A VARIANT OF NOETHER NORMALISATION

Dedicated to Professor Paolo Salmon on his sixtieth birthday

Abstract. Let \( X \) be an affine variety over an infinite field \( k \), together with a collection of finite morphisms \( f_i : X \to \mathbb{A}^{n_i} \). We prove that for the general 'product' linear projection \( \prod_i p_i : \prod_i \mathbb{A}^{n_i} \to \prod_i \mathbb{A}^{s_i} \), the composite \( p \circ (\prod_i f_i) : X \to \mathbb{A}^{\sum i s_i} \) is finite, provided \( \sum_i s_i \geq \dim X \). This generalizes the Noether Normalisation theorem, in a manner analogous to Nori's generalisation of the 'Whitney embedding theorem' for smooth affine varieties. It also extends Nori's theorem (and its generalisation to non-smooth varieties) to more than 2 factors.

The aim of this note is to prove the following variant of Noether normalization.

Theorem 1. Let \( X \) be an affine variety of dimension \( d \) over an infinite field \( k \). Let \( f_i : X \to \mathbb{A}^{n_i}, 1 \leq i \leq m \) be finite morphisms, and let \( h = (f_1, \ldots, f_m) : X \to \mathbb{A}^{n}, n = n_1 + \cdots + n_m \), be the product morphism. Let \( V_i, 1 \leq i \leq m \) respectively denote the \( k \)-vector spaces of linear homogeneous functions on \( \mathbb{A}^{n_i}, 1 \leq i \leq m \).

Suppose that \( r_i, 1 \leq i \leq m \), are chosen with \( 0 < r_i \leq n_i \) and \( r_1 + \cdots + r_m = d \), and let \( G_i \) denote the Grassmannian of \( r_i \)-dimensional subspaces of \( V_i \); set \( G = G_1 \times \cdots \times G_m \). Then there exists a non-empty Zariski open subset \( U \subset G \) such that for \( x \in U(k) \), if \( W_i \subset V_i \) are the corresponding subspaces, and \( g_i : \mathbb{A}^{n_i} \to \mathbb{A}^{r_i} \) the projection determined by
This was motivated by an argument of Nori, used to compare two embeddings of a smooth, \( n \)-dimensional affine variety over an infinite field \( k \) in \( \mathbb{A}^{2n+2}_k \) (see [S] for an extension of Nori's result to the case of an arbitrary affine \( k \)-scheme of finite type). The above form of Noether normalisation implies a result about embeddings, generalising Theorem 1 of [S].

To state this generalisation, we recall some notation from [S]. If \( X = \text{Spec} \ A \) is an affine \( k \)-scheme of finite type, and \( M \) is a finite \( A \)-module, define

\[
\eta(M) = \dim \text{Spec} \ S_A(M),
\]

where \( S_A(M) \) is the symmetric algebra of \( M \) over \( A \). We then have an expression for \( \eta(M) \),

\[
\eta(M) = \sup_{P \in \text{Spec} \ R} \left\{ \mu_P(M) + \dim A/P \right\},
\]

here \( \mu_P(M) = \dim_{k(P)} M \otimes_A k(P) \), where \( k(P) \) is the residue field of \( A_P \). If \( M \) is supported at all minimal primes of \( A \), then the number \( \eta(M) \) may also be interpreted as the bound on the number of generators of \( M \) as an \( A \)-module given by Forster (see [F]). The above formula for \( \eta(M) \) is easily proved by considering the dimensions of fibres of the morphism \( \text{Spec} \ S_A(M) \to \text{Spec} A \) — indeed, for \( P \in \text{Spec} A \), the (scheme theoretic) fibre of \( \text{Spec} S_A(M) \to \text{Spec} A \) over the point \( P \) is the affine space of dimension \( \mu_P(M) \) over the residue field of \( P \), and hence the dimension of the Zariski closure in \( \text{Spec} S_A(M) \) of this fibre is \( \dim A/P + \mu_P(M) \). But clearly the dimension of \( \text{Spec} S_A(M) \) is the supremum of the dimensions of these Zariski closures. The above expression for \( \eta(M) \) has been obtained earlier by C. Huneke and M. Rossi [HR] (with fewer assumptions on the ring \( A \)); we rank the referee for providing us with this reference.
**Theorem 2.** Let $X = \text{Spec } A$ be an affine variety of dimension $d$ over an infinite field $k$. Let $f_i : X \rightarrow \mathbb{A}^{n_i}, 1 \leq i \leq m$ be closed embeddings, and let $h = (f_1, \cdots, f_m) : X \rightarrow \mathbb{A}^n, n = n_1 + \cdots + n_m$, be the product embedding. Let $V_i, 1 \leq i \leq m$ respectively denote the $k$-vector spaces of linear homogeneous functions on $\mathbb{A}^{n_i}, 1 \leq i \leq m$.

Let $s_i, 1 \leq i \leq r$ be non-negative integers such that $s_i \leq n_i$, and $N = \sum_i s_i \geq s$, where

$$s = \sup\{2d + 1, \eta(\Omega^1_{A/k})\}.$$

Let $G_i$ be the Grassmannian of $s_i$ dimensional subspaces in $V_i$; set $G = G_1 \times \cdots \times G_r$. Then there is a non-empty Zariski open set $U \subset G$ such that if $x \in U$ is a $k$-rational point, $W_i \subset V_i$ are the corresponding subspaces, and $g_i : X \rightarrow A^{s_i}$ is the projection determined by $W_i$, then

$$(g_1 \times \cdots \times g_m) \circ h : X \rightarrow \prod_{i=1}^m A^{s_i} = A^N$$

is a closed embedding.

This result was proved in [S] with the additional hypothesis that $\sup_i s_i \geq d$, using the standard Noether Normalisation lemma. If instead we use Theorem 1, we obtain a proof of Theorem 2. The details are left to the reader.

1. **Proof of Theorem 1**

By the standard Noether normalisation lemma, we reduce easily to the special case when $n_i = d$ for all $1 \leq i \leq m$, and the product map $h : X \rightarrow \mathbb{A}^{md}$ is an embedding. Let $\overline{X} \subset \mathbb{P}^{md}$ denote the Zariski closure of $X$ in the corresponding projective space. Let $H \cong \mathbb{P}^{md-1}$ be the hyperplane at infinity in $\mathbb{P}^{md}$, and let $Y = \overline{X} \cap H$.

Any linear projection $p : \mathbb{A}^{md} \rightarrow A^{s}$ extends uniquely to a linear projection $\mathbb{P}^{md} - L \rightarrow \mathbb{P}^{s}$ for a linear subspace $L \subset H$ of dimension $md - s - 1$; this restricts to the linear projection $H - L \rightarrow H'$ where $H' = \mathbb{P}^{s} - A^{s} \cong \mathbb{P}^{s-1}$ is the hyperplane at infinity in $\mathbb{P}^{s}$.
If \( \{L_j\}_{j=1}^r \) is any finite collection of linear subspaces of \( \mathbb{P}^{md} \), we denote their span by \( <L_1, \ldots, L_r> \) (the span is the smallest linear subspace containing the union \( \bigcup_{j=1}^r L_j \)).

Let \( A_i \subseteq \mathbb{A}^{md} \) be the affine subspace defined by

\[
A_i = \{(x_1, \ldots, x_m) \in (\mathbb{A}^d)^m = A^{md} | x_j = 0 \text{ for } j \neq i\},
\]

and let \( H_i \subseteq H \) be the hyperplane at infinity of \( A_i \); thus \( H_i \cong \mathbb{P}^{d-1} \). Let \( H^i \) be the span of \( \{H_j\}_{j \neq i} \). Then \( H_i, H^i \) are disjoint linear subspaces of \( H \) of dimensions \( d - 1 \) and \( (m-1)d - 1 \) respectively, which span \( H \). We are given that the projections \( \mathbb{P}^{md} - H^i \rightarrow \mathbb{P}^d \) restrict to finite morphisms on \( X \) for \( 1 \leq i \leq m \), since the linear projection from \( H^i \) restricts to the morphism \( f_i : X \rightarrow \mathbb{A}^d \).

We may identify \( G_i \) with the Grassmannian of \( d - r_i - 1 \)-dimensional linear subspaces of \( H^i \), i.e., of linear subspaces of codimension \( r_i \). The theorem amounts to the following statement: there is a non-empty Zariski open set \( U \subseteq G_i = G_1 \times \cdots \times G_m \) such that for \( x = (x_1, \ldots, x_m) \in U(k) \), if \( L_i \subseteq H_i \) are the linear subspaces corresponding to \( x_i \in G_i \), then the projection \( p_L \) from \( L = <L_1, \ldots, L_m> \), the span of the \( L_i \), restricts to a finite morphism on \( X \). Observe that if \( L^i \) is the span of \( \bigcup_{j \neq i} L_j \), then \( L_i \cap L^i \subseteq H_i \cap H^i = \emptyset \). This implies that \( L \) has dimension \( (m-1)d - 1 \); also, since each \( L_i \subseteq H, L \) is contained in \( H \). Further, \( \dim L^i = (m-2)d + r_i - 1 \).

In the following two lemmas, fix linear subspaces \( L_i \subseteq H_i \) as above. Let \( \overline{L_i} = p_L(L_i) \subseteq \mathbb{P}^{2d-r_i} \), where \( p_L : \mathbb{P}^{md} - L_i \rightarrow \mathbb{P}^{2d-r_i} \) is the projection from \( L^i \). Then \( p_L \) restricts to an isomorphism \( \overline{L_i} \cong \overline{L_i} \).

**Lemma 1.** Let \( \overline{\mathbb{P}^{md}} \) be the Zariski closure in \( \mathbb{P}^{md} \times \prod_{i=1}^m \mathbb{P}^{2d-r_i} \) of the graph of the product linear projection

\[
(p_{L1}, \ldots, p_{Lm}) : \mathbb{P}^{md} - \bigcup_i L^i \rightarrow \prod_{i=1}^m \mathbb{P}^{2d-r_i},
\]

and let \( \overline{\gamma} : \overline{\mathbb{P}^{md}} \rightarrow \prod_{i=1}^m \mathbb{P}^{2d-r_i} \) be the induced morphism. Let \( \overline{X} \subseteq \overline{\mathbb{P}^{md}} \) be the strict transform of \( \overline{X} \). Suppose that

\[
\overline{p(\overline{X})} \cap (\overline{L_1} \times \cdots \times \overline{L_m}) = \emptyset.
\]

Then the linear projection \( p_L : \mathbb{P}^{md} - L \rightarrow \mathbb{P}^d \) restricts to a finite morphism on \( X \).
Proof. Let \( p_i \) be the composite

\[
p_i : \mathbb{P}^{md} \xrightarrow{\tilde{\rho}} \prod_{i=1}^{m} \mathbb{P}^{2d-r_i} \xrightarrow{\pi_i} \mathbb{P}^{2d-r_i}
\]

where \( \pi_i \) is the projection onto the \( i \)th factor. Let

\[
U_i = \mathbb{P}^{md} - \tilde{\rho}^{-1}(L_i).
\]

We claim that the linear projection \( p_L : \mathbb{P}^{md} - L \to \mathbb{P}^d \), regarded as a rational map \( \tilde{\rho}_L : \mathbb{P}^{md} \to \mathbb{P}^d \), is actually a morphism on

\[
\bigcup_{i=1}^{m} U_i = \mathbb{P}^{md} - \tilde{\rho}^{-1}(L_1 \times \cdots \times L_i).
\]

By hypothesis, \( \tilde{X} \subset \bigcup U_i \). Hence, granting the above claim, \( \tilde{\rho}_L \) yields a morphism defined in a Zariski open neighbourhood of \( \tilde{X} \).

To prove the claim, note that there is a composite morphism

\[
\theta_i : U_i \xrightarrow{\tilde{\rho}} \prod_{i=1}^{m} \mathbb{P}^{2d-r_i} - \pi_i^{-1}(L_i) \xrightarrow{\theta_i} \mathbb{P}^{2d-r_i} - L_i \xrightarrow{\rho_{L_i}} \mathbb{P}^d
\]

such that the restriction to \( \mathbb{A}^{md} \subset U_i \subset \mathbb{P}^{md} \) is just the linear projection \( p_L : \mathbb{A}^{md} \to \mathbb{A}^d \). Hence the maps \( \theta_i \) and \( \theta_j \) agree on a dense open subset of \( U_i \cap U_j \) for all \( i,j \). By separatedness, \( \theta_i \) and \( \theta_j \) agree on \( U_i \cap U_j \) for all \( i,j \), and hence determine a well defined morphism \( \tilde{\rho}_L \) on \( \bigcup_{i=1}^{m} U_i \) as claimed.

Now \( X \subset \mathbb{A}^{md} \subset \mathbb{P}^{md} \). Let \( \tilde{Y} = \tilde{X} - X \). To prove the lemma, it suffices to show that

\[
\tilde{\rho}_L(\tilde{Y}) \subset M = \mathbb{P}^{d-1},
\]

where \( M \) is the hyperplane at infinity in \( \mathbb{P}^d \) (\( M \) is the image of \( H - L \) under \( p_L \)). To verify this inclusion, it suffices to show that

\[
\theta_i(\tilde{Y} \cap U_i) \subset M
\]

for all \( i \). If \( x \in \tilde{Y} \cap U_i \), then

\[
\pi_i \circ \tilde{\rho}(x) = y \in \mathbb{P}^{md} - L_i.
\]
Since the projection $p_{H^i} : \mathbb{P}^{md} - H^i \rightarrow \mathbb{P}^d$ restricts to a finite morphism on $X$, and $p_{H^i}$ factors through $p_{L^i} : \mathbb{P}^{md} - H^i$, we see that $p_{L^i}$ restricts to a finite morphism on $X$. Hence $y \in M_i \approx \mathbb{P}^{2d-r_i-1}$, the hyperplane at infinity in $\mathbb{P}^{2d-r_i}$. Thus

$$y \in M_i - L_i \subset \mathbb{P}^{2d-r_i} - L_i.$$ 

The projection $p_{L_i} : \mathbb{P}^{2d-r_i} - L_i \rightarrow \mathbb{P}^d$ evidently maps $M_i - L_i$ onto the hyperplane at infinity $M \subset \mathbb{P}^d$. Hence $\theta_i(x) = p_{L_i}(y) \in M$, as desired.

**Lemma 2.** Let $\widehat{P}^{md}$ be the Zariski closure in $\mathbb{P}^{md} \times \prod_{i=1}^m \mathbb{P}^d$ of the graph of the product linear projection

$$(p_{H^1}, \ldots, p_{H^m}) : \mathbb{P}^{md} - \cup H^i \rightarrow \prod_{i=1}^m \mathbb{P}^d,$$

and let $\widehat{p} : \mathbb{P}^{md} \rightarrow \prod_{i=1}^m \mathbb{P}^d$ be the induced morphism. Let $\widehat{X} \subset \mathbb{P}^{md}$ be the strict transform of $X$, and let $\widehat{L_i} = p_{H^i}(L_i)$ so that $\widehat{L_i} \cong L_i$. Suppose that

$$\widehat{p}(\widehat{X}) \cap (\widehat{L_1} \times \cdots \times \widehat{L_m}) = \phi.$$

Then the linear projection $p_L : \mathbb{P}^{md} - L \rightarrow \mathbb{P}^d$ restricts to a finite morphism on $X$.

**Proof.** Let $\overline{H_i} = p_{L_i}(H^i - L^i) \subset \mathbb{P}^{2d-r_i}$. Let

$$Z_i = (\pi_i \circ \widehat{p})^{-1}(\overline{H^i}) \subset \mathbb{P}^{md}.$$

The composite morphism

$$\overline{\mathbb{P}^{md}} - Z_i \xrightarrow{\overline{\pi_i \circ \widehat{p}}} \mathbb{P}^{2d-r_i} - H_i \xrightarrow{p_{H^i}} \mathbb{P}^d$$

restricts to the linear projection $p_{H^i}$ on $\mathbb{A}^{md} \subset \mathbb{P}^{md}$. Hence there is a natural morphism $\mu : \overline{\mathbb{P}^{md}} - \bigcup_{i=1}^m Z_i \rightarrow \mathbb{P}^{md}$, and a commutative diagram
Further, \( A^{md} \subset \widetilde{\mathbb{P}}^{md} - \cup_i Z_i \) maps isomorphically to its image in \( \widetilde{\mathbb{P}}^{md} \), so that \( \mu|_X \) is an isomorphism onto its image. Hence \( \mu(\tilde{X} - \cup_i Z_i) \subset \hat{X} \).

Since \( L^i \subset H^i \), and \( H^i \cap L_i = \phi \), we have \( \overline{L_i} \cap \overline{H_i} = \phi \). Hence \( Z_i \subset U_i \), and so \( \mu \) is defined in a Zariski open neighbourhood of \( T = \mathbb{P}^{md} - \cup_{i=1}^m U_i \). If \( \tilde{T} = \tilde{p}^{-1}(\prod_{i=1}^m \overline{L_i}) \), then there is a commutative diagram

\[
\begin{array}{ccc}
T \cap \tilde{X} & \xrightarrow{\mu} & \tilde{T} \cap \hat{X} \\
\downarrow \tilde{p} & & \downarrow \hat{p} \\
(\prod_i \overline{L_i}) \cap \tilde{p}(\tilde{X}) & \xrightarrow{\prod \overline{H_i}} & (\prod_i \overline{L_i}) \cap \hat{p}(\hat{X}).
\end{array}
\]

By hypothesis, \( (\prod_i \overline{L_i}) \cap \tilde{p}(\tilde{X}) = \phi \). Hence \( (\prod_i \overline{L_i}) \cap \hat{p}(\hat{X}) = \phi \), and by lemma 1, \( p_L \) restricts to a finite morphism on \( X \).

We now complete the proof of the theorem. Let \( \tilde{M}_i \approx \mathbb{P}^{d-1} \) be the hyperplane at infinity in the \( i^{th} \) factor of \( \prod_{i=1}^m \mathbb{P}^d \), the target of \( \tilde{p} \). Then \( \overline{L_i} \subset \tilde{M}_i \). Clearly \( \tilde{p}(\tilde{X}) \cap (\prod \tilde{M}_i) = \phi \) (where \( X \) regarded as an open subset of \( \hat{X} \)). Hence to verify the condition

\[
\tilde{p}(\tilde{X}) \cap \left( \prod_{i=1}^m \overline{L_i} \right) = \phi,
\]

it suffices to verify that

\[
\tilde{p}(\tilde{X} - X) \cap \left( \prod_{i=1}^m \overline{L_i} \right) = \phi.
\]

Let \( S = \tilde{p}(\tilde{X} - X) \subset \prod_{i=1}^m \tilde{M}_i \). Then \( \dim S \leq d - 1 \). Let

\[
\Gamma = \{ (x_1, \ldots, x_m, t_1, \ldots, t_m) \in G \times S | t_i \in \overline{L_i} \text{ for all } i \}.
\]
where \( L_i \) is the subspace associated to \( x_i \in G_i \). Then each fibre of \( \Gamma \rightarrow S \) is a product of sub-Grassmannians of \( G \) of codimension \( \sum_{i=1}^{m} r_i = d > \dim S \). Hence the projection \( \Gamma \rightarrow G \) is not dominant. By lemma 2, the Zariski open set \( U = G - \overline{\text{im}\Gamma} \) has the property described in the statement of the theorem. ■

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