Affine lines on $\mathbb{Q}$-homology planes

By

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1. Introduction

An algebraic surface $X$ defined over $\mathbb{C}$ is called a $\mathbb{Q}$ (respectively $\mathbb{Z}$) homology plane if $H_i(X, \mathbb{Q}) = 0$ (resp. $H_i(X, \mathbb{Z}) = 0$) for all $i > 0$. By a result of T. Fujita, a $\mathbb{Q}$-homology plane is an affine surface. $\mathbb{Q}$-homology planes occur naturally and "abundantly" as follows. Let $Z$ be a smooth rational surface and $D$ a simply connected curve on $Z$ whose irreducible components generate $H_2(Z; \mathbb{Q})$ freely. Then $X := Z - D$ is a $\mathbb{Q}$-homology plane (cf. Lemma 5).

Following results about the existence of contractible algebraic curves on $\mathbb{Q}$-homology planes are known.

(i) If $\kappa(X) = \infty$, then there is a morphism $\phi: X \to B$ where $B$ is a nonsingular curve, such that a general fibre of $\phi$ is isomorphic to $\mathbb{C}$, and hence there are infinitely many contractible curves on $X$ (cf. [M], Chapter I, Theorem 3.13).

(ii) If $\kappa(X) = 1$, then $X$ contains at least one and at most two contractible curves (cf. [M-S], Lemma 2.15). If $X$ is a $\mathbb{Z}$-homology plane with $\kappa(X) = 1$, then $X$ contains a unique contractible curve and it is smooth (cf. [G-M]).

(iii) If $\kappa(X) = 2$, then $X$ contains no contractible algebraic curve (cf. [M-T2]).

In this paper we complete the picture by proving the following (somewhat unexpected) result. For the terminology used in the statement of the theorem, see §1.

**Theorem.** Let $X$ be a $\mathbb{Q}$-homology plane with $\kappa(X) = 0$. Then the following assertions are true.

(i) If $X$ is not NC-minimal, then $X$ contains a unique contractible curve $C$. Moreover $C$ is smooth with $\kappa(X - C) = 0$.

(ii) If $X$ is NC-minimal and not the surface $H[k, -k]$ in Fujita's classification, then $X$ has no contractible curves.

(iii) If $X$ is NC-minimal and is isomorphic to $H[k, -k]$ with $k \geq 2$, then there is a unique contractible curve $C$ on $X$ and it is smooth. Further, $\kappa(X - C) = 0$.

(iv) The surface $X = H[1, -1]$ has exactly two contractible curves, say $C$.
and \( L \). Further, both the curves are smooth, \( \bar{\kappa}(X-C) = 0 \) and \( \bar{\kappa}(X-L) = 1 \). The curves \( C \) and \( L \) intersect each other transversally in exactly two points.

It should be remarked that by a beautiful result of Fujita, there does not exist a \( \mathbb{Z} \)-homology plane \( X \) with \( \bar{\kappa}(X) = 0 \). This follows from the complete classification of NC-minimal \( \mathbb{Q} \)-homology planes with \( \bar{\kappa}(X) = 0 \) due to Fujita (cf. [F, §8.64]). A direct and short proof of this was recently found by the first author and M. Miyanishi. In this paper we use this classification of Fujita in a crucial way.

Combining the results in this paper with the earlier known results, we get the following.

**Corollary.** A \( \mathbb{Q} \)-homology plane with three contractible curves is of logarithmic Kodaira dimension \(-\infty\).

### 2. Notations and preliminaries

All algebraic varieties considered in this paper are defined over the field of complex numbers \( \mathbb{C} \).

For any topological space \( X \), \( e(X) \) denotes its topological Euler characteristic.

Given a connected, smooth, quasiprojective variety \( V \), \( \bar{\kappa}(V) \) denotes the logarithmic Kodaira dimension of \( V \) as defined by S. Iitaka (cf. [I]).

By a \((-n)\)-curve on a smooth algebraic surface we mean a smooth rational curve with self-intersection \(-n\). By a normal crossing divisor on a smooth algebraic surface we mean a reduced algebraic curve \( C \) such that every irreducible component of \( C \) is smooth, no three irreducible components pass through a common point and all intersections of the irreducible components of \( C \) are transverse. For brevity, we will call a normal crossing divisor an n.c. divisor. Let \( D \) be an n.c. divisor on a smooth surface. We say that \( D \) is a minimal normal crossing divisor if any \((-1)\)-curve in \( D \) intersects at least three other irreducible components of \( D \). A minimal normal crossing divisor will be called an m.n.c. divisor for brevity.

Following Fujita, we call a divisor \( D \) on a smooth projective surface \( Y \) pseudo-effective if \( H \cdot D \geq 0 \) for every ample divisor \( H \) on \( Y \).

For the convenience of the reader, we now recall some basic definitions which are used in the results about Zariski-Fujita decomposition of a pseudo-effective divisor (cf. [F], §6; [M-T], Chapter 1).

Let \( (Y,D) \) be a pair of a nonsingular surface \( Y \) and a normal crossing divisor \( D \). A connected curve \( T \) consisting of irreducible curves in \( D \) (a connected curve in \( D \), for short) is a twig if the dual graph of \( T \) is a linear chain and \( T \) meets \( D-T \) in a single point at one of the end points of \( T \); the other end of \( T \) is called a tip of \( T \). A connected curve \( R \) (resp. \( F \)) in \( D \) is a club (resp. an abnormal club) if \( R \) (resp. \( F \)) is a connected component of \( D \) and the
homology planes

The dual graph of $R$ (resp. $F$) is a linear chain (resp. the dual graph of the exceptional curves of a minimal resolution of singularities of a non-cyclic quotient singularity). A connected curve $B$ in $D$ is rational (resp. admissible) if each irreducible component of $B$ is rational (resp. if none of the irreducible components of $B$ is a $(-1)$-curve and the intersection matrix of $B$ is negative definite). An admissible rational twig $T$ is maximal if $T$ is not contained in an admissible rational twig with more irreducible components.

Let $|T_d|$ (resp. $|R_d|$ and $|F_d|$) be the set of all admissible rational maximal twigs (resp. admissible rational maximal clubs and admissible rational maximal abnormal clubs). Then there exists a decomposition of $D$ into a sum of effective $\mathbb{Q}$-divisors, $D = D^* + Bk(D)$, such that $\text{Supp}(Bk(D)) = \bigcup_i T_i \cup \bigcup_\mu R_\mu \cup \bigcup_\nu F_\nu$ and $(K_Y + D^*) \cdot Z = 0$ for every irreducible component $Z$ of $\text{Supp}(Bk(D))$. The divisor $Bk(D)$ is called the bark of $D$, and we say that $K_Y + D^*$ is produced by the peeling of $D$. For details of how $Bk(D)$ is obtained from $D$, see [M-T].

The Zariski–Fujita decomposition of $K_Y + D$, in case $K_Y + D$ is pseudo-effective, is as follows:

There exist $\mathbb{Q}$-divisors $P, N$ such that $K_Y + D \approx P + N$ where, $\approx$ denotes numerical equivalence, and

(a) $P$ is numerically effective (nef, for short). If $\kappa(Y - D) = 0$, then $P \approx 0$ by a fundamental result of Kawamata (cf. [Ka2]).

(b) $N$ is effective and the intersection form on the irreducible components of $N$ is negative definite

(c) $P \cdot D_i = 0$ for every irreducible component $D_i$ of $N$.

$N$ is unique and $P$ is unique up to numerical equivalence. If some multiple of $K_Y + D$ is effective, then $P$ is also effective.

The following result from [F, Lemma 6.20] is very useful.

**Lemma 1.** Let $(Y, D)$ be as above. Assume that all the maximal rational twigs, maximal rational clubs and maximal abnormal rational clubs of $D$ are admissible. Let $\kappa(Y - D) \geq 0$. As above, let $P + N$ be the Zariski decomposition of $K_Y + D$. If $N \neq Bk(D)$, then there exists a $(-1)$-curve $L$, not contained in $D$, such that one of the following holds:

(i) $L$ is disjoint from $D$
(ii) $L \cdot D = 1$ and $L$ meets an irreducible component of $Bk(D)$
(iii) $L \cdot D = 2$ and $L$ meets two different connected components of $D$ such that one of the connected components is a maximal rational club $R_\nu$ of $D$ and $L$ meets a tip of $R_\nu$

Further, $\kappa(V - D - L) = \kappa(Y - D)$.

Following Fujita, we will say that a smooth affine surface $V$ with $\kappa(V) \geq 0$ is NC-minimal if it has a smooth projective completion $\bar{V}$ such that $D := \bar{V} - V$ is an m.n.c. divisor and $N = Bk(D)$, where $P + N$ is the Zariski–Fujita decomposition of $K_{\bar{V}} + D$. 
The following results proved by Kawamata will be used frequently.

**Lemma 2.** (cf. [Ka1]). Let Y be a smooth quasi-projective algebraic surface and $f: Y \to B$ be a surjective morphism to a smooth algebraic curve such that a general fibre $F$ of $f$ is irreducible. Then $\kappa(Y) \geq \kappa(B) + \kappa(F)$.

**Lemma 3.** (cf. [Ka2]). Let $Y$ be a smooth quasi-projective algebraic surface with $K(Y) = 1$. Then there is a Zariski-open subset $U$ of $Y$ which admits a morphism $f: U \to B$ onto a smooth algebraic curve $B$ such that a general fibre of $f$ is isomorphic to either $\mathbb{C}^*$ or an elliptic curve.

We call such a fibration a $\mathbb{C}^*$-fibration or an elliptic fibration respectively.

Similarly, we can define a $\mathbb{C}$-fibration and a $\mathbb{P}^1$-fibration on a smooth projective surface.

As mentioned in the introduction, the next result follows from R. Kobayashi’s inequality and plays an important role in the proof of the theorem.

**Lemma 4.** (cf. [M-T2]). Let $V$ be a smooth affine surface with $e(V) \leq 0$. Then $\kappa(V) \leq 1$.

We begin with some properties of $\mathbb{Q}$-homology planes.

Let $X$ be a smooth affine surface and $X \subset Z$ be a smooth projective compactification with $D := Z - X$.

**Lemma 5.** Assume that the irregularity $q(Z) = 0$. Then $X$ is a $\mathbb{Q}$-homology plane if and only if the irreducible components of $D$ generate $H_2(Z; \mathbb{Q})$ freely and $H_1(D; \mathbb{Q}) = 0$.

**Proof.** We use the long exact cohomology sequence with $\mathbb{Q}$-coefficients of the pair $(X, D)$. By Poincaré duality, $H^i(Z; D; \mathbb{Q}) = H_{4-i}(X)$. Hence $H_i(X) = 0$ for $i > 0$ if and only if the restriction map $H^i(Z; \mathbb{Q}) \to H^i(D; \mathbb{Q})$ is an isomorphism for $i < 4$. Since $H_1(Z; \mathbb{Q}) = H_3(Z; \mathbb{Q}) = 0$ by assumption, it follows that $X$ is a $\mathbb{Q}$-homology plane if and only if $H_1(D; \mathbb{Q}) = 0$ and the irreducible components of $D$ generate $H_2(Z; \mathbb{Q})$ freely.

Now let $X$ be an affine surface with either a $\mathbb{C}$-fibration or a $\mathbb{C}^*$-fibration, $\Phi: X \to B$. For a suitable smooth compactification $X \subset Z$ we get a $\mathbb{P}^1$-fibration $\Phi: Z \to \overline{B}$, where $\overline{B}$ is a smooth compactification of $B$. We will need the following result due to Gizatullin.

**Lemma 6.** Let $F$ be a scheme-theoretic fibre of $\Phi$. Then we have:

1. $F_{\text{red}}$ is a connected normal crossing divisor all whose irreducible components are isomorphic to $\mathbb{P}^1$.
2. If $F$ is not isomorphic to $\mathbb{P}^1$, then $F_{\text{red}}$ contains a $(-1)$-curve. If a $(-1)$-curve occurs with multiplicity 1 in $F$, then $F_{\text{red}}$ contains another $(-1)$-curve.
Note that from (1) it follows that a \((-1)\)-curve in \(F_{\text{red}}\) meets at most two other irreducible components of \(F\).

Let \(\phi: X \rightarrow B\) be a \(\mathbb{C}^*\)-fibration and \(\Phi: Z \rightarrow \overline{B}\) be an extension as above. Then \(D\) contains either one or two irreducible components which map onto \(\overline{B}\) by \(\Phi\). We will call these components as horizontal. All other irreducible components of \(D\) are contained in the fibres of \(\Phi\). An irreducible component of \(D\) will be called a \(D\)-component for the sake of brevity. We say that \(\phi\) is twisted if there is only one horizontal \(D\)-component (in \([F]\), such a fibration is called a gyozâ). Otherwise we say that \(\phi\) is untwisted (in \([F]\), such a fibration is called a sandwich). In the untwisted case the horizontal \(D\)-components are cross-sections of \(\Phi\) and in the twisted case the horizontal \(D\)-component is a \(2\)-section.

The next result follows by an easy counting argument using the fact that the irreducible components of the divisor at infinity in a smooth compactification of a \(\mathbb{Q}\)-homology plane generate the Picard group, \(\text{Pic}(X)\), freely over \(\mathbb{Q}\).

**Lemma 7.** (cf. [G-M], Lemma 3.2). Let \(\phi: X \rightarrow B\) be a \(\mathbb{C}^*\)-fibration on a \(\mathbb{Q}\)-homology plane \(X\). Then we have:

1. If \(\phi\) is twisted, then \(B \cong \mathbb{C}\), all the fibres of \(\phi\) are irreducible, there is a unique fibre \(F_0\) of \(\phi\) such that \(F_{\text{red}}\) is isomorphic to \(\mathbb{C}\) and all other fibres are isomorphic to \(\mathbb{C}^*\), if taken with reduced structure.

2. If \(\phi\) is untwisted and \(B \cong \mathbb{P}^1\), then all the properties of the fibres of \(\phi\) are the same as (1) above.

3. If \(\phi\) is untwisted and \(B \cong \mathbb{C}\), then \(\phi\) has exactly one fibre \(F_0\) with two irreducible components and all the other fibres are isomorphic to \(\mathbb{C}^*\), if taken with reduced structure. Either both the components of \(F_0\) are isomorphic to \(\mathbb{C}\) which intersect transversally in one point or they are disjoint with one isomorphic to \(\mathbb{C}\) and the other one isomorphic to \(\mathbb{C}^*\).

In order to avoid repetitive arguments in the proof of the theorem, we give detailed proof of the next result and use such arguments without details later on.

**Lemma 8.** Let \(X\) be a \(\mathbb{Q}\)-homology plane with \(\kappa(X) = 0\) and \(\phi: X \rightarrow B\) be a \(\mathbb{C}^*\)-fibration. Let \(F_0\) be the reducible fibre of \(\phi\) (cf. lemma 7) which contains a contractible irreducible curve \(C\). Consider a smooth completion \(Z \supset X\) with \(D := Z - X\) an n.c. divisor and \(\Phi: Z \rightarrow \mathbb{P}^1\) a \(\mathbb{P}^1\)-fibration which extends \(\phi\).

1. Suppose \(\phi\) is twisted.

If \(\kappa(X - C) = 0\), then the morphism \(X - C \rightarrow \mathbb{C}^*\) has no singular fibres. If \(\kappa(X - C) = 1\), then the morphism \(X - C \rightarrow \mathbb{C}^*\) has at least one multiple fibre.

In both the cases, the fibre over the point \(p_\infty := \mathbb{P}^1 - B\) can be assumed to have the dual graph.
and the horizontal component $D_h$ intersects the $(-1)$-curve transversally in a single point.

(2) Suppose $\phi$ is untwisted and $B \cong \mathbb{C}$.

Then the fibre $F_\infty$ over $p_\infty$ is a regular fibre of $\Phi$ and the two horizontal $D$-components meet this fibre in two distinct points. The morphism $X - C \to \mathbb{C}$ has at least one multiple fibre.

(3) Suppose $\phi$ is untwisted and $B \cong \mathbb{P}^1$.

If $\overline{k}(X - C) = 0$, then $\phi': X - C \to \mathbb{C}$ has at least one and at most two multiple fibres. If $\phi'$ has two multiple fibres, then their multiplicities are 2 each. If $\overline{k}(X - C) = 1$, then $\phi'$ has at least two multiple fibres.

Proof. (1) Let $\phi = \phi|_{X - C}$. Suppose $\phi'$ has a multiple fibre, say $m_1 F_1$, with $m_1 \geq 2$. Denote by $p_0, p_1$ the points $\phi(C), \phi(F_1)$ respectively. Using lemma 9, we can construct a finite ramified covering $\tau: A \to \mathbb{C}$, ramified only over $p_0, p_1$ such that the ramification index over $p_i$ is $m_i$ for $i = 0, 1$, where $m_0$ is a large integer. Then the normalization of the fibre product $A \times C$ contains a Zariski-open subset $U$ which is a finite étale covering of $X - C$. Since $\overline{k}(A) = 1$ for large $m_0$, by lemma 2, $\overline{k}(U) = 1$. But then $\overline{k}(X - C) = 1$, since $\overline{k}$ does not change under finite étale coverings by a result of Iitaka (cf. [1]). This contradiction shows that $\phi'$ has no multiple fibre, if $\overline{k}(X - C) = 0$. Hence $\phi'$ has no singular fibre.

If $\phi'$ has no multiple fibre, then $X - C$ has a 2-sheeted étale cover which is isomorphic to $\mathbb{C}^* \times \mathbb{C}^*$. Hence $\overline{k}(X - C) = 0$.

The assertion about the fibre $F_\infty$ is proved by Fujita in [F], lemma 7.5(2).

(2) The assertion about $F_\infty$ is proved in [F], lemma 7.6(1). If $\phi'$ has no multiple fibre, then $X - C$ is isomorphic to $\mathbb{C} \times \mathbb{C}^*$, contradicting the assumption that $\overline{k}(X) = 0$.

(3) Suppose $\overline{k}(X - C) = 0$. If $\phi'$ has no multiple fibre, then $X - C$ is isomorphic to $\mathbb{C} \times \mathbb{C}^*$, a contradiction. If $\phi'$ has two multiple fibres $m_1 F_1, m_2 F_2$, then letting $p_i$ be the points $\phi(F_i)$ for $i = 0, 1, 2$, we can construct a finite galois covering $\tau: A \to \mathbb{P}^1$ which is ramified only over $p_i$ and the ramification index at any point over $p_i$ is $m_i$ for $i = 0, 1, 2$. If one of the $m_1, m_2$ is strictly bigger than 2, then for large $m_0$, $A$ is non-rational. But then we see that $\overline{k}(X - C) \geq 1$. Hence $m_1 = m_2 = 2$.

The proof for the case $\overline{k}(X - C) = 1$ is similar.

The next result follows from R. H. Fox's solution of Fenchel's conjecture (cf. [Fo] and [C]).

**Lemma 9.** Let $a_1, \ldots, a_r$ be distinct points in $\mathbb{P}^1$ with $r \geq 3$ and $m_1, \ldots, m_r$ be integers $\geq 2$. Then there is a finite Galois covering $\tau: B \to \mathbb{P}^1$ such that the rami-
Lemma 10. Let $C_1, C_2$ be two distinct contractible curves on a $\mathbb{Q}$-homology plane $X$ with $\kappa(X) \geq 0$. Then $C_1 \cap C_2 \neq \emptyset$ and if the intersection is a single point then it is transverse.

Proof. Since $e(X - C_1) = 0$, by lemma 4, $\kappa(X - C_1) \leq 1$. Clearly, $\kappa(X - C_1) \geq 0$.

Consider the case $\kappa(X - C_1) = 0$. Since $\text{Pic}(X)$ is finite, there exists a regular function $f$ of $X$ such that $(f) = mC_1$ for some integer $m$. We can assume that the morphism given by $f: X - C_1 \to \mathbb{C}^*$ has connected general fibres. Then by lemma 2, a general fibre of this morphism is isomorphic to $\mathbb{C}^*$. Thus, $X$ has a $\mathbb{C}^*$-fibration such that $C_1$ is contained in a fibre. Suppose $C_1 \cap C_2 = \emptyset$. Since $C_2$ does not contain any non-constant units, the image of $C_2$ is a point. This contradicts lemma 7.

Suppose $\kappa(X - C_1) = 1$. If $C_1 \cap C_2 = \emptyset$, then $e(X - (C_1 \cup C_2)) = -1$ and hence by lemma 4, $\kappa(X - (C_1 \cup C_2)) = 1$. Then by lemma 3 we see that $X - (C_1 \cup C_2)$ has a $\mathbb{C}^*$-fibration. Since $X$ does not contain any complete curves, this morphism extends to a $\mathbb{C}^*$-fibration on $X$. Then $C_1$ and $C_2$ are mapped to points, otherwise the fibration is a $\mathbb{C}$-fibration. Again by lemma 7, both $C_1, C_2$ lie in the same fibre and hence $C_1, C_2$ intersect transversally in a single point by part (3) of lemma 7.

Now we know that $C_1 \cap C_2 \neq \emptyset$. Suppose $C_1 \cap C_2$ is a single point. Then $e(C_1 \cup C_2) = 1$, $e(X - C_1 \cup C_2) = 0$, and hence $\kappa(X - (C_1 \cup C_2) \leq 1$ by lemma 4. Arguing as above, we see that $X$ admits a $\mathbb{C}^*$-fibration such that $C_1 \cup C_2$ is contained in a single fibre and hence they intersect transversally in a single point, again by lemma 7.

3. Fujita's classification

In this section we describe the classification of NC-minimal $\mathbb{Q}$-homology planes with $\kappa = 0$ due to Fujita (cf. [F], 8.64). There are four types of such surfaces. We also describe Fujita's surfaces $H[-1, 0, -1]$, which are NC-minimal surfaces with $\kappa = 0, e = 0$ and $b_1 = 1$.

Type 1 (cf. [F], §8.26). $H[k, -k]$ with $k \geq 1$

The dual graph of the divisor $D$ at infinity for an m.n.c. compactification is given by

\[ \begin{array}{c}
T_1 & \bullet \\
B_1 & \bullet \\
T_3 & \bullet \\
T_2 & \bullet \\
B_2 & \bullet \\
T_4 & \bullet \\
\end{array} \]
Here \( B_i^2 = k, B_i^3 = -k \) and \( T_i^2 = -2 \) for all \( i \). There is a \((-1)\)-curve \( E_1 \) meeting the tips \( T_1, T_2 \) transversally in a single point and no other point of \( D \). Similarly, there is a \((-1)\)-curve \( E_2 \) meeting \( T_3, T_4 \) transversally in a single point and no other point of \( D \). The divisor \( F_1 = T_1 + 2E_1 + T_2 \) is a fibre of a \( \mathbb{P}^1 \)-fibration \( \Phi \) on \( X \) and \( F_2 = T_3 + 2E_2 + T_4 \) is another fibre of \( \Phi \). The curves \( B_1 \) and \( B_2 \) are cross sections of \( \Phi \). Let \( F_0 \) be the fibre of \( \Phi \) through \( B_1 \cap B_2 \). Clearly \( C = F_0 - (B_1 \cap B_2) \cong \mathbb{C} \), hence \( C \) is a contractible curve in \( X \).

**Lemma 11.** \( \kappa (X-C) = 0 \).

**Proof.** The \( \mathbb{C}^* \)-fibration \( \phi : X-C \to \mathbb{C} \) has exactly two multiple fibres corresponding to \( 2E_1 \) and \( 2E_2 \). Let \( p_i = \phi (F_i) \) for \( i = 0, 1, 2 \). Using lemma 9 we can construct a degree 2 galois covering \( \tau : B \to \mathbb{P}^1 \) such that the ramification index over \( p_i \) is 2 for each \( i \). By Riemann–Hurwitz formula, \( B \cong \mathbb{P}^1 \). Then \( X \times \tau B \to B \) is a \( \mathbb{C}^* \)-fibration and \( X \times \tau B - \tau^{-1} (C) \) is an étale cover of \( X-C \) isomorphic to \( \mathbb{C}^* \times \mathbb{C}^* \). Hence \( \kappa (X-C) = 0 \).

Types 2, 3 and 4 are denoted by \( Y[3, 3, 3], Y[2, 4, 4] \) and \( Y[2, 3, 6] \) respectively by Fujita (§8.37, 8.53, 8.54, 8.59, 8.61). The dual graphs of each of these have a unique branch point. There are three maximal twigs \( T_1, T_2 \) and \( T_3 \) for each of them and \( \sum_1^3 1/d (T_i) = 1 \), where \( d (T_i) \) is the absolute value of the determinant of the intersection matrix of \( T_i \).

Fujita has shown that \( \pi_1 (X) \) is a finite cyclic group for any NC-minimal \( \mathbb{Q} \)-homology plane with \( \kappa (X) = 0 \). This result will be used effectively in the next section.

Now we will describe the surfaces \( H[-1, 0, -1] \) (cf. [F], §8.5).

The dual graph of an m.n.c. divisor at infinity is given by

![Diagram](image)

Here, \( B_1^2 = B_2^3 = -1, D_0^2 = 0 \) and \( T_i^2 = -2 \).

**4. Proof of the Theorem (Non NC-minimal case)**

Let \( X \) be a \( \mathbb{Q} \)-homology plane with \( \kappa (X) = 0 \). In this section we prove the following.

**Proposition.** Suppose \( X \) does not have an NC-minimal compactification, then \( X \) contains a unique contractible curve.

**Proof.** Suppose \( L \) is a contractible curve in \( X \). Then \( \kappa (X-L) \leq 1 \) and there is a \( \mathbb{C}^* \)-fibration \( \phi' : X-L \to \mathbb{P}^1 \) which extends to a \( \mathbb{C}^* \)-fibration \( \phi : X \to \mathbb{P}^1 \).
and $\phi(L)$ is a point (cf. proof of lemma 10). We choose a smooth compactification $X \subset \mathbb{Z}$ such that $D = Z - X$ is a normal crossing divisor and $\phi$ extends to a $\mathbb{P}^1$-fibration $\Phi: Z \to \mathbb{P}^1$. We now consider the three cases given by lemma 7.

Case 1. $\phi$ is twisted. By lemma 7 (1), $B \cong \mathbb{C}$ and every fibre of $\phi$ is irreducible. The fibre $F_\infty = \Phi^{-1}(p_\infty)$ has the dual graph as described in lemma 8 (1) and the 2-section $D_h$ meets the $(-1)$-curve in $F_\infty$ transversally in a single point.

First consider the case $\kappa(X - L) = 0$. The surface $X - L$ has the following properties.

(i) $X - L$ is affine
(ii) $\kappa(X - L) = 0$
(iii) $e(X - L) = b_g(X - L) = 0$ and $b_1(X - L) = 1$
(iv) $X - L$ is NC-minimal.

The property (iii) follows from the long exact cohomology sequence with compact support of the pair $(X, L)$ and duality. The property (iv) follows from the observation that if $X - L$ is not NC-minimal, then by lemma 1, $X - L$ contains a curve $C \cong \mathbb{C}$. But then $C$ is closed in $X$ and disjoint from $L$, contradicting lemma 10.

Now the surface $X - L$ is isomorphic to $H[-1, 0, -1]$. Let $F_0$ be the fibre of $\Phi$ containing $L$. We may assume that any $(-1)$-curve in $D$ contained in $F_0$ meets at least two other $D$-components in $F_0$. Since $D$ is a connected tree of $\mathbb{P}^1$'s, either $F_{\text{red}} = \overline{L}$ or the horizontal component $D_h$ meets an irreducible component $D_0$ of $D$ which occurs with multiplicity 2 in $F_0$ (observe that $F_0 - \overline{L}$ is connected). Suppose $D_1 \subset D$ is a $(-1)$-curve in $F_0$ which is disjoint from $D_h$. Then by lemma 6 (1), $D_1$ meets at most two other $D$-components contained in $F_0$. Hence we can contract $D_1$ to a smooth point and get another compactification $Z_1$ which satisfies the same properties as $Z$.

Repeating this argument we can assume that $\overline{L}$ and $D_0$ are the only possible $(-1)$-curves in $F_0$. Moreover, if $D_0$ is a $(-1)$-curve then it meets two other $D$-components. We claim that $D_h$ is not a $(-1)$-curve. Otherwise, the m.n.c. divisor obtained from $D \cup \overline{L}$ by succession of contractions of $(-1)$-curves cannot be of the type described by Fujita. Now we see that $D$ is an m.n.c. divisor.

Since $X$ is not NC-minimal and $D$ is m.n.c., there exists a $(-1)$-curve $\overline{C}$ given by lemma 1. Let $C = \overline{C} \cap X$. If $\overline{C} \neq \overline{L}$ then $\overline{C}$ is horizontal as it has to meet $L$. Hence $\overline{C}$ meets one of the tip components $T_i$ of $F_\infty$. As above, $X - C$ is also of the type $H[-1, 0, -1]$. By contracting $C$ and then the image of $T_i$, we obtain a compactification divisor of $X - C$ which is not of type $H[-1, 0, -1]$. Hence $C = L$.

By lemma 8 (1), $\kappa(X - L) = 1$ if and only if $\phi$ has at least one multiple fibre other than $L$. Now assume that $\kappa(X - L) = 1$. Then we can see that $D_h$
meets at least three $D$-components and hence $D$ can be assumed to be m.n.c. as above. By lemma 1, there is a $(-1)$-curve $\tilde{C}$ in $Z$ satisfying the properties stated there. We arrive at a contradiction as above by first contracting $C$ and then $T_i$.

Case 2. $\phi$ is untwisted and $B \cong \mathbb{C}$. Now $\phi$ has a unique fibre which contains two irreducible components, say $L$ and $L'$. Any other fibre of $\phi$ is isomorphic to $C^*$, if taken with reduced structure. The fibre $F_\infty$ is a smooth fibre of $\phi$ and the two horizontal components of $D$ meet $F_\infty$ in distinct points. The divisor $D$ may not be m.n.c., but it is obtained from an m.n.c. divisor by successive blow-ups. By lemma 8(2), the morphism $X \to \mathbb{C}$ has at least one multiple fibre. From this we can see as above that $D$ can be assumed to be m.n.c. Again since $X$ is not NC-minimal, we get a $(-1)$-curve $\tilde{C} \cong \mathbb{P}^1$ on $Z$ which meets only a twig component of $D$. If $\tilde{C} \neq L$, then we get a contradiction as above.

Case 3. $\phi$ is untwisted and $B \cong \mathbb{P}^1$. Then every fibre of $\phi$ is irreducible. Any fibre of $\phi$ other than $L$ is isomorphic to $C^*$, if taken with reduced structure. By lemma 7.6 of [F], we can assume that every fibre of $\Phi$ other than the fibre $F_0$ containing $L$ is a linear chain such that the two horizontal components of $D$ meet the tip components of the fibre. From the connectivity of $D$ we see that the union of $D$-components in $F_0$ is connected. Denote by $D_1$, $D_2$ the horizontal components. Let $D_0$ be a $D$-component contained in $F_0$ which meets $D_1$ or $D_2$. Then $D_0$ occurs with multiplicity 1 in $F_0$. If $D_0$ is a $(-1)$-curve it can meet at most one more $D$-componet in $F_0$. Hence we can contract $D_0$ to get a smaller compactification of $X$. Consequently we can assume that $L$ is the unique $(-1)$-curve in $F_0$.

Now $(K_Z + D) \cdot \tilde{L} = 0$. On the other hand, if $K_Z + D \cong P + N$ is the Zariski-Fujita decomposition then $P \cong 0$ by the properties of the Zariski decomposition. Hence $N \cdot L = 0$. From the assumption that $X$ is not NC-minimal, we know that there exists a curve $C \subset X$ such that $C \cong \mathbb{C}$ and its closure $\bar{C}$ occurs in $N$. But by lemma 10 if $L \neq C$ then $L \cdot C > 0$.

If $\kappa(X-L) = 1$, then by lemma 8, the morphism $X \to \mathbb{C}$ has at least two multipie fibres. Then both $D_1$ and $D_2$ are branch points for the dual graph of $D$ and hence $D$ is m.n.c. The curve $\bar{C}$ above can be assumed to be a $(-1)$-curve. Since $\bar{C} \cdot \tilde{L} > 0$, the intersection form on the subspace of Pic $Z \otimes \mathcal{O}$ generated by $\bar{C}$ and $\tilde{L}$ is not negative definite. Hence $\tilde{L}$ does not occur in $N$ and $N \cdot \tilde{L} > 0$ as $\bar{C} \subset N$, a contradiction. If $\kappa(X-L) = 0$, then we have a morphism $X \to \mathbb{C}$ with one fibre $mL$ and general fibre isomorphic to $C^*$, as in the proof of lemma 10. This is a twisted fibration by lemma 7. Then we are reduced to the case 1 and hence $L$ is the unique contractible curve. This completes the proof of the proposition.
5. Proof of the Theorem (NC-minimal case)

We begin with the following general result.

**Lemma 12.** Let $\Gamma$ be a connected normal crossing divisor on a smooth projective surface $Y$. Assume the following conditions.

(i) Every irreducible component of $\Gamma$ is isomorphic to $\mathbb{P}^1$.

(ii) The dual graph of $\Gamma$ has at most one branch point.

(iii) If the dual graph has a branch point, then $\Gamma$ has exactly three maximal twigs $T_1$, $T_2$ and $T_3$ and $\sum 1/d(T_i) > 1$.

(iv) $\Gamma$ supports a divisor $G$ with $G \cdot G > 0$.

Then $\overline{K}(Y-\Gamma) = -\infty$.

**Proof.** Suppose that $\overline{K}(Y-\Gamma) \geq 0$. We will give the proof when $\Gamma$ has a branch point. Then $K_Y + \Gamma$ has a Zariski-decomposition $P + N$. First assume that $(Y, \Gamma)$ is NC-minimal. Then $N = B_k(\Gamma)$. Let $C_1$, $C_2$ and $C_3$ be the irreducible components of the maximal twigs $T_1$, $T_2$ and $T_3$ respectively meeting $C_0$, the $\Gamma$-component corresponding to the branch point. By lemma 6.16 of [F], the coefficients of $C_i$ in $B_k(\Gamma)$ are $1/d(T_i)$. Hence $P = K_Y + C_0 + \sum_{i=1}^{3} (1 - \frac{1}{d(T_i)}) C_i + \cdots$. But then $P \cdot C_0 = -2 + \sum (1 - 1/d(T_i)) < 0$, contradicting the fact that $P$ is nef.

If $(Y, \Gamma)$ is not NC-minimal, by lemma 1 we can reduce to the case when there is a $(-1)$-curve $E$ on $Y$ which occurs in $N$, $E$ is not contained in $\Gamma$ and $E \cdot \Gamma = 1$, where $E$ meets a component of $B_k(\Gamma)$. Then $\overline{K}(Y-\Gamma) = \overline{K}(Y-\Gamma \cup E)$. By contracting $E$ and any $(-1)$-curves in the maximal twigs successively we reduce to the situation when either the image of $\Gamma$ becomes linear or a maximal twig has a vertex with non-negative weight or the NC-minimal case occurs. If a maximal twig has a vertex with non-negative weight then by lemma 6.13 of [F], we get $\overline{K}(Y-\Gamma) = -\infty$, a contradiction. This proves the result.

Let $X$ be an NC-minimal $\mathbb{Q}$-homology plane with $\overline{K}(X) = 0$. Then $\pi_1(X)$ is a finite cyclic group by Fujita.

**Lemma 13.** Assume that $X$ contains a contractible curve $C$. Then $X$ is of type $H[k, -k], k \geq 1$.

**Proof.** As before, there is a $\mathbb{C}^*$ fibration $\phi: X \rightarrow B$ with $\phi(C)$ a point and $B \cong \mathbb{C}$ or $\mathbb{P}^1$. We consider the three cases depending on the type of $\phi$.

Case 1. $\phi$ is twisted.

Then $B \cong \mathbb{C}$ and all the fibres of $\phi$ are irreducible. We claim that $\phi$ has at most one multiple fibre. Let $p_1, \ldots, p_r$ be the points in $B$ corresponding to the multiple fibres and $p_\infty = \mathbb{P}^1 - B$. If $r \geq 2$, then we can construct a suitable non-cyclic covering $A \rightarrow \mathbb{P}^1$, ramified over $p_1, \ldots, p_r, p_\infty$. Then we get a connected étale cover $\widetilde{X} \rightarrow X$ with non-cyclic galois group. This is not possible.
Hence $r \leq 1$.

As before, $\phi$ extends to a $\mathbb{P}^1$-fibration $\Phi: Z \rightarrow \mathbb{P}^1$ on a smooth compatification $Z$ of $X$. Let $D = Z - X$. As in lemma 8, we see that $\overline{\kappa}(X - C) = 0$ if the morphism $X - C \rightarrow \mathbb{C}^*$ has no multiple fibre. Let $F_0$ be the fibre of $\Phi$ containing $C$.

Using the lemma 12, we now see that the dual graph of $D$ has at least one branch point. But the fibre $F_\infty$ has the form

```
-2 -1 -2
```

by lemma 8 (1). Hence by lemma 12 again $D$ has at least two branch points and $D$ is obtained from an NC-minimal divisor of the form $H[k, -k]$ for $k \geq 1$.

If the morphism $X - C \rightarrow \mathbb{C}^*$ has a multiple fibre with multiplicity $m > 1$ and $F_0 \neq C$ then the divisor $D$ is m.n.c and the 2-section $D_h$ meets at least four other curves in $D$. This contradicts Fujita’s classification. Hence either the morphism $X - C \rightarrow \mathbb{C}^*$ has no multiple fibre or $C = F_0$. In the later case, $X - C \rightarrow \mathbb{C}^*$ has one multiple fibre by lemma 12 and $\overline{\kappa}(X - C) = 1$. Further, $D_h$ is a branch point of $D$.

**Case 2.** $\phi$ is untwisted and $B = \mathbb{C}$.

We claim that this case does not occur. First we observe that the fibre $F_\infty$ is a regular fibre of $\Phi$ and the two horizontal components meet $F_\infty$ in two distinct points. It is easy to see that $D$ cannot be obtained from any of the surfaces Fujita has described by a finite succession of blowing-ups.

**Case 3.** $\phi$ is untwisted and $B \cong \mathbb{P}^1$

The fibration $\phi$ has at most two multiple fibres by lemma 8. The curve $F_0 - C$ is connected. The morphism $\phi': X - C \rightarrow \mathbb{C}$ has at least one multiple fibre by lemma 8 (3). If $\phi'$ has only one multiple fibre, then $X - C$ contains $\mathbb{C}^* \times \mathbb{C}^*$ as a Zariski open subset and hence $\overline{\kappa}(X - C) = 0$. Suppose $\phi'$ has two multiple fibres. Then $D$ is m.n.c. and we see that the horizontal $D$-components $D_1$ and $D_2$ intersect in a point on $\overline{C}$. This shows that $X$ is of type $H[k, -k]$. Further, the multiple fibres have multiplicity 2 each (otherwise $D$ cannot be of type $H[-1, 0, -1]$) and $\overline{\kappa}(X - C) = 0$, as in the proof of lemma 8 (3).

Next we prove the following.

**Lemma 14.** Let $X$ be of type $H[k, -k]$ and $X$ contains a contractible curve $L$ with $\overline{\kappa}(X - L) = 1$. Then $k = 1$.

**Proof.** From the proof of lemma 10, we know that there is a twisted $\mathbb{C}^*$-fibration $\phi: X \rightarrow \mathbb{C}$ with $\phi(L)$ a point. Further, $\phi'$ has exactly one multiple fibre, where $\phi': X - L \rightarrow \mathbb{C}^*$ is the restriction. The horizontal component $D_h$ is a branch point for $D$ and the fibre $F_\infty$ has the dual graph.
\( Q \)-homology planes

\[
\begin{array}{c}
-2 & -1 & -2
\end{array}
\]

\( \tilde{L} \) is a reduced fibre of \( \phi \) by the proof of case 1 of lemma 13. Using lemma 6 repeatedly we see that \( \tilde{L} \) can be assumed to be the full fibre of \( \phi \). From Fujita's description of \( D \), we see that \( k=1 \) because the branch points intersect and one of them is a \((-1)\)-curve.

To complete the proof of the theorem, it remains to prove the following result.

**Lemma 15.** (1) On the surface \( X \) of type \( H[k, -k] \), there is a unique contractible curve \( C \) with \( \kappa(X-C)=0 \).

(2) On \( H[1, -1] \) there is a unique contractible curve \( L \) with \( \kappa(X-L)=1 \).

(3) If \( k=1 \) and \( C \) and \( L \) are the contractible curves as above then \( C \cdot L = 2 \) and they meet transversally.

**Proof.** (1) Let \( C \) be a contractible curve on \( X \) with \( \kappa(X-C)=0 \). There is a \( C^* \)-fibration \( \phi: X \to C \) such that for some \( m \geq 1 \), \( mC \) is a fibre of \( \phi \). Then \( \phi \) is a twisted fibration. Let \( X \subset Z \) be a smooth projective compactification such that \( \phi \) extends to a \( \mathbb{P}^1 \)-fibration \( \Phi: Z \to \mathbb{P}^1 \). By lemma 8 (1) there is no multiple fibre for the map \( X \to C \to C^* \). The fibre \( F_\infty \) has the dual graph,

\[
\begin{array}{c}
\vdots & 
\vdots & 
\vdots
\end{array}
\]

and \( D_h \) meets the \((-1)\)-curve in \( F_\infty \). Let \( F_0 \) be the fibre of \( \phi \) containing \( \tilde{C} \) and \( D_0 \) be the \( D \)-component of \( F_0 \) that meets \( D_h \). We claim that \( D_0 \) meets only one other \( D \)-component in \( F_0 \). If not, \( D_0 \) is a branch point of \( D \) and from Fujita's classification, we deduce that \( D_h \) is a \((-1)\)-curve and after contracting \( D_h \), we get an NC-minimal completion of \( X \). But this is not of type \( H[k, -k] \) with \( k \geq 1 \). Hence we may even assume that \( D_0 \) is not a \((-1)\)-curve.

As before, we may assume that \( \tilde{C} \) is the only \((-1)\)-curve in \( F_0 \). Since an NC-minimal completion of \( X \) is obtained from contracting suitable \((-1)\)-curves in \( D \), we conclude that \( D_h \) is a \((-1)\)-curve. Then \( D_0 \) is a \((-2)\)-curve. By repeating this argument, we infer that the dual graph of \( \tilde{C} \cup D \) is

\[
\begin{array}{c}
\vdots & 
\vdots & 
\vdots & 
\vdots
\end{array}
\]
By successive contractions of \((-1)\)-curves starting with \(D_h\), we get an m.n.c. compactification divisor of \(X\) such that the dual graph of the image of \(\bar{C} \cup D\) looks like \(H[k, -k]\), with the image of \(\bar{C}\) passing through the intersection of the two branching curves. From this it is easy to see that the curve \(\bar{C}\) is unique.

(2) Let \(L\) be a contractible curve on \(X\) with \(\bar{\kappa}(X-L) = 1\). By the proof of case 1 of lemma 13 and lemma 14, we can assume that \(\bar{L} \cup D\) looks like

```
  1  \(\bar{L}\)  \(-2\)
    \(-2\)
    \(-2\)  \(-1\)  \(-2\)
```

Clearly, \(\bar{L}\) is a full fibre of the \(\mathbb{P}^1\)-fibration on \(Z\) given by the linear system \([T_2+2B_2+T_4]\). Therefore \(L\) is unique.

(3) We have seen that \(\bar{C}\) passes through the intersection of \(B_1\) and \(B_2\) and meets transversally with both. Hence \(\bar{C} \cdot \bar{L} = 2\). Now by lemma 10, \(C \cap L\) consists of 2 distinct points as \(\bar{L}\) does not pass through \(B_1 \cap B_2\). This completes the proof of the theorem.

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