bennett *et al.* [6], Deutsch *et al.* [7], and Gisin [8, 9]. An important question that presented itself at an early stage of this development was this: Can every entangled state be distilled? The Horodecki family answered this question affirmatively in a significant particular case: Every inseparable state of a pair of qubits can, given sufficiently many copies, be distilled into a singlet [10].

Subsequently they proved a result that applies to arbitrary $m\times n$ bipartite systems [11]: A state that does not violate the Peres-Horodecki PPT (positive partial transpose) criterion [12, 13] can not be distilled. Since inseparable PPT states can not be distilled, they are said to possess bound (*i.e.*, undistillable) entanglement. The first examples of such undistillably entangled PPT states were constructed by Horodecki [14], and several families of PPT bound-entangled states have been presented since then. The role PPT bound-entanglement could play as a quantum resource has also been studied.

The Problem of NPPT Bound Entanglement: Given a bipartite state ρ, let $(\rho^{T_B})^{\otimes n} = (\rho^{\otimes n})^{T_B}$ denote the partial transpose of the state of n identical copies. Then the necessary and sufficient condition for distillability of ρ is that the inequality [11]

$$\langle\psi|\left(\rho^{T_B}\right)^{\otimes n}|\psi\rangle \geq 0, \qquad (1)$$

be violated by a Schmidt rank two vector $|\psi\rangle$ in the n-copy Hilbert space, for some n. Thus in order to be distillable, the state should be NPPT (*i.e.*, the state should have non-positive partial transpose). What has remained an open problem is this: Are there undistillable NPPT states? Does undistillability imply PPT [15]?

Clearly, violation of the above inequality for a particular $n = n_0$ implies violation for all $n > n_0$. Given an NPPT state ρ, if it satisfies this inequality for all Schmidt rank two states, for a particular n, we say that the state is n-copy undistillable or, equivalently, that the partial transpose ρ^{T_B} is n-copy 2-positive. Thus ρ is undistillable if ρ^{T_B} is n-copy 2-positive for all n.

While Horodecki *et al.* pointed out in a subsequent work [16] that NPPT bound entangled states, if they exist in nature, should be found among the one-copy undistillable NPPT Werner states [17], it may be fair to say that this problem attained its present state of fame only when two leading groups produced independently, and about the same time, analytical and numerical evidence [18, 19] for its existence in the context of the Werner family of states [More recent numerical attempt has been undertaken in Ref. [20]]. As the analytic part of the evidence it was shown, for every n, that there is a corresponding range of parameter values over which the NPPT Werner state remains provably n-copy undistillable. But the range itself becomes rapidly smaller with increasing n, and was not proved to remain nonzero as $n\to\infty$. It should be added that no one-copy undistillable NPPT Werner state was shown to be n-copy distillable either. Similar evidence in the multipartite case has been considered by explicit construction of n-copy undistillable NPPT states, for every $n\geq 1$ [21].

It should be noted for completeness that some systems are known not to support NPPT bound entangled states at all. These include $2\times n$ dimensional systems [10, 19] and all bipartite Gaussian states [22].

Shor *et al.* [23] have used the conjectured existence of NPPT bound entanglement to prove nonadditivity of bipartite distillable entanglement. And Eggeling *et al.* [24] have related the existence of NPPT bound entanglement to the connection between the sets of separable superoperators and PPT-preserving channels. Further, Vollbrecht and Wolf [25] have shown that an additional resource in the form of infinitesimal amount of PPT bound-entanglement can render any one-copy undistillable NPPT state one-copy distillable.

In a more recent paper Watrous [26] has constructed, for every $n\geq 1$, states which are n-copy undistillable, and yet are distillable. This is a surprising and important result, for it shows that n-copy undistillability, even if n is very large, does not by itself prove undistillability. The burden this finding places on numerical evidence for NPPT bound entanglement is evident.

We conclude this brief summary of the present status of the conjectured existence of NPPT bound entanglement by noting that the conjecture itself seems to enjoy the confidence of researchers in quantum information theory, even though the evidence presented so far has been assessed differently by different authors [23, 24, 26].

In this Letter we prove that any $d \times d$ bipartite system with $d \geq 3$ has a fairly large family of NPPT states which are undistillable. As will be seen, this family is much larger than the Werner family of conjectured NPPT bound entangled states, but it turns out to be convenient to begin our proof with the Werner family.

Proof of Existence of NPPT Bound Entanglement: We may define the one-parameter family of Werner state ρ_α in $d \times d$ dimensions through the partial transpose of ρ_α:

$$\rho_\alpha^{\mathrm{T_B}} = \mathrm{Id} - d\alpha P. \tag{2}$$

Here P is the projection on the standard maximally entangled state:

$$P = |\Psi\rangle\langle\Psi|, \quad |\Psi\rangle = \frac{1}{\sqrt{d}}\sum_{k=1}^{d}|k,k\rangle. \tag{3}$$

These states as defined are not normalized to unit trace, but this does in no way affect our considerations below.

Nonnegativity of ρ_α forces on α the restriction $-1 \leq \alpha \leq 1$. This allowed range for α divides into three interesting regions [18, 19]:

$$\begin{aligned} -1 \leq \alpha \leq \frac{1}{d}, &\quad \text{PPT}, \quad \text{separable}, \\ \frac{1}{2} < \alpha \leq 1, &\quad \text{NPPT}, \quad \text{one-copy distillable}, \\ \frac{1}{d} < \alpha \leq \frac{1}{2}, &\quad \text{NPPT}, \quad \text{one-copy undistillable}. \end{aligned} \tag{4}$$

Clearly, it is the last mentioned range which is of interest for the issue on hand. In this range ρ_α is NPPT, yet ${\rho_\alpha}^{T_B}$ is 2-positive. Since ρ_α being NPPT implies that $\rho_\alpha^{\otimes n}$ is NPPT, for all n, the issue really is whether $({\rho_\alpha}^{T_B})^{\otimes n}$ is 2-positive as well. We show that it indeed is, for all n.

Let us denote by $\mathcal{S}^{(1)}$ the collection of all pure and mixed states of Schmidt rank (number) ≤ 2 [27]. This is a convex set whose extremals are all pure states of Schmidt rank 1 or 2. Let $\theta = (\theta_1, \theta_2, \cdots, \theta_d)$ be a d-tuple of angles, and consider the subgroup of $U(d)$, the d-dimensional unitary group, consisting of diagonal matrices U_θ:

$$(U_\theta)_{kl} = \delta_{kl} e^{i\theta_k}. \tag{5}$$

This is the standard maximal abelian subgroup U_A of $U(d)$. Now average each element σ of $\mathcal{S}^{(1)}$ over the local group $U_A \otimes U_A^*$:

$$\sigma \in \mathcal{S}^{(1)} \rightarrow \sigma' = \int d\theta \, U_\theta \otimes U_\theta^* \, \sigma \, U_\theta^\dagger \otimes U_\theta^T. \tag{6}$$

Here $d\theta = d\theta_1 d\theta_2 \cdots d\theta_d$, with θ_k running over the interval $[0, 2\pi]$, independently for every k. This group-averaging process is analogous to the diagonal twirl operation [18]. The resulting images of elements of $\mathcal{S}^{(1)}$ constitute a new convex set $\Omega^{(1)}$, where the superscript on $\mathcal{S}$ and Ω reflects our intention to extend these considerations to several copies of the Werner state.

The reduction in complexity achieved by going from $\mathcal{S}^{(1)}$ to $\Omega^{(1)}$ should be appreciated. For instance, whereas $\mathcal{S}^{(1)}$ has a $CP^{d-1} \times CP^{d-1}$ worth of product states among its extremals, the product states among the extremals of $\Omega^{(1)}$ are precisely d^2 in number; these are the standard computational basis states $|k,l\rangle$. Similarly the pure states of Schmidt rank $= 2$ among the extremals of $\Omega^{(1)}$ necessarily have one of the forms $\alpha_1|11\rangle + \alpha_2|22\rangle$, $\beta_2|22\rangle + \beta_3|33\rangle$, and $\gamma_1|11\rangle + \gamma_3|33\rangle$; each of these three sets is a Bloch-sphere in worth. And, therefore the maximally entangled rank-two states are precisely three circles worth in number, being of the form $\frac{1}{\sqrt{2}}(|11\rangle + e^{i\delta_1}|22\rangle)$, $\frac{1}{\sqrt{2}}(|22\rangle + e^{i\delta_2}|33\rangle)$, or $\frac{1}{\sqrt{2}}(|33\rangle + e^{i\delta_3}|11\rangle)$.

Now 2-positivity of $\rho_\alpha^{T_B}$ is equivalent to the demand that $\mathrm{Tr}(\rho^{T_B}\sigma) \geq 0$, for all $\sigma \in \mathcal{S}^{(1)}$. In view of the $U_A \otimes U_A^*$ symmetry of ρ^{T_B}, this is equivalent to the demand that $\mathrm{Tr}(\rho^{T_B}\sigma) \geq 0$, for all $\sigma \in \Omega^{(1)}$ and in view of the convexity of $\Omega^{(1)}$, it is sufficient for the extremals of $\Omega^{(1)}$ to meet this demand. This convexity argment is in essence a recognition of the fact that the real-valued expression $\mathrm{Tr}(\rho_\alpha^{T_B} sigma)$ is linear in σ.

We thus find that the minimum of $\mathrm{Tr}(\rho^{T_B}\sigma)$ over $\Omega^{(1)}$ equals $(1 - 2\alpha)$, showing that ρ^{T_B} is 2-positive for all $\alpha \leq \frac{1}{2}$. The minimum is achieved by $\frac{1}{\sqrt{2}}(|11\rangle + |22\rangle)$, $\frac{1}{\sqrt{2}}(|22\rangle + |33\rangle)$, and $\frac{1}{\sqrt{2}}(|33\rangle + |11\rangle)$, and by no other states.

We may note in passing that the local symmetry of ρ^{T_B} is not $U_A \otimes U_A^*$, but the full group $U(d) \otimes U(d)^*$. We wish to go beyond the Werner states later in this Letter, and for this reason we have based our analysis on the subgroup $U_A \otimes U_A^*$, rather than on the full group.

We now move on to consider $\rho_\alpha \otimes \rho_\alpha$, two copies of the Werner state ρ_α. The set $\mathcal{S}^{(2)}$ is constructed as the collection of $(d^2 \times d^2)$-dimensional states of Schmidt number ≤ 2. From the convex set $\mathcal{S}^{(2)}$ we obtain $\Omega^{(2)}$ by averaging each $\sigma \in \mathcal{S}^{(2)}$ over the local group $(U_A \otimes U_A^*) \otimes (U_A \otimes U_A^*)$. It is to be understood that the first $(U_A \otimes U_A^*)$ factor acts on the Hilbert space of the first copy and the second on that of the second copy, independently.

The extremals of $\Omega^{(2)}$ are readily enumerated. The rank-1 extremals of $\Omega^{(2)}$ are necessarily of the form $|k,l\rangle_1 \otimes |i,j\rangle_2$. They are d^4 in number. The rank two states are of two types.

$$\begin{aligned} \text{Type I}: \quad & (\alpha|kk\rangle_1 + \beta|ll\rangle_1) \otimes |ij\rangle_2, \\ & |kl\rangle_1 \otimes (\alpha|ii\rangle_2 + \beta|jj\rangle_2); \\ \text{Type II}: \quad & \alpha|kk\rangle_1 \otimes |ii\rangle_2 + \beta|ll\rangle_1 \otimes |jj\rangle_2. \end{aligned} \tag{7}$$

At the risk of sounding repetitive, we emphasize that this is a complete enumeration of the extremals of $\Omega^{(2)}$.

As with the single copy case 2-positivity of $\rho_\alpha \otimes \rho_\alpha$ is equivalent to the demand $\mathrm{Tr}[(\rho_\alpha^{T_B} \otimes \rho_\alpha^{T_B})\sigma] \geq 0$, for all $\sigma \in \Omega^{(2)}$, which in turn is equivalent to the demand that this condition be met by all the extremal states of $\Omega^{(2)}$. The type-I rank-2 states being products across the two copies, cannot bring out genuinely two-copy properties if any, and thus we are left with only the type-II states to examine:

$$\begin{aligned} \langle \Psi_{II}|\rho_\alpha^{T_B} \otimes \rho_\alpha^{T_B}|\Psi_{II}\rangle & \\ = \quad & |\alpha|^2 \langle kk|\rho_\alpha^{T_B}|kk\rangle\langle ii|\rho_\alpha^{T_B}|ii\rangle \\ & + |\beta|^2 \langle ll|\rho_\alpha^{T_B}|ll\rangle\langle jj|\rho_\alpha^{T_B}|jj\rangle \\ & + \alpha^*\beta \langle kk|\rho_\alpha^{T_B}|ll\rangle\langle ii|\rho_\alpha^{T_B}|jj\rangle \\ & + \alpha\beta^* \langle ll|\rho_\alpha^{T_B}|kk\rangle\langle jj|\rho_\alpha^{T_B}|ii\rangle. \end{aligned} \tag{8}$$

Now, 2-positivity of ρ^{T_B} is equivalent to the Schwartz inequality

$$\langle kk|\rho_\alpha^{T_B}|ll\rangle \leq [\langle kk|\rho_\alpha^{T_B}|kk\rangle\langle ll|\rho_\alpha^{T_B}|ll\rangle]^{\frac{1}{2}}, \tag{9}$$

for rank-1 states. Use of this inequality in Eq.(8) proves

$$\langle \Psi_{II}|\rho_\alpha^{T_B} \otimes \rho_\alpha^{T_B}|\Psi_{II}\rangle \geq 0. \tag{10}$$

That is one-copy 2-positivity of the $(U_A \otimes U_A^*)$ invariant operator ρ^{T_B} implies its two-copy 2-positivity. And we have proved

Theorem 1: All one-copy undistillable Werner states, that is all ρ_α's in the entire range $\frac{1}{d} \leq \alpha \leq \frac{1}{2}$, are two-copy undistillable.

To move on to the n-copy case, assume that $\rho_\alpha^{T_B}$ is $(n-1)$-copy 2-positive. That is $\langle \psi (\rho_\alpha^{T_B})^{\otimes(n-1)}|\psi\rangle \geq 0$ for all rank-2 states of the $(n-1)$-copy Hilbert space. We wish to prove that this implies $(\rho_\alpha^{T_B})^{\otimes(n)}$ is 2-positive.

To this end, form $\Omega^{(n)}$ by averaging each $\sigma \in \mathcal{S}^{(n)}$ over the local group $(U_A \otimes U_A^*)^{\otimes n}$, with one $(U_A \otimes U_A^*)$ factor acting on the Hilbert space of each copy independently. Again, the extremals of $\Omega^{(n)}$ consist of rank-1 and rank-2 pure states. The rank-1 states are of the form $|i_1, j_1\rangle \otimes |i_2, j_2\rangle \otimes ....|i_n, j_n\rangle$. That is, these are tensor products of computational basis states, one picked from each copy, and thus are d^{2n} in number.

The rank-2 extremal (or pure) states in $\Omega^{(n)}$ are of two types, as in the two-copy case:

$$\begin{aligned} \text{Type I}: \quad & |\Psi_I\rangle = |\psi\rangle \otimes |i, j\rangle, \\ \text{Type II}: \quad & |\Psi_{II}\rangle = |\phi_1\rangle \otimes |i, i\rangle + |\phi_2\rangle \otimes |j, j\rangle, \end{aligned} \tag{11}$$

where $|\psi\rangle$ is a $(n-1)$-copy rank-2 state, and $|\phi_1\rangle$, $|\phi_2\rangle$ are $(n-1)$-copy rank-1 states. Since type-I states have a product structure across the copies, nonnegative expectation values of $(\rho_\alpha^{T_B})^{\otimes n}$ in respect of type-I rank-2 states follows directly from the assumed $(n-1)$-copy 2-positivity, the additional copy offering nothing new in the type-I case. The same $(n-1)$-copy 2-positivity is equivalent to the validity of the Schwartz inequality for $(n-1)$-copy rank-1 states, and this in turn implies the nonnegativity of the expectation values of $(\rho_\alpha^{T_B})^{\otimes n}$ for type-II states. We have thus proved, by induction,

Theorem 2: The one-copy 2-positive $(\rho_\alpha^{T_B})^{\otimes n}$'s in the entire parameter range $\frac{1}{d} \leq \alpha \leq \frac{1}{2}$ are n-copy 2-positive, for all n. That is, these one-copy undistillable NPPT Werner states ρ_α are n-copy undistillable, for all n, and hence are bound entangled.

Finally, it should be evident that the only property of ρ_α, apart from its NPPT and one-copy 2-positivity properties, used in our analysis is its $(U_A \otimes U_A)$ invariance or, equivalently, the $(U_A \otimes U_A^*)$ invariance of its partial transpose. It follows that our conclusions apply to all states with these properties. That is,

Theorem 3: Every one-copy undistillable NPPT state in $d \times d$ dimensions is bound entangled if it possesses $(U_A \otimes U_A)$ symmetry.

This is the main result of this Letter. It shows that the family of NPPT bound entangled states in $d \times d$ dimensions, for any $d \geq 3$, is much larger than what might have been anticipated. This point is worth illustrating.

It is easily seen that in 3×3 dimensions the most general $(U_A \otimes U_A^*)$ invariant ρ^{T_B} has the form

$$\rho^{T_B} = \begin{bmatrix} \rho_{11} & 0 & 0 & 0 & -z_{12} & 0 & 0 & 0 & -z_{31}^* \\ 0 & \rho_{12} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \rho_{13} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \rho_{21} & 0 & 0 & 0 & 0 & 0 \\ -z_{12}^* & 0 & 0 & 0 & \rho_{22} & 0 & 0 & 0 & -z_{23} \\ 0 & 0 & 0 & 0 & 0 & \rho_{23} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \rho_{31} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \rho_{32} & 0 \\ -z_{31} & 0 & 0 & 0 & -z_{23}^* & 0 & 0 & 0 & \rho_{33} \end{bmatrix}$$

Clearly, ρ has to be positive semidefinite in order to be a valid density matrix. This demand is equivalent to the conditions: (i) all the diagonal elements $\rho_{ij} \geq 0$, and $\rho_{ij}\rho_{ji} \geq |z_{ij}|^2$ for all $i \neq j$. The NPPT requirement demands that the 3×3 submatrix

$$\begin{bmatrix} \rho_{11} & -z_{12} & -z_{31}^* \\ -z_{12}^* & \rho_{22} & -z_{23} \\ -z_{31} & -z_{23}^* & \rho_{33} \end{bmatrix}$$

should be nonpositive, and the 2-positivity demand is equivalent to the three inequalities $\rho_{11}\rho_{22} \geq |z_{12}|^2$, $\rho_{22}\rho_{33} \geq |z_{23}|^2$, and $\rho_{33}\rho_{11} \geq |z_{31}|^2$. The NPPT demand and the 2-positivity demands thus involve only the six parameters ρ_{11}, ρ_{22}, ρ_{33} and z_{12}, z_{23}, z_{31}. The phases of the complex z-parameters can be tuned by (local) change of phases of the basis vectors on the A and B sides (to be precise it is sufficient to carry out the changes on one side only), but the argument of $z_{12}\ z_{23}\ z_{31}$ is invariant under such gauge transformations.

Thus our family of NPPT bound entangled states in 3×3 involves, when normalized to unit trace, 12 parameters; eight coming from the diagonals ρ_{ij}, three coming from the magnitudes of the z-parameters, and one gauge-invariant phase. These are canonical parameters, and do not take into consideration parameters arising from local unitary transformations.

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