ON THE SPECTRUM OF ASYMPTOTIC SLOPES

A. J. PARAMESWARAN AND S. SUBRAMANIAN

Abstract. The slopes of maximal subbundles of rank s divided by the degree of the map under various pull backs form a bounded collection of numbers called the s-spectrum of the bundle. We study the supremum of the s-spectrum and determine it in terms of the Harder Narasimhan filtration of the bundle.

1. Introduction

Line subbundles of maximal degree in a rank 2 bundle on a curve have been studied in [LN] and many other subsequent papers. In [BP] it was shown that a vector bundle is strongly semistable if and only if the slope of the maximal subbundle of a given rank in a finite pull back is bounded by the slope of the pull back bundle.

In this note we study the behaviour of maximal subbundles of vector bundles on curves after finite pull backs. These slopes of maximal subbundles of rank s divided by the degree of the map under various pull backs form a bounded collection of numbers called the s-spectrum of the bundle. We study the supremum of the s-spectrum and determine it in terms of the Harder Narasimhan filtration of the bundle. We also give a criterion for a spectrum value to be isolated.

2. Preliminaries

Let k be an algebraically closed field of arbitrary characteristic. Let C be an irreducible smooth projective curve over k of genus g. For a vector bundle V over C we will denote by $\mathbb{G}r(s,V)$ the Grassmann bundle over C defined by the space of all s dimensional quotients of V. The tautological line bundle over $\mathbb{G}r(s,V)$ will be denoted by $\mathcal{O}_{\mathbb{G}r(s,V)}(1)$. This is the determinant of the universal quotient bundle on $\mathbb{G}r(s,V)$. Tensor powers of this tautological line bundle are denoted by $\mathcal{O}_{\mathbb{G}r(s,V)}(n)$. The slope of V is defined as

$$\mu(V) := \frac{\deg(V)}{\mathrm{rank}(V)}$$

V is called *semistable* if for all subbundles $W \subset V$ of positive rank $\mu(W) \leq \mu(V)$. V is called *strongly semistable* if the iterated Frobenius pull back, $F_C^{n*}(V)$, is semistable for all $n > 0$, where $F_C : C \to C$ denotes the Frobenius morphism. If V is a strongly semistable vector bundle of rank r, then for any representation of GL_r into GL_n, the induced rank n vector bundle is strongly semistable (cf. [RR]).

Given a vector bundle V, there is a unique filtration (the Harder Narasimhan filtration) $V_\bullet := \{0 = V_0 \subset \cdots \subset V_l = V\}$ such that each V_i/V_{i-1} is semistable and the slopes of successive quotients are strictly decreasing, i.e., $\mu_i := \mu(V_i/V_{i-1}) > \mu_{i+1} := \mu(V_{i+1}/V_i)$. The subbundles V_i are defined inductively as the inverse image of the maximal subbundle of maximal slope in V/V_{i-1}. We call the bundles V_i/V_{i-1}, the *Harder Narasimhan factors* of the bundle V. The bundle V_1 is called the *maximal subbundle* of V and denoted by $V_{\max}$. It's slope $\mu(V_1) = \mu(V_{\max})$ is called the maximal slope of V and denoted by $\mu_{\max}(V)$. We recall the following result whose proof we will omit.

Lemma 2.1. *If V and W are semistable vector bundles over a smooth curve C, with $\mu(V) > \mu(W)$, then $Hom_C(V, W) = H^0(C, V^* \otimes W) = 0$.*

Now we have the following result which gives enough complete intersection curves in the projective bundle.

Lemma 2.2. *Let V be a strongly semistable vector bundle of rank r over C and L a line bundle of degree $> 2g - ns\mu(V)$ with $n > 0$. Then $\mathcal{O}_{\mathbb{G}r(s,V)}(n) \otimes \pi^* L$ separates points on $\mathbb{G}r(s, V)$.*

Proof. Since $\pi : \mathbb{G}r(s, V) \to C$ is a smooth fibration with Grassmann varieties as fibres and higher cohomologies of ample bundles vanish on Grassmannians, it follows that

$$H^1(\mathbb{G}r(s,V)\ , \mathcal{O}_{\mathbb{G}r(s,V)}(n) \otimes \pi^* L) \cong H^1(C, \pi_*(\mathcal{O}_{\mathbb{G}r(s,V)}(n) \otimes \pi^* L)) \cong H^1(C, V_{s,n} \otimes L)$$

where $V_{s,n}$ is the vector bundle associated to V by the Weyl module with highest weight $n\omega_s$ with ω_s as the fundamental weight corresponding to the s-th exterior power representation. Notice that the slope of $\wedge^s V$ is equal to $s\mu(V)$.

Hence for any two points $x, y \in C$, Serre duality implies

$$H^1(C,\ V_{s,n} \otimes L(-x - y)) \ = \ H^0(C,\ V^*_{s,n} \otimes L^{-1}(x + y) \otimes K_C)^*$$

To show this cohomology group vanishes, it suffices to show the semistable bundle $V^*_{s,n} \otimes L^{-1}(x + y) \otimes K_C$ has negative slope. Now we have:

$$\mu(V^*_{s,n} \otimes L^{-1}(x + y) \otimes K_C) = \mu(V^*_{s,n}) - \deg\ L + 2g = -ns\mu(V) - \deg\ L + 2g$$

which is negative if and only if $\deg\ L > 2g - ns\mu(V)$. Hence for two distinct points $x, y \in C$,

$$H^0(C,\ V_{s,n} \otimes L) \to H^0(C,\ V_{s,n}\ |_{x+y})$$

is surjective whenever $\deg\ L > 2g - ns\mu(V)$. This implies $\mathcal{O}_{\mathbb{G}r(s,V)}(n) \otimes \pi^* L$ separate points and surjects onto sections of $\mathcal{O}_{\mathbb{G}r(V_x)}(n)$ on the Grassmannian $\mathbb{G}r(s, V_x)$. $\square$

Consider the universal exact sequence on $\mathbb{G}r(s, V)$:

$$0 \to \mathcal{S} \to \pi^* V \to \mathcal{Q} \to 0 \tag{2.1}$$

Then det $\mathcal{Q} \cong \mathcal{O}_{\mathbb{G}r(s,V)}(1)$ is the tautological line bundle. For any line bundle on $\mathbb{G}r(s,V)$ let [-] denote the corresponding cycle class. The Chow group of 0 cycles on $\mathbb{G}r(s,V)$ is canonically isomorphic to the Chow group of 0 cycles of the curve C (cf. [F], Prop. 14.6.5,). Then we have the following:

Lemma 2.3. *Given any line bundle L on C and any fibre F of $\pi : \mathbb{G}r(s,V) \to C$ we have*

$$[\mathcal{O}_{\mathbb{G}r(s,V)}(1)]^{s(r-s)+1} = (s(r-s)+1)s\mu(V)([\mathcal{O}_{\mathbb{G}r(s,V)}(1)]^{s(r-s)} \cdot F) \tag{2.2}$$

$$[\mathcal{O}_{\mathbb{G}r(s,V)}(1)]^{s(r-s)} \cdot [\pi^*L] = \mathrm{degL}([\mathcal{O}_{\mathbb{G}r(s,V)}(1)]^{s(r-s)} \cdot F) \text{ and } [\pi^*L]\cdot[\pi^*L] = 0 \tag{2.3}$$

Proof. If we pull back the bundle under a finite map, both sides of the formulae gets multiplied by the degree of the map. Hence we may assume, after a finite pull back if necessary, that there exists a line bundle L such that $L^{\otimes r} \cong \det V$., i.e., det V has an r^{th} root. Then $\oplus_1^r L \cong \mathcal{O}_C^{\oplus r} \otimes L$. Let $\mathcal{L}^{-1}$ be a very ample line bundle on C such that $V \otimes \mathcal{L}^{-1}$ and $L \otimes \mathcal{L}^{-1}$ are globally generated and have vanishing first cohomology.

Let $Quot_P(\mathcal{O}_C^{\oplus N} \otimes \mathcal{L})$ denote the quot scheme over C of quotient of fixed Hilbert polynomial P (the degree and rank determine the polynomial) of the trivial vector bundle of rank $N >> 0$ twisted by the line bundle $\mathcal{L}$. Let $U \subset Quot_P(\mathcal{O}_C^{\oplus N} \otimes \mathcal{L})$ be the open set where the universal quotient sheaf is locally free and $H^0(C, V \otimes \mathcal{L}^{-1}) \cong H^0(C, \mathcal{O}_C^{\oplus N})$. Then by [Ne] (Remark 5.5, page 140) U is smooth and irreducible and hence connected. Then $\oplus_1^r L \cong \mathcal{O}_C^{\oplus r} \otimes L$ and V belong to the same quot scheme, in fact U.

Let $\mathcal{V}$ be the universal bundle over $C \times U$ and $\mathbb{G}r(s,\mathcal{V})$ by the corresponding Grassmannian bundle over $C \times U$. For each vector bundle W representing a closed point $[W] \in U$, the restriction $\mathcal{V} \mid_{C\times[W]}$ is canonically isomorphic to W. Hence the restriction of the Grassmannian bundle $\mathbb{G}r(s,\mathcal{V})$ is canonically $\mathbb{G}r(s,W)$. Since the isomorphism of $CH_0(\mathbb{G}r(s,V))$ with $CH_0(C)$ is canonical, it suffices to check the formula for any closed point (vector bundle) of the open subset U of the quot scheme $Quot_P(\mathcal{O}_C^{\oplus N} \otimes \mathcal{L})$.

Hence we obtain,

$$[\mathcal{O}_{\mathbb{G}r(s,V)}(1)]^{s(r-s)+1} \cong [\mathcal{O}_{\mathbb{G}r(s,\oplus L)}(1)]^{s(r-s)+1} \cong [\mathcal{O}_{\mathbb{G}r(s,\oplus \mathcal{O}_C \otimes L)}(1)]^{s(r-s)+1}$$
$$\cong \{[\mathcal{O}_{\mathbb{G}r(s,\oplus \mathcal{O}_C)}(1)] + [L]^{\otimes s}\}^{s(r-s)+1} = (s(r-s)+1)s\deg L(\mathcal{O}_{\mathbb{G}r(s,\oplus \mathcal{O}_C)}(1).F)$$
$$= (s(r-s)+1)s\mu(V)(\mathcal{O}_{\mathbb{G}r(s,\oplus L)}(1).F) = (s(r-s)+1)s\mu(V)(\mathcal{O}_{\mathbb{G}r(s,V)}(1).F)$$ □

Choose constants $0 < \epsilon_n \leq 1$ and line bundles L_n on C such that deg $L_n = 2g - ns\mu(V) + \epsilon_n$. Then $\mathcal{O}_{\mathbb{G}r(s,V)}(n) \otimes \pi^* L_n$ separates points by Lemma 2.2 and hence defines enough smooth complete intersection curves in $\mathbb{G}r(s,V)$ by Bertini's theorem. Now we have the following result.

Lemma 2.4. *Let D be an irreducible complete intersection curve in $\mathbb{G}r(s,V)$ defined by sections of $\mathcal{O}_{\mathbb{G}r(s,V)}(n)\otimes\pi^*L_n$. Then*

$$\mu(\mathcal{S}\mid_D)=n^{s(r-s)}([\mathcal{O}_{\mathbb{G}r(s,V)}(1)]^{s(r-s)}\cdot F)(\mu(V)-\frac{s(2g+\epsilon_n)}{n})$$

Proof. Since D is a complete intersection, the degree of $\mathcal{O}_{\mathbb{G}r(s,V)}(1)$ on D can be calculated as the cup product (denoted by $\cdot$ in the Chow ring) of the cycle classes of the corresponding divisors (line bundles) with the class of $\mathcal{O}_{\mathbb{G}r(s,V)}(1)$. Note that $[\mathcal{O}_{\mathbb{G}r(s,V)}(n)]=n[\mathcal{O}_{\mathbb{G}r(s,V)}(1)]$ as the tensor product of line bundles gives the sum of the corresponding classes. Hence we can interpret deg $\mathcal{O}_{\mathbb{G}r(s,V)}(1)\mid_D$ as

$$\begin{aligned}\deg\,\mathcal{O}_{\mathbb{G}r(s,V)}(1)\mid_D = [D]\cdot[\mathcal{O}_{\mathbb{G}r(s,V)}(1)] &= ([\mathcal{O}_{\mathbb{G}r(s,V)}(n)]+[\pi^*L_n])^{s(r-s)}\cdot[\mathcal{O}_{\mathbb{G}r(s,V)}(1)]\\ &=\{[\mathcal{O}_{\mathbb{G}r(s,V)}(n)]^{s(r-s)}+s(r-s)[\mathcal{O}_{\mathbb{G}r(s,V)}(n)]^{s(r-s)-1}\cdot[\pi^*L_n]\}\cdot[\mathcal{O}_{\mathbb{G}r(s,V)}(1)]\\ &=n^{s(r-s)}[\mathcal{O}_{\mathbb{G}r(s,V)}(1)]^{s(r-s)+1}+s(r-s)n^{s(r-s)-1}[\mathcal{O}_{\mathbb{G}r(s,V)}(1)]^{s(r-s)}\cdot[\pi^*L_n]\end{aligned}$$

Now the degree of the universal subbundle on D,

$$\deg\,\mathcal{S}\mid_D=\deg\,\pi^*V\mid_D\;-\deg\,\mathcal{Q}\mid_D$$

$$=n^{s(r-s)}([\mathcal{O}_{\mathbb{G}r(s,V)}(1)]^{s(r-s)}\cdot F)\deg V\;-\;n^{s(r-s)}(s(r-s)+1)s\mu(V)([\mathcal{O}_{\mathbb{G}r(s,V)}(1)]^{s(r-s)}\cdot F)$$
$$-n^{s(r-s)-1}s(r-s)\deg\,L_n\;([\mathcal{O}_{\mathbb{G}r(s,V)}(1)]^{s(r-s)}\cdot F)$$

Now by substituting for deg L_n in the above and simplifying this expression we get

$$\mu(\mathcal{S}\mid_D)=n^{s(r-s)}([\mathcal{O}_{\mathbb{G}r(s,V)}(1)]^{s(r-s)}\cdot F)(\mu(V)-\frac{s(2g+\epsilon_n)}{n})$$

□

Note that the degree of $\pi_D:D\to C$ is equal to cardinality of a general fibre of π_D over x which equals $[\mathcal{O}_{\mathbb{G}r(s,V_x)}(n)]^{s(r-s)}\cdot F=n^{s(r-s)}([\mathcal{O}_{\mathbb{G}r(s,V)}(1)]^{s(r-s)}\cdot F)$.

3. Genuinely ramified maps

Let us begin with the following definition.

Definition 3.1. *Let $f:D\to C$ be a finite morphism of integral curves. Then f is said to be* genuinely ramified *if f is separable and does not factor through an étale cover of C.*

Lemma 3.2. *A separable morphism $f:D\to C$ is genuinely ramified if and only if $(f_*\mathcal{O}_D)_{\max}\cong\mathcal{O}_C$.*

Proof. By projection formula we have

$$H^0(D,f^*W)=H^0(C,W\otimes f_*\mathcal{O}_D)$$

Hence for a semistable bundle S on C of positive slope

$$Hom(S,f_*\mathcal{O}_D)=Hom(f^*S,\mathcal{O}_D)=0$$

because f^*S remains semistable of positive slope as f is separable. This shows that $\mu_{\max} f_*\mathcal{O}_D = 0$.

If rank $(f_*\mathcal{O}_D)_{\max} > 1$, then it forms a sheaf of subalgebras of $f_*\mathcal{O}_D$ on C. Hence by taking the spectrum of $f_*\mathcal{O}_{D\max}$ we obtain an étale cover of C factoring $f : D \to C$. Hence f is not genuinely ramified. $\square$

In [N], Madhav Nori has constructed the fundamental group scheme $\pi(X)$ of any complete variety X. This is constructed as a Tannaka category whose objects are essentially finite vector bundles on X. Further he has also constructed a principal $\pi(X)$ bundle $\tilde{X} \to X$. Let $\pi_1^{\text{alg}}(X)$ denote the étale fundamental group of a complete scheme X. Then one can show that $\pi_1^{\text{alg}}(X)$ is a quotient of $\pi(X)$ whose objects in the Tannaka category are the vector bundles that are trivial on a finite étale cover of X. This quotient morphism induces a $\pi_1^{\text{alg}}(X)$-bundle $\mathcal{X} \to X$ over X.

Lemma 3.3. *A separable morphism of curves $D \to C$ is genuinely ramified if and only if the induced map $\pi_1^{\text{alg}}(D) \to \pi_1^{\text{alg}}(C)$ is an epimorphism.*

Proof. Clearly an epimorphism on the fundamental group implies genuine ramification. This follows from the fact that for any finite étale morphism $f : D \to C$ of degree d the index of the image of the étale fundamental group is equal to d.

Any finite morphism $f : D \to C$ induces the map on étale fundamental groups whose image $f_*(\pi_1^{\text{alg}}(D)) = \Gamma$ is of finite index. Define $\tilde{D} := \mathcal{C}/\Gamma$. Then the induced map $\tilde{D} \to C$ is étale such that $f : D \to C$ factors through this. This proves the converse. $\square$

Lemma 3.4. *If D is a general complete intersection curve in $\mathbb{G}r(s, V)$ defined by an ample line bundle, then the induced projection $D \to C$ is genuinely ramified*

Proof. Notice that the algebraic fundamental group of $\mathbb{G}r(s, V)$ is naturally isomorphic to the algebraic fundamental group of C as Grassmannians are algebraically simply connected. If D is a complete intersection curve in $\mathbb{G}r(s, V)$, then the algebraic fundamental group of D surject onto the algebraic fundamental group of $\mathbb{G}r(s, V)$ by the algebraic analogue of Lefshetz Theorem. Hence $\pi_1^{\text{alg}}(D) \to \pi_1^{\text{alg}}(C)$ is surjective. Now the result follows from Lemma 3.3. $\square$

Lemma 3.5. *Let $f : D \to C$ be genuinely ramified morphism of smooth projective curves. Then*

(a) If V and W are two semistable bundles on C of same slope, then

$$Hom_C(V, W) \cong Hom_D(f^*V, f^*W)$$

(b) If V is a stable bundle on C, then f^*V is stable on D.

(c) If V is a semistable bundle on C and $F \subset f^*V$ is a subbundle of same slope as f^*V, then F is isomorphic to the pull back of a subbundle of V.

Proof. (a) Given two semistable bundles V, W of same slope on C, we have

$$Hom_D(f^*V, f^*W) \cong Hom_C(V, f_*f^*W) \cong Hom_C(V, W \otimes f_*\mathcal{O}_D) \cong Hom_C(V, W)$$

The last equality follows from the fact that $Hom_C(F, f_*\mathcal{O}_D/\mathcal{O}_C) = 0$ for any semistable bundle F of slope ≥ 0 as genuine ramification of f implies that $f_*\mathcal{O}_D/\mathcal{O}_C$ has negative maximal slope (see Lemma 3.2).

(b) Since the socle (maximal subbundle that is a direct sum stable bundles (cf. [MR])) is unique, it follows that the socle of f^*V descends to the socle of V when f is separable. Since V is stable, this descended bundle has to be V itself. This shows that the pull back of a stable bundle is polystable under any finite separable map. Now the stability of f^*V for genuinely ramified maps follows from (a) since projections to direct summands are endomorphisms which do not come from below.

(c) Let V be a semistable bundle over C. Let $F \subset f^*V$ be a subbundle of same slope. Then the socle S_F of F is contained in the socle S_{f^*V} of f^*V and hence a direct summand of S_{f^*V}. But by uniqueness of the socle, S_{f^*V} is $f^*(S_V)$. Since stable bundles pull back to stable bundles, S_F coincides with some factors of $f^*(S_V)$ and hence is a pull back. Now the assertion follows by induction on the rank applied to the bundle $F/S_F \subset f^*V/S_F$. □

Corollary 3.6. *Let $f : D \to C$ be a finite separable morphism of smooth projective irreducible curves. Then for any semistable vector bundle W on D we have*
(a) $\mu_{\max}(f_*W) \leq \frac{\mu(W)}{\deg\ f}$
(b) If $\mu_{\max}(f_*W) = \frac{\mu(W)}{\deg\ f}$, *then* $\mathrm{rank}(f_*W)_{\max} \leq \mathrm{rank}\ W \cdot \mathrm{rank}(f_*\mathcal{O}_D)_{\max}$

Proof. First assertion follows from the fact that $Hom_C(F, f_*W) \cong Hom_D(f^*F, W)$. Hence semistable bundles of slope $> \frac{\mu(W)}{\deg\ f}$ have no morphism to f_*W.

If $\mathrm{rank}(f_*\mathcal{O}_D)_{\max} > 1$, then the morphism $f : D \to C$ factors through an étale morphism $\pi : \tilde{C} \to C$ such that $\tilde{f} : D \to \tilde{C}$ satisfies $\tilde{f}_*\mathcal{O}_D = \mathcal{O}_{\tilde{C}}$ (see Lemma 3.2).

If W' is any semistable bundle on $\tilde{C}$, then $\chi(W') = \chi(\pi_*W')$ as the cohomologies do not change by taking direct images under finite maps. By Riemann-Roch theorem we get

$$\chi(W') = (\mathrm{rank}\ (W'))(1 - g_{\tilde{C}} + \mu(W')) = (\mathrm{rank}\ (W'))(\deg\ (\pi)(1 - g_C) + \mu(W'))$$

and

$$\chi(\pi_*(W')) = \mathrm{rank}\ (\pi_*(W'))(1 - g_C + \mu(\pi_*(W'))) = (\deg\ (\pi)\mathrm{rank}\ (W'))(1 - g_C + \mu(\pi_*(W')))$$

since π is étale, $1 - g_{\tilde{C}} = (\deg\ \pi)(1 - g_C)$. Hence by comparing the terms above we conclude that $\mu(W') = (\deg\ (\pi))(\mu(\pi_*(W')))$. That π_*W' is semistable of slope $\frac{\mu(W')}{\deg\ \pi}$ follows from (a).

Hence it suffices to prove (b) for the case when f is genuinely ramified.

On the contrary, assume there is a semistable bundle W with $\mu_{\max}(f_*W) = \frac{\mu(W)}{\deg\ f}$ and $\mathrm{rank}(f_*W)_{\max} > \mathrm{rank} W$, then consider the natural map, $f^*(f_*W)_{\max} \to Q \subset W$ with image Q. By taking direct image we obtain $(f_*Q)_{\max} = (f_*W)_{\max}$. Since $f^*(f_*W)_{\max}$ and W are semistable of same slope, Q is a semistable vector bundle of same slope $\frac{\mu(W)}{\deg\ f}$. By Lemma 3.5(c), $Q \cong f^*Q'$ is itself a pull back. Then $(f_*Q)_{\max} = (f_*f^*Q')_{\max} = Q'$, hence has the same rank as Q, which is at most rank W, a contradiction. □

4. Asymptotic Slopes and Strong Semistability

Let C be a smooth curve defined over an algebraically closed field k of arbitrary characteristic. Let V be a vector bundle of rank r over C. For each $1 \le s < r$, we denote the slope of maximal subbundle by $e_s(V)$.

$$e_s(V) := Max\ \{\ \frac{\deg(W)}{s}\ |\ W \subset V\ \text{ is a subbundle of rank } s\ \}$$

Define the asymptotic s-spectrum $\mathcal{AS}_s(V)$ and the asymptotic s-slope $\nu_s(V)$ as follows:

$$\mathcal{AS}_s(V) := \{\ \frac{e_s(f^*(V))}{\deg\ f}\ \}$$

$$\nu_s(V) := Limsup\ \frac{e_s(f^*(V))}{\deg\ f} = Limsup\ \mathcal{AS}_s(V)$$

where the supremum is taken over all finite morphisms $f : D \to C$. Now we have the following criterion for strong semistability in terms of the asymptotic slopes.

Theorem 4.1. *A vector bundle V is strongly semistable if and only if $\nu_s(V) = \mu(V)$ for some s. Then the asymptotic s-slopes $\nu_s(V)$ are equal to the usual slope $\mu(V)$ for all s.*

Proof. Let s be a given integer such that $1 \le s < r$. In [BP], it is proved that a vector bundle is strongly semistable if and only if for every morphism $f : D \to C$, and for every subbundle $W \subset f^*V$ of rank s, the slope of W is at most the slope of f^*V. Hence if V is not strongly semistable then there exists a finite morphism $f : D \to C$ and a subbundle $W \subset f^*V$ of rank s such that $\mu(W) > \mu(f^*(V))$. Hence $\nu_s(V) > \mu(V)$.

Assume V is strongly semistable. Then for any given finite map $f : D \to C$, f^*V is semistable and hence for every subbundle $W \subset f^*V$ of rank s, the slope of W is at most the slope of f^*V. Hence $\nu_s(V) \le \mu(V)$.

Given $\epsilon > 0$, choose $n \gg 0$ such that $s(2g + \epsilon_n) < n\epsilon$. Then for the line bundle L_n, $\mathcal{O}_{\mathbb{G}r(s,V)}(n) \otimes \pi^*L$ separate points by Lemma 2.2. The kernel of the universal sequence on a general complete intersection D on $\mathbb{G}r(s,V)$ defined by $\mathcal{O}_{\mathbb{G}r(s,V)}(n) \otimes \pi^*L$ (which exists as it separates points) has slope $n^{s(r-s)}([\mathcal{O}_{\mathbb{G}r(s,V)}(1)]^{s(r-s)} \cdot F)(\mu(V) - \frac{s(2g+\epsilon_n)}{n})$, by Lemma 2.4. The degree of $\pi_D : D \to C$ is $n^{s(r-s)}([\mathcal{O}_{\mathbb{G}r(s,V)}(1)]^{s(r-s)} \cdot F)$. Hence dividing by the degree of $D \to C$, we obtain a number whose difference with $\mu(V)$ is less than ϵ. Hence $\nu_{r-s}(V) = \mu(V)$ for semistable bundles. □

Remark 4.2. *Let V be a strongly semistable vector bundle over a smooth curve C. Then there is a sequence of genuinely ramified maps $f_i : D_i \to C$ such that*

$$Lim \frac{e_s(f_i^*(V))}{\deg\ f_i} \ = \ \mu(V)$$

follows from Lemma 3.4.

Let $V_\bullet := \{0 = V_0 \subset \cdots \subset V_l = V\}$ be the Harder Narasimhan filtration of V. Let d_i, r_i denote the degree and rank of V_i for $i = 0, 1, 2, \cdots l$. Assume that the Harder Narasimhan factors V_i/V_{i-1} are strongly semistable if the characteristic is positive. By [La], any vector bundle on a curve in positive characteristic has such a Harder Narasimhan Filtration after a finite number of Frobenius pull backs. Now we can determine the asymptotic slopes of V for each s.

Theorem 4.3. *$s\nu_s = d_i + (s - r_i)\mu_{i+1}$ where $r_i < s \leq r_{i+1}$*

Proof. First we note that if L is any line bundle, then $e_s(V \otimes L) = e_s(V) + \deg\ L$ and the spectrum $\mathcal{AS}_s(V \otimes L) = \mathcal{AS}_s(V) + \deg L$. Hence $\nu_s(V \otimes L) = \nu_s(V) + \deg L$. Since the Harder-Narasimhan filtration of $V \otimes L$ is given by $V_i \otimes L$, $d_i(V \otimes L) = d_i(V) + r_i \cdot \deg L$. Now assuming the formula for $V \otimes L$, we get:

$$s(\nu_s(V \otimes L)) = d_i(V \otimes L) + (s - r_i)\mu_{i+1}(V \otimes L)$$
$$s(\nu_s(V)) + s(\deg\ L) = d_i(V) + r_i(\deg\ L) + (s - r_i)(\mu_{i+1}(V)) + (s - r_i)(\deg\ L)$$

By simplifying, we obtain the formula for V.

Hence by taking deg L to be sufficiently large we may assume that all μ_i's are positive.

Let $f : D \to C$ be any finite map and $W \subset f^*(V)$ be a subbundle of rank s. Let $W_j \subset f^*(V_j/V_{j-1})$ be the saturation of the image of $W \cap f^*V_j$ in $f^*(V_j/V_{j-1})$. Let s_j be the rank of W_j and δ_j be equal to $\frac{\deg\ W_j}{\deg\ f}$. Then we have $s_j \leq r_j - r_{j-1}$, $r_i < \sum s_j = s \leq r_{i+1}$ and $\frac{\text{degree } W}{\deg\ f} \leq \sum \delta_j$. Since V_j/V_{j-1} is strongly semistable, we also have $\delta_j \leq s_j\mu_j \leq (r_j - r_{j-1})\mu_j$ for all $j \geq 1$. Now by comparing the first i terms and the rest of the terms we get the inequality:

$$\frac{\deg\ W}{\deg\ f} \leq \sum_{j=1}^{l} s_j\mu_j = \sum_{j=1}^{i} s_j\mu_j + \sum_{j=i+1}^{l} s_j\mu_j \leq \sum_{j=1}^{i} (r_j - r_{j-1})\mu_j + \sum_{j=i+1}^{l} s_j\mu_{i+1} \leq d_i + (s - r_i)\mu_{i+1}$$

To show the equality we produce subbundles of rank s in coverings with degree divided by the degree of the covering arbitrarily close to $d_i + (s - r_i)\mu_{i+1}$ where $r_i < s \leq r_{i+1}$. By Theorem 4.1, we can find a covering $f : D \to C$ and a subbundle $W_i \subset f^*(V_i/V_{i-1})$ of rank $s - r_i$ such that $\mu(V_i/V_{i-1}) - \mu(W_i) < \epsilon$. Let $W := \pi^{-1}(W_i)$ be the inverse image of W_i in $V_i \subset V$ by the projection $\pi : V_i \to V_i/V_{i-1}$. Then it shows that $d_i + (s - r_i)\mu_{i+1} - \mu(W)/\deg$ f $< \epsilon$. Hence the theorem. □

Theorem 4.4. *Let $f : Y \to X$ be a morphism and $W \subset f^*V$ be a subbundle of rank s with $r_i < s \leq r_{i+1}$, such that*

$$\frac{\deg\ (W)}{\deg\ f} > d_i + (s - r_i)\mu_{i+1} - (\mu(V_i) - \mu_{i+1})$$

Then $f^*V_i \subset W \subset f^*V_{i+1}$.

Proof. From the proof of Theorem 4.3, $s\nu_s = d_i + (s - r_i)\mu_{i+1}$, the inequalities becomes equality if and only if $s_j = r_j - r_{j-1}$ for all j, and hence $W \cap V_i = V_i$, proving $V_i \subset W$. □

5. The geometry of the spectrum

Notice that the s-spectrum $\mathcal{AS}_s(V) \subset [e_s(V), \overline{\mu}_{\max}(V)] \subset \mathbb{R}$ is a subset of the bounded interval. Hence it has maximum, minimum, and cluster points. We have described the supremum (asymptotic slopes) of the spectrum for each s. This leads to the following natural question.

Question: Are asymptotic s-slopes the only limit points of the spectrum? Or is it likely to be dense in the interval $[e_s(V), \mu_{\max}(V)]$?

Now we give a criterion for the asymptotic slopes to be an isolated value for strongly semistable vector bundles.

Lemma 5.1. *Let V be a strongly semistable vector bundle. Then $\mu(V)$ is an isolated point of the asymptotic s-spectrum $\mathcal{AS}_s(V)$ if and only if $\mathcal{AS}_s(V) = \{\ \mu(V)\ \}$, i.e., there exists a subbundle $W \subset V$ of rank s such that $\mu(W) = \mu(V)$.*

Proof. Assume $W \subset V$ is a subbundle of slope $\mu(V)$. Then for any map $f : D \to C$, $f^*(W) \subset f^*(V)$ is a maximal subbundle and hence $\frac{\mu(f^*W)}{\deg\ f} = \mu(W) = \mu(V)$. Hence the spectrum is a singleton, proving $\mathcal{AS}_s(V) = \{\ \mu(V)\ \}$.

Let V be a strongly semistable vector bundle such that $\mu(V)$ is isolated in the spectrum $\mathcal{AS}_s(V)$.

From Remark 4.2 and the hypothesis that $\mu(V)$ is isolated, it follows that there is a genuinely ramified map such that the pull back of V has a subbundle of same slope. By Lemma 3.5 (c), this subbundle descends to a subbundle of same slope as V. □

References

[BP] I. Biswas and A. J. Parameswaran, *A criterion for strongly semistable principal bundles over a curve in positive characteristic*, Bull. Sci. Math. 128(9), 761-773.

[F] W. Fulton, *Intersection Theory*, Springer Verlag, New York (1984)

[L] A. Langer, *Semistable sheaves in positive characteristic*, Ann. of Math. 159 (2004), 251–276.

[LN] H. Lange and M.S. Narasimhan, *Maximal subbundles of rank two vector bundles on curves*, Math. Ann. 266 (1983), 55-72.

[MR] V. B. Mehta and A. Ramanathan, *Restriction of stable sheaves and representations of the fundamental group*, Inv. Math. 77 (1984), 163-172.

[M] J. P. Murre, *Lectures on an introduction to Grothendieck's theory of the fundamental group*, T.I.F.R. Lecture notes (1967).

[Ne] Newstead, *Lectures on Introduction to moduli problems and orbit spaces*, TIFR Lecture notes, Narosa Publishing House, New Delhi (1978).

[N] M. V. Nori, *The fundamental group scheme*, Proc. Indian Acad. Sci. (Math. Sci.) 91 (1982), 73-122.

[RR] S, Ramanan and A. Ramanathan, *Some remarks on the instability flag*, Tohoku Math. Jour. 36(1984), 269-291.

School of Mathematics, Tata Institute of Fundamental Research, Homi Bhabha Road, Mumbai 400005, India

E-mail address: param@math.tifr.res.in

E-mail address: subramnn@math.tifr.res.in