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A.J. Parameswaran – V. Srinivas

A VARIANT OF NOETHER NORMALISATION

Dedicated to Professor Paolo Salmon on his sixtieth birthday

Abstract. Let X be an affine variety over an infinite field k, together with a collection of finite morphisms $f_i : X \to \mathbf{A}^{n_i}$. We prove that for the general 'product' linear projection $\prod_i p_i : \prod_i \mathbf{A}^{n_i} \to \prod_i \mathbf{A}^{s_i}$, the composite $p \circ (\prod_i f_i) : X \to \mathbf{A}^{\sum_i s_i}$ is finite, provided $\sum_i s_i \geq \dim X$. This generalizes the Noether Normalisation theorem, in a manner analogous to Nori'a generalisation of the 'Whitney embedding theorem' for smooth affine varieties. It also extends Nori's theorem (and its generalisation to non-smooth varieties) to more than 2 factors.

The aim of this note is to prove the following variant of Noether normalization.

THEOREM 1. *Let X be an affine variety of dimension d over an infinite field k. Let $f_i : X \longrightarrow \mathbf{A}^{n_i}$, $1 \leq i \leq m$ be finite morphisms, and let $h = (f_1, \cdots, f_m) : X \longrightarrow \mathbf{A}^n, n = n_1 + \cdots + n_m$, be the product morphism. Let $V_i, 1 \leq i \leq m$ respectively denote the k-vector spaces of linear homogeneous functions on $\mathbf{A}^{n_i}, 1 \leq i \leq m$.*

Suppose that $r_i, 1 \leq i \leq m$, are chosen with $0 < r_i \leq n_i$ and $r_i + \cdots + r_m = d$, and let $\mathbf{G}_i$ denote the Grassmannian of r_i-dimensional subspaces of V_i; set $\mathbf{G} = \mathbf{G}_1 \times \cdots \times \mathbf{G}_m$. Then there exists a non-empty Zariski open subset $U \subset \mathbf{G}$ such that for $x \in U(k)$, if $W_i \subset V_i$ are the corresponding subspaces, and $g_i : \mathbf{A}^{n_i} \longrightarrow \mathbf{A}^{r_i}$ the projection determined by

W_i, then

$$(g_1 \times \cdots \times g_m) \circ h : X \longrightarrow \prod_{i=1}^{m} \mathbf{A}^{r_i} = \mathbf{A}^d$$

is finite.

This was motivated by an argument of Nori, used to compare two embeddings of a smooth, n-dimensional affine variety over an infinite field k in $\mathbf{A}_k^{2n+2}$ (see [S] for an extension of Nori's result to the case of an arbitrary affine k-scheme of finite type). The above form of Noether normalisation implies a result about embeddings, generalising Theorem $1'$ of [S].

To state this generalisation, we recall some notation from [S]. If $X = \operatorname{Spec} A$ is an affine k-scheme of finite type, and M is a finite A-module, define

$$\eta(M) = \dim \operatorname{Spec} S_A(M)\,,$$

where $S_A(M)$ is the symmetric algebra of M over A. We then have an expression for $\eta(M)$,

$$\eta(M) = \sup_{\mathcal{P} \in \operatorname{Spec} R} \{\mu_{\mathcal{P}}(M) + \dim A/\mathcal{P}\}\,;$$

here $\mu_{\mathcal{P}}(M) = \dim_{k(\mathcal{P})} M \otimes_A k(\mathcal{P})$, where $k(\mathcal{P})$ is the residue field of $A_{\mathcal{P}}$. If M is supported at all minimal primes of A, then the number $\eta(M)$ may also be interpreted as the bound on the number of generators of M as an A-module given by Forster (see [F]). The above formula for $\eta(M)$ is easily proved by considering the dimensions of fibres of the morphism $\operatorname{Spec} S_A(M) \longrightarrow \operatorname{Spec} A$ – indeed, for $\mathcal{P} \in \operatorname{Spec} A$, the (scheme theoretic) fibre of $\operatorname{Spec} S_A(M) \longrightarrow \operatorname{Spec} A$ over the point $\mathcal{P}$ is the affine space of dimension $\mu_{\mathcal{P}}(M)$ over the residue field of $\mathcal{P}$, and hence the dimension of the Zariski closure in $\operatorname{Spec} S_A(M)$ of this fibre is $\dim A/\mathcal{P} + \mu_{\mathcal{P}}(M)$. But clearly the dimension of $\operatorname{Spec} S_A(M)$ is the supremum of the dimensions of these Zariski closures. The above expression for $\eta(M)$ has been obtained earlier by C.Huneke and M.Rossi [HR] (with fewer assumptions on the ring A); we rank the referee for providing us with this reference.

THEOREM 2. *Let $X = Spec\ A$ be an affine variety of dimension d over an infinite field k. Let $f_i : X \longrightarrow \mathbf{A}^{n_i}, 1 \leq i \leq m$ be closed embeddings, and let $h = (f_1, \cdots, f_m) : X \longrightarrow \mathbf{A}^n, n = n_1 + \cdots + n_m$, be the product embedding. Let $V_i, 1 \leq i \leq m$ respectively denote the k-vector spaces of linear homogeneous functions on $\mathbf{A}^{n_i}, 1 \leq i \leq m$.*

Let $s_i, 1 \leq i \leq r$ be non-negative integers such that $s_i \leq n_i$, and $N = \sum_i s_i \geq s$, where

$$s = \sup\{2d+1, \eta(\Omega^1_{A/k})\}\,.$$

Let $\mathbf{G}_i$ be the Grassmannian of s_i dimensional subspaces in V_i; set $\mathbf{G} = \mathbf{G}_1 \times \cdots \times \mathbf{G}_r$. Then there is a non-empty Zariski open set $U \subset \mathbf{G}$ such that if $x \in U$ is a k-rational point, $W_i \subset V_i$ are the corresponding subspaces, and $g_i : X \longrightarrow A^{s_i}$ is the projection determined by W_i, then

$$(g_1 \times \cdot \times g_m) \circ h : X \longrightarrow \prod_{i=1}^{m} \mathbf{A}^{s_i} = \mathbf{A}^N$$

is a closed embedding.

This result was proved in [S] with the additional hypothesis that $\sup_i s_i \geq d$, using the standard Noether Normalisation lemma. If instead we use Theorem 1, we obtain a proof of Theorem 2. The details are left to the reader.

1. Proof of Theorem 1

By the standard Noether normalisation lemma, we reduce easily to the special case when $n_i = d$ for all $1 \leq i \leq m$, and the product map $h : X \longrightarrow \mathbf{A}^{md}$ is an embedding. Let $\overline{X} \subset \mathbf{P}^{md}$ denote the Zariski closure of X in the corresponding projective space. Let $H \cong \mathbf{P}^{md-1}$ be the hyperplane at infinity in $\mathbf{P}^{md}$, and let $Y = \overline{X} \cap H$.

Any linear projection $p : \mathbf{A}^{md} \longrightarrow \mathbf{A}^s$ extends uniquely to a linear projection $\mathbf{P}^{md} - L \to \mathbf{P}^s$ for a linear subspace $L \subset H$ of dimension $md-s-1$; this restricts to the linear projection $H - L \to H'$ where $H' = \mathbf{P}^s - \mathbf{A}^s \cong \mathbf{P}^{s-1}$ is the hyperplane at infinity in $\mathbf{P}^s$.

If $\{L_j\}_{j=1}^r$ is any finite collection of linear subspaces of $\mathbf{P}^{md}$, we denote their span by $< L_1, \cdots, L_r >$ (the *span* is the smallest linear subspace containing the union $\cup_{j=1}^r L_j$).

Let $\mathbf{A}_i \subset \mathbf{A}^{md}$ be the affine subspace defined by

$$\mathbf{A}_i = \{(x_1, \cdots, x_m) \in (\mathbf{A}^d)^m = \mathbf{A}^{md} | x_j = 0 \text{ for } j \neq i\},$$

and let $H_i \subset H$ be the hyperplane at infinity of $\mathbf{A}_i$; thus $H_i \cong \mathbf{P}^{d-1}$. Let H^i be the span of $\{H_j\}_{j\neq i}$. Then H_i, H^i are disjoint linear subspaces of H of dimensions $d-1$ and $(m-1)d-1$ respectively, which span H. We are given that the projections $\mathbf{P}^{md} - H^i \to \mathbf{P}^d$ restrict to finite morphisms on X for $1 \leq i \leq m$, since the linear projection from H^i restricts to the morphism $f_i : X \longrightarrow \mathbf{A}^d$.

We may identify $\mathbf{G}_i$ with the Grassmannian of $d - r_i - 1$-dimensional linear subspaces of H_i i.e., of linear subspaces of codimension r_i. The theorem amounts to the following statement: there is a non-empty Zariski open set $U \subset \mathbf{G} = \mathbf{G}_1 \times \cdots \times \mathbf{G}_m$ such that for $x = (x_1, \cdots, x_m) \in U(k)$, if $L_i \subset H_i$ are the linear subspaces corresponding to $x_i \in \mathbf{G}_i$, then the projection p_L from $L = < L_1, \cdots, L_m >$, the span of the L_i, restricts to a finite morphism on X. Observe that if L^i is the span of $\cup_{j\neq i} L_j$, then $L_i \cap L^i \subset H_i \cap H^i = \phi$. This implies that L has dimension $(m-1)d-1$; also, since each $L_i \subset H$, L is contained in H. Further, dim $L^i = (m-2)d + r_i - 1$.

In the following two lemmas, fix linear subspaces $L_i \subset H_i$ as above. Let $\overline{L_i} = p_{L^i}(L_i) \subset \mathbf{P}^{2d-r_i}$, where $p_{L^i} : \mathbf{P}^{md} - L_i \longrightarrow \mathbf{P}^{2d-r_i}$ is the projection from L^i. Then p_{L^i} restricts to an isomorphism $L_i \cong \overline{L_i}$.

LEMMA 1. *Let $\widetilde{\mathbf{P}^{md}}$ be the Zariski closure in $\mathbf{P}^{md} \times \prod_{i=1}^m \mathbf{P}^{2d-r_i}$ of the graph of the product linear projection*

$$(p_{L^1}, \ldots, p_{L^m}) : \mathbf{P}^{md} - \cup L^i \longrightarrow \prod_{i=1}^m \mathbf{P}^{2d-r_i},$$

and let $\widetilde{p} : \widetilde{\mathbf{P}^{md}} \longrightarrow \prod_{i=1}^m \mathbf{P}^{2d-r_i}$ be the induced morphism. Let $\widetilde{X} \subset \widetilde{\mathbf{P}^{md}}$ be the strict transform of $\overline{X}$. Suppose that

$$\widetilde{p}(\widetilde{X}) \cap (\overline{L_1} \times \cdots \times \overline{L_m}) = \phi.$$

Then the linear projection $p_L : \mathbf{P}^{md} - L \longrightarrow \mathbf{P}^d$ restricts to a finite morphism on X.

Proof. Let p_i be the composite

$$p_i : \widetilde{\mathbf{P}^{md}} \xrightarrow{\widetilde{p}} \prod_{i=1}^{m} \mathbf{P}^{2d-r_i} \xrightarrow{\pi_i} \mathbf{P}^{2d-r_i}$$

where π_i is the projection onto the i^{th} factor. Let

$$U_i = \widetilde{\mathbf{P}^{md}} - p_i^{-1}(\overline{L_i}) .$$

We claim that the linear projection $p_L : \mathbf{P}^{md} - L \longrightarrow \mathbf{P}^d$, regarded as a rational map $\widetilde{p_L} : \widetilde{\mathbf{P}^{md}} \cdots > \mathbf{P}^d$, is actually a morphism on

$$\cup_{i=1}^{m} U_i = \widetilde{\mathbf{P}^{md}} - \widetilde{p}^{-1}(\overline{L_1} \times \cdots \times \overline{L_i}) .$$

By hypothesis, $\widetilde{X} \subset \cup U_i$. Hence, granting the above claim, $\widetilde{p_L}$ yields a morphism defined in a Zariski open neighbourhood of $\widetilde{X}$.

To prove the claim, note that there is a composite morphism

$$\theta_i : U_i \xrightarrow{\widetilde{p}} \prod_{i=1}^{m} \mathbf{P}^{2d-r_i} - \pi_i^{-1}(\overline{L_i}) \xrightarrow{i_i'} \mathbf{P}^{2d-r_i} - \overline{L_i} \xrightarrow{p_{\overline{L_i}}} \mathbf{P}^d$$

such that the restriction to $\mathbf{A}^{md} \subset U_i \subset \widetilde{\mathbf{P}^{md}}$ is just the linear projection $p_L : \mathbf{A}^{md} \longrightarrow \mathbf{A}^d$. Hence the maps θ_i and θ_j agree on a dense open subset of $U_i \cap U_j$ for all i, j. By separatedness, θ_i and θ_j agree on $U_i \cap U_j$ for all i, j, and hence determine a well defined morphism $\widetilde{p_L}$ on $\cup_{i=1}^{m} U_i$ as claimed.

Now $X \subset \mathbf{A}^{md} \subset \widetilde{\mathbf{P}^{md}}$. Let $\widetilde{Y} = \widetilde{X} - X$. To prove the lemma, it suffices to show that

$$\widetilde{p_L}(\widetilde{Y}) \subset M \cong \mathbf{P}^{d-1} ,$$

where M is the hyperplane at infinity in $\mathbf{P}^d$ (M is the image of $H - L$ under p_L). To verify this inclusion, it suffices to show that

$$\theta_i(\widetilde{Y} \cap U_i) \subset M$$

for all i. If $x \in \widetilde{Y} \cap U_i$, then

$$\pi_i \circ \widetilde{p}(x) = y \in \mathbf{P}^{md} - \overline{L_i} .$$

Since the projection $p_{H^i} : \mathbf{P}^{md} - H_i \longrightarrow \mathbf{P}^d$ restricts to a finite morphism on X, and p_{H^i} factors through $p_{L^i}|_{\mathbf{P}^{md}-H_i}$, we see that p_{L^i} restricts to a finite morphism on X. Hence $y \in M_i \approx \mathbf{P}^{2d-r_i-1}$, the hyperplane at infinity in $\mathbf{P}^{2d-r_i}$. Thus

$$y \in M_i - \overline{L_i} \subset \mathbf{P}^{2d-r_i} - \overline{L_i}\,.$$

The projection $p_{\overline{L_i}} : \mathbf{P}^{2d-r_i} - \overline{L_i} \longrightarrow \mathbf{P}^d$ evidently maps $M_i - \overline{L_i}$ onto the hyperplane at infinity $M \subset \mathbf{P}^d$. Hence $\theta_i(x) = p_{\overline{L_i}}(y) \in M$, as desired.

∎

LEMMA 2. *Let $\widetilde{\mathbf{P}^{md}}$ be the Zariski closure in $\mathbf{P}^{md} \times \prod_{i=1}^m \mathbf{P}^d$ of the graph of the product linear projection*

$$(p_{H^1}, \ldots, p_{H^m}) : \mathbf{P}^{md} - \cup H^i \longrightarrow \prod_{i=1}^m \mathbf{P}^d\,,$$

and let $\widetilde{p} : \widetilde{\mathbf{P}^{md}} \longrightarrow \prod_{i=1}^m \mathbf{P}^d$ be the induced morphism. Let $\widehat{X} \subset \widetilde{\mathbf{P}^{md}}$ be the strict transform of $\overline{X}$, and let $\widehat{L_i} = p_{H^i}(L_i)$ so that $\widehat{L_i} \cong L_i$. Suppose that

$$\widetilde{p}(\widehat{X}) \cap (\widehat{L_1} \times \cdots \times \widehat{L_m}) = \phi\,.$$

Then the linear projection $p_L : \mathbf{P}^{md} - L \longrightarrow \mathbf{P}^d$ restricts to a finite morphism on X.

Proof. Let $\overline{H_i} = p_{L^i}(H^i - L^i) \subset \mathbf{P}^{2d-r_i}$. Let

$$Z_i = (\pi_i \circ \widetilde{p})^{-1}(\overline{H^i}) \subset \widetilde{\mathbf{P}^{md}}\,.$$

The composite morphism

$$\widetilde{\mathbf{P}^{md}} - Z_i \xrightarrow{\pi_i \circ \widetilde{p}} \mathbf{P}^{2d-r_i} - \overline{H_i} \xrightarrow{p_{\overline{H_i}}} \mathbf{P}^d$$

restricts to the linear projection p_{H^i} on $\mathbf{A}^{md} \subset \widetilde{\mathbf{P}^{md}}$. Hence there is a natural morphism $\mu : \widetilde{\mathbf{P}^{md}} - \cup_{i=1}^m Z_i \longrightarrow \widehat{\mathbf{P}^{md}}$, and a commutative diagram

$$\begin{array}{ccc}
\widetilde{\mathbf{P}^{md}} - \cup_i Z_i & \xrightarrow{\mu} & \widehat{\mathbf{P}^{md}} \\
\tilde{p}\Big\downarrow & & \Big\downarrow\hat{p} \\
\prod(\mathbf{P}^{2d-r_i} - \overline{H^i}) & \xrightarrow{\prod p_{\overline{H_i}}} & \prod \mathbf{P}^d .
\end{array}$$

Further, $\mathbf{A}^{md} \subset \widetilde{\mathbf{P}^{md}} - \cup_i Z_i$ maps isomorphically to its image in $\widehat{\mathbf{P}^{md}}$, so that $\mu|_X$ is an isomorphism onto its image. Hence $\mu(\widetilde{X} - \cup_i Z_i) \subset \widehat{X}$.

Since $L^i \subset H^i$, and $H^i \cap L_i = \phi$, we have $\overline{L_i} \cap \overline{H_i} = \phi$. Hence $Z_i \subset U_i$, and so μ is defined in a Zariski open neighbourhood of $T = \widetilde{\mathbf{P}^{md}} - \cup_{i=1}^m U_i$. If $\widehat{T} = \widehat{p}^{-1}(\prod_{i=1}^m \widehat{L_i})$, then there is a commutative diagram

$$\begin{array}{ccc}
T \cap \widetilde{X} & \xrightarrow{\mu} & \widetilde{T} \cap \widehat{X} \\
\tilde{p}\Big\downarrow & & \Big\downarrow\hat{p} \\
(\prod_i \overline{L_i}) \cap \tilde{p}(\widetilde{X}) & \xrightarrow{\prod p_{\overline{H_i}}} & (\prod_i \widehat{L_i}) \cap \hat{p}(\widehat{X}) .
\end{array}$$

By hypothesis, $(\prod_i \widehat{L_i}) \cap \widehat{p}(\widehat{X}) = \phi$. Hence $(\prod_i \overline{L_i}) \cap \widetilde{p}(\widetilde{X}) = \phi$, and by lemma 1, p_L restricts to a finite morphism on X. ∎

We now complete the proof of the theorem. Let $\widehat{M_i} \approx \mathbf{P}^{d-1}$ be the hyperplane at infinity in the i^{th} factor of $\prod_{i=1}^m \mathbf{P}^d$, the target of $\widehat{p}$. Then $\widehat{L_i} \subset \widehat{M_i}$. Clearly $\widehat{p}(X) \cap (\prod \widehat{M_i}) = \phi$ (where X regarded as an open subset of $\widehat{X}$). Hence to verify the condition

$$\widehat{p}(\widehat{X}) \cap \left(\prod_{i=1}^m \widehat{L_i}\right) = \phi ,$$

it suffices to verify that

$$\widehat{p}(\widehat{X} - X) \cap \left(\prod_{i=1}^m \widehat{L_i}\right) = \phi .$$

Let $S = \widehat{p}(\widehat{X} - X) \subset \prod_{i=1}^m \widehat{M_i}$. Then $\dim S \leq d - 1$. Let

$$\Gamma = \{(x_1, \ldots, x_m, t_1, \cdots, t_m) \in \mathbf{G} \times S | t_i \in \widehat{L_i} \text{ for all } i\} ,$$

where L_i is the subspace associated to $x_i \in \mathbf{G}_i$. Then each fibre of $\Gamma \longrightarrow S$ is a product of sub-Grassmannians of $\mathbf{G}$ of codimension $\sum_{i=1}^{m} r_i = d >$ dim S. Hence the projection $\Gamma \longrightarrow \mathbf{G}$ is not dominant. By lemma 2, the Zariski open set $U = \mathbf{G} - \overline{\mathrm{im}\Gamma}$ has the property described in the statement of the theorem. ∎

REFERENCES

[F] O. FORSTER, *Über die Anzahl der Erzeugenden eines Ideals in einem noetherschen Ring*, Math. Zeit. **84** (1964), 80-87.

[HR] C. HUNECKE, M. ROSSI, *The dimension and components of symmetric algebras*, J. Alg. **98** (1986), 200-210.

[S] V. SRINIVAS, *On the embedding dimension of an affine variety*, Math. Ann. **289** (1991), 125-132.

A.J. PARAMESWARAN, V. SRINIVAS, School of Mathematics,
Tata Institute of Fundamental Research, Homi Bhabha Road,
Bombay-400005, India.

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