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Optical Phases and the Symplectic Group

R. SIMON

Institute of Mathematical Sciences,
Madras 600 113, India.

I consider it an honour and privilege to be able to speak on this historic occasion. Both waves and symmetry were dear to Raman, and dynamically changing phases which manifest themselves as optical frequency shifts, a lifelong preoccupation. Since geometric phases arise basically from the action of symmetry groups of transformations on waves, it seems most appropriate that we have this Workshop on Geometric Phases in Optics as a special part of the Raman Centenary Symposium on Waves and Symmetry, at this Institute where Pancharatnam[1] discovered more than three decades ago the phase which is central to the theme of the workshop. It may not be out of place to note that this work of Pancharatnam was communicated by Sir C.V. Raman himself.

Ever since Berry[2] uncovered five years ago the 'adiabatic phase' and its geometric nature in full generality and in a form applicable to a wide variety of situations, the subject of geometric phases has been pursued vigorously, both theoretically[3,4] and experimentally[5,6]. Attention to Pancharatnam's work was drawn first by Ramaseshan and Nityananda[7], and subsequently by Berry[8] himself in a paper written with the express purpose of "bringing out the full originality of Pancharatnam's contribution by expressing his optics in quantum mechanical language and clarifying the relationship between his phase and (Berry's) adiabatic phase." Indeed experiments involving the Pancharatnam type of geometry have recently played an important role[9–11] in experimentally verifying various aspects of the theory; this has been made amply transparent in the other talks in this workshop[12–14].

Symmetry plays an important role in both classical and quantum optics and this has been eloquently brought out in Prof. Mukunda's talk[15], to which I shall make repeated reference. The very construct of a light ray which has both a position and direction brings in the Euclidean group E_3, also called the inhomogeneous rotation group ISO(3)[16]. We can change the direction of the ray and hence we have the rotation group SO(3), and move the ray from one location to another resulting in the translation group T_3; the group E_3 = ISO(3) is the combination (semi-direct product) of these two groups. While translation of a light ray does not affect its polarization, the constraint part of the Maxwell system of wave equations, div $\vec{E}$ = div $\vec{B}$ = 0, when transcribed to the ray picture, implies that as the direction of the light ray changes the (transverse) polarization changes in a definite way: the polarization vector undergoes a rotation as prescribed by the law of parallel transport[17]. Since a rotation matrix in the linear polarization basis identically equals a diagonal SU(2) matrix in the circular polarization (helicity) basis, it follows that such a rotation of the polarization vector is the same as introducing a relative phase between left and right circular polarizations. The helical fiber experiment of Tomita and Chiao[18] brings out the geometric nature of this phase associated with the group SO(3), forming a beautiful verification of Berry's ideas: the geometric phase suffered by the two helicity states have opposite signature, with magnitude equal to the solid angle that the closed circuit the direction of propagation traces on the sphere of directions subtends at its centre. This change in polarization is brought in due to purely geometric reasons by systems which are basically 'insensitive' to polarization. It should be noted that this phase is independent of wave length and hence survives even in the ray picture which is the short wavelength approximation to the true wave picture.

Given a light ray, it is also possible to change its transverse polarization directly even without changing its direction. This is what polarization sensitive systems like plates made of birefringent or optically active material do. With the propagation direction thus fixed to be along the z-axis we can describe the transverse polarization as a linear combination of x and y polarizations, or right and left circular polarizations. If the coefficients of the expansion are written as a column vector with two complex entries, it is immediately seen that intensity preserving optical systems act as SU(2) transformations on this column vector [With the restriction on the conservation of intensity relaxed, we will have the larger group SL(2,C) rather than SU(2); I shall return to this aspect later in the talk]. With fixed intensity and the over-all phase suppressed, the column vectors can be mapped onto points on the unit sphere S^2 called the Poincaré sphere. On this state space S^2, the SU(2) systems act as rotations[13], and we get the Pancharatnam geometry. For cyclic evolution of polarization state (closed circuit on the Poincaré sphere), the geometric phase equals half the solid angle, rather than the solid angle itself, in view of the fact that SU(2) is the double covering of SO(3).

There is yet another class of Lie groups which plays a dominant role in optics. This is the symplectic group and its cousins[15]. For the rest of the talk I will be concerned with optical phases in situations wherein this class of groups plays a role. Let us first concentrate on one degree of freedom so that the relevant group of linear homogeneous canonical transformations is Sp(2,R) = SL(2,R), which is the same as SU(1,1). Indeed, as the position momentum operators $\hat{x}$, $\hat{p}$ undergo Sp(2,R) transformation, the creation annihilation operators $\hat{a}^\dagger$, $\hat{a}$ undergo SU(1,1) transformation:

$$S = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{Sp}(2,\mathrm{R}),\ S^T\sigma_2 S = \sigma_2;$$

$$\begin{pmatrix} \hat{x} \\ \hat{p} \end{pmatrix} \rightarrow \begin{pmatrix} \hat{x}' \\ \hat{p}' \end{pmatrix} = S\begin{pmatrix} \hat{x} \\ \hat{p} \end{pmatrix} \Leftrightarrow$$

$$\begin{pmatrix} \hat{a} \\ \hat{a}^\dagger \end{pmatrix} \rightarrow \begin{pmatrix} \hat{a}' \\ \hat{a}'^\dagger \end{pmatrix} = V\begin{pmatrix} \hat{a} \\ \hat{a}^\dagger \end{pmatrix},\quad V^\dagger\sigma_3\ V = \sigma_3;$$

$$V = \frac{1}{2}\begin{pmatrix} a + d - i(b - c) & a - d + i(b + c) \\ a - d - i(b + c) & a + d + i(b - c) \end{pmatrix} \in \mathrm{SU}(1,1). \tag{1}$$

Sp(2,R) plays an important role both in the theory of squeezed states (evolution under Hamiltonians quadratic in the position and momentum operators) and in the ray and wave optical description of first order paraxial optical systems[15]. Just as SU(2) is the double covering of SO(3), the noncompact group Sp(2,R) covers twice the pseudo-orthogonal group SO(2,1) of Lorentz transformations in a $(2 + 1)$ dimensional Minkowski space $M_{2,1}$.

Since we are concerned with phases, we cannot afford to ignore the delicate fact that the unitary representation we have in quantum mechanics or first order wave optics is not a faithful representation of Sp(2,R), but that of the Metaplectic group Mp(2) which is a double covering of Sp(2,R). In other words we have a two-valued representation of Sp(2,R) and hence a four-fold covering of SO(2,1).

SQUEEZED STATES AND THE METAPLECTIC GEOMETRIC PHASE

If $|\Psi_0\rangle$ is the ground state of a harmonic oscillator, coherent states are obtained by acting on $|\Psi_0\rangle$ with the Heisenberg-Weyl unitary displacement operators $\hat{D}(\alpha) = \exp[\alpha\hat{a}^\dagger - \alpha^*\hat{a}]$. While the generators of $\hat{D}(\alpha)$ are linear in $\hat{a}$, $\hat{a}^\dagger$ those of the unitary squeeze operators are metaplectic operators quadratic in $\hat{a}$, $\hat{a}^\dagger$:

$$\hat{S}(Z) = \exp\left[\tfrac{1}{4}(Z\hat{a}^2 - Z^*\hat{a}^{\dagger 2})\right]. \tag{2}$$

The squeezed states $|\alpha,Z\rangle$ are then defined as[19]

$$|\alpha,Z\rangle = \hat{D}(\alpha)\ \hat{S}(Z)|\Psi_0\rangle. \tag{3}$$

Since the displacement operator $\hat{D}(\alpha)$ is easier to handle[20], I will concentrate on $\hat{S}(Z)$ and define squeezed states as ($\alpha = 0$)

$$|Z\rangle = \hat{S}(Z)|\Psi_0\rangle. \tag{4}$$

Using the Euler parametrization for Mp(2) we can write the squeeze operator in the convenient form

$$\hat{S}(Z) = \exp\left[-i\frac{\theta}{4}(\hat{a}^\dagger\hat{a} + \hat{a}\hat{a}^\dagger)\right]\exp\left[\frac{r}{4}(\hat{a}^2 - \hat{a}^{\dagger 2})\right]\exp\left[i\frac{\theta}{4}(\hat{a}^\dagger\hat{a} + \hat{a}\hat{a}^\dagger)\right], \tag{5}$$

where r, θ are given by the polar decomposition $Z = r\exp(i\theta)$. Since the right most exponential acting on $|\psi_0\rangle$ introduces just a phase, we can parametrize the manifold of squeezed states as

$$|Z\rangle = |\,r,\theta\rangle = \exp\left[-i\frac{\theta}{4}(\hat{a}^\dagger\hat{a} + \hat{a}\hat{a}^\dagger)\right]\exp\left[\frac{r}{4}(\hat{a}^2 - \hat{a}^{\dagger 2})\right]|\Psi_0\rangle. \tag{6}$$

With $x_0 = \cosh r$, $x_1 = \sinh r\cos\theta$, $x_2 = \sinh r\sin\theta$, it is immediately seen that squeezed states are in one-to-one correspondence with points on the positive timelike unit hyperboloid $x_0^2 - x_1^2 - x_2^2 = 1$, $x_0>0$.

Let $|\Psi\rangle$ be taken over a closed circuit C of squeezed states on this hyperboloid. Then the associated geometric phase β can be immediately computed[20] with the help of (6):

$$\beta = i\oint_C\left[\langle r,\theta|\frac{\partial}{\partial r}|r,\theta\rangle\mathrm{d}r + \langle r,\theta|\frac{\partial}{\partial\theta}|r,\theta\rangle\,\mathrm{d}\theta\right]$$

$$= i\oint_C\Big\{\langle\Psi_0|\,(\hat{a}^2 - \hat{a}^{\dagger 2})/4\,|\Psi_0\rangle\,\mathrm{d}r + \langle\Psi_0|\exp\left[-\frac{r}{4}(\hat{a}^2 - \hat{a}^{\dagger 2})\right]$$

$$(-i/4)\,(\hat{a}^\dagger\hat{a} + \hat{a}\hat{a}^\dagger)\exp\left[+\frac{r}{4}(\hat{a}^2 - \hat{a}^{\dagger 2})\right]|\Psi_0\rangle\mathrm{d}\theta\Big\} \tag{7}$$

$$= \tfrac{1}{4}\oint_C\cosh r\,\mathrm{d}\theta = \tfrac{1}{4}\oiint_\Sigma\sinh r\,\mathrm{d}r\,\mathrm{d}\theta,$$

where Σ is that part of the surface on the hyperboloid for which C is the boundary. Thus, the metaplectic geometric phase is one fourth the area of the circuit on the hyperboloid. This is consistent with the fact that Mp(2) is a four-fold covering of SO(2,1) [This result may be compared with claims to the contrary in Ref. (21)].

The configuration space wave function of a squeezed state ought to have the simple form

$$\Psi(q) = \pi^{-1/4}\,W^{-1/2}\exp(i\chi)\exp(i\rho q^2)\exp(-q/2W^2), \tag{8}$$

where W is the width of the wave function and ρ its phase curvature. χ is the phase at $q = 0$ and, borrowing the terminology from Gaussian (laser) beam propagation, we will call it the "on-axis" phase. The form (8) follows from the Iwazawa decomposition[15]

$$\hat{S}(Z) = \exp\left(-\frac{i}{2}\xi\,\hat{x}^2\right)\exp\left[-i\frac{\eta}{4}(\hat{x}\,\hat{p} + \hat{p}\,\hat{x})\right]\exp\left[-\frac{i\zeta}{4}(\hat{x}^2 + \hat{p}^2)\right] \tag{9}$$

available for any metaplectic operator and hence for the squeeze operator: acting on $|\Psi_0\rangle$ the right most operator introduces the on axis phase, the middle one changes the width and finally the left most one introduces the phase curvature.

Now consider the evolution of a state which at $t = 0$ is highly squeezed:

$$\Psi(q,\, t = 0) = \pi^{-1/4}\, W_0^{-1/2} \exp\left[-\frac{q^2}{2W_0^2}\right],\; W_0^2 \ll 1. \tag{10}$$

On the hyperboloid it is represented by the point $(x_0,\; x_1,\; x_2) = (\cosh r,\; \sinh r,\; 0)$, $e^{-r} = W_0^2$, while $|\Psi_0\rangle$ is represented by $(1,0,0)$. Let this squeezed state evolve under the harmonic oscillator Hamiltonian, which acts on the hyperboloid as a rotation about the x_0 axis. The width, curvature and on axis phase evolve in time to $W(t)$, $\rho(t)$ and $\chi(t)$. We are interested in $\chi(t)$. To compute it geometrically we need a closed circuit on the hyperboloid. It is convenient to close the circuit as follows: actual evolution from $|\Psi(t = 0)\rangle$ to $|\Psi(t)\rangle$ followed by $\exp[i\rho(t)\hat{x}^2]$ to kill the phase curvature and finally $\exp[(-i\eta/4)(\hat{x}\hat{p} + \hat{p}\hat{x})]$ for suitable value of η to bring the width $W(t)$ back to the initial value W_0. The latter two actions *do not change the on axis phase,* and hence $\chi(t)$ of our interest equals the total phase picked up by the state during this cyclic evolution.

The dynamical phase is easy to compute, for metaplectic Hamiltonians produce SO(2,1) Lorentz rotations on the hyperboloid. For a Hamiltonian which produces Lorentz rotation about a vector $\vec{y}$ in $M_{2,1}$ the dynamical phase suffered by a state represented by a vector $\vec{x}$ on the hyperboloid is simply related to the Lorentz inner product $\vec{x}\cdot\vec{y}$. As a consequence the dynamical phase is zero only for states moving on geodesics (intersections of the hyperboloid with planes containing the origin in $M_{2,1}$) in exactly the same way as with SU(2) where the dynamical phase is zero only for great circle arcs.

The geometric phase is given by the area formula (7), and adding it with the dynamical phase one obtains the total phase. But I do not know of any simple analytic way of computing area of a circuit on the hyperboloid. I have, however, verified, by numerically computing the area, that the sum of the dynamical and geometric phases for this circuit indeed gives the correct value of $\chi(t)$. The correct value of $\chi(t)$ is, of course, determined analytically by another interesting consideration to which I now turn.

The idea is to use the Pancharatnam prescription for comparing phases of two distinct states[1]. Let us choose the phases of all the squeezed states in such a way that they are "in phase" with $|\Psi_0\rangle$ according to the Pancharatnam prescription. That is for every state $|\Psi\rangle$ on the hyperboloid its arbitrary phase should be chosen such that $\langle\Psi_0|\Psi\rangle$ is real positive. Such a choice is always possible to implement, for the squeezed states are generalized coherent states[22] (of the group Sp(2,R)), and hence no two of them are orthogonal.

Let us combine ρ and W into a complex parameter Ω so that the squeezed state has the wave function

$$\Psi_\Omega(q) = N(\Omega)\, \exp\left[\frac{iq^2}{2\Omega}\right],$$

$$\frac{1}{\Omega} = \rho + \frac{i}{W^2} = \frac{x_0 + x_1}{x_2 - i}\,, \tag{11}$$

where $(x_0,\; x_1,\; x_2)$ is the point on the hyperboloid representing $\Psi_\Omega(q)$, and $N(\Omega)$ is a complex normalization constant whose phase $\delta(\Omega)$ we have to now fix by demanding that $\langle\Psi_0(q)|\Psi_\Omega(q)\rangle$ is real positive with $\Psi_0(q) = \pi^{-1/4}\exp[-q^2/2]$. [An attractive property of Ω is that when x_0, x_1, x_2 undergo SO(2,1) Lorentz rotation, Ω undergoes a Mobius transformation]. Evaluation of the Gaussian integral representing this inner product fixes $\delta(\Omega)$ to be equal and opposite to the phase of $\left[1 - i\dfrac{x_0 + x_1}{x_2 - i}\right]^{1/2}$. Thus in terms of the coordinates on the hyperboloid we have

$$\delta(\Omega) = \delta(r,\theta) = \frac{\theta}{4} - \frac{1}{2}\arctan\,(e^r \tan\,(\theta/2))\,. \tag{12}$$

With the arbitrary phase of every state on the hyperboloid adjusted to be "in phase" with $|\Psi_0\rangle$, we note that under the harmonic oscillator evolution $|\Psi_0\rangle$ picks up a phase of $\exp[-i\omega t/2]$. Further θ increases linearly as $2\omega t$ (rather than as ωt, since Sp(2,R) is a double covering of SO(2,1)), and r remains constant. Since the inner product does not change under any unitary evolution, we see that the squeezed state evolves in such a way that it continues to be in phase with $\exp[-i\omega t/2]|\Psi_0\rangle$. Thus, the time-dependent parameters of the squeezed state under the harmonic oscillator evolution are

$$r(t) = \text{constant},$$

$$\theta(t) = 2\omega t \ (\text{mod } 2\pi); \tag{13}$$

$$\chi(t) = \frac{\theta(t)}{4} - \frac{1}{2} \arctan\,[e^r \tan\,(\theta(t)/2)] - \omega t/2. \tag{14}$$

It is this expression for the on-axis phase $\chi(t)$ with which our earlier computation of the sum of the geometric and dynamical phase agrees. For a highly squeezed state $r \gg 1$, and it is seen from (14) that for such a state the on-axis phase jumps by $\pi/2$ at values of $t = n\pi/\omega$. These are precisely the values of t at which the width returns to its original squeezed minimum value. These are also the instants of time at which the squeezed state passes through the point ($\cosh r$, $\sinh r$, 0) on the geodesic curve $\theta = 0$ on the hyperboloid. This phase jump is reminiscent of the Guoy effect[23] where the phase of a highly focussed cylindrical beam jumps by $\pi/2$ (for spherical beam the jump is π, since two degrees of freedom) as it crosses the focus. Indeed we can call the phenomenon we have in (14) the *Guoy effect for squeezed states.* Our earlier consideration shows that part of this metaplectic phase jump is dynamical with the other part being geometric and given by one fourth the area on the hyperboloid. Even for small values of r there is a *phase bunching* near the $\theta = 0$ curve on the hyperboloid, and this bunching increases exponentially with r; for large values of r the phase bunching saturates and becomes a Dirac delta function of amplitude $\pi/2$.

A classical coherent Gaussian beam also picks up under free propagation an "extra phase" at and near its waist. Since wave optics is the metaplectic representation of ray optics, it follows that this extra phase can also be understood as a metaplectic geometric phase.

There is one situation, other than the one involving squeezed states, where the behaviour of the on-axis phase $\chi(t)$ can be experimentally verified. This is the parabolic index fiber or square law medium. It has a Gaussian eigen mode of characteristic width. If we launch into the fiber a Gaussian beam whose width is much smaller than this characteristic width, then its behaviour as it propagates down the axis of the fiber will be identical to the configuration space evolution of a highly squeezed state. In particular the on axis phase will be governed by (14) and the metaplectic phase jump may be easier to verify experimentally in this case.

Partial polarizers and the symplectic geometric phase

Earlier in the talk I considered intensity preserving SU(2) systems in polarization optics. If we do not require our linear optical system to preserve intensity, then its action on the two-component transverse vector will be through a matrix γS where S is an SL(2,C) matrix and γ a scalar. Since γ does not play any role in the geometric considerations, we can consider our optical systems to be SL(2,C) systems. Further, since intensity is now a state variable the state space is no longer the unit sphere S^2; the correct state space is the positive time-like unit hyperboloid in a 3 + 1 dimensional Minkowski space and it is parametrized by the well-known Stokes parameters[24]. If we restrict attention to partial linear polarizers then we have the sub group SL(2,R) = Sp(2,R) = SU(1,1) $\subset$ SL(2,C) rather than the full SL(2,C), and the analysis can be carried out on a submanifold of the 3 + 1 hyperboloid. This is so for the following reasons: if we start with real x,y components, they remain real under all SL(2,R) action. Thus, we are back to our old 2 + 1 hyperboloid. But, this time the group SL(2,R) = Sp(2,R) itself, and not its double covering, acts on this hyperboloid. And hence the geometric phase will be half the area on the hyperboloid. This is supported by experiment[25].

The effect of the geometric phase in this case will mean the following. The effect of two partial polarizers acting in sequence will not be just a partial polarizer, but one followed by a rotation.

It is this rotation which is given by the Sp(2,R) geometric phase + dynamical phase. This is reminiscent of the Wigner rotation. In fact it is: both are SL(2,R) problems. Partial polarizers are in one-to-one correspondence with symmetric SL(2,R) matrices and so also are boosts. And the product of two symmetric matrices is not a symmetric matrix, but one followed by a rotation. The point being made is that with three partial linear polarizers we can synthesize an optical rotator, the angle of rotation being given, apart from the dynamical phase, by half the area on the 2 + 1 hyperboloid.

We have seen metaplectic optical phases, given by one fourth the area on the hyperboloid, in quantum optics and classical wave optics; and symplectic phases, given by half the area on the hyperboloid, in polarization optics. The question that naturally arises is this: are there optical phases of the SO(2,1) type given by the area on the hyperboloid, rather than half of it or one fourth of it? There may be, I do not know.

Skew rays in lens systems and rotation of polarization

The passage of light rays through lens systems is governed by Fermat's principle, and hence by a Hamiltonian structure. While we have linear canonical transformations constituting the symplectic group for paraxial rays, for nonparaxial rays the transformation is not symplectic. It is, however, a canonical transformation – a nonlinear one.

For simplicity, let us consider lens systems which have rotational symmetry about the system axis. A meridional ray remains meridional, and on the sphere of propagation directions it traverses a circuit which lies in a plane containing the origin of the sphere (great circle). Hence when the ray direction regains its original value we have on the sphere of directions a circuit with zero area; left and right circular polarization components suffer the same phase change and hence no change in the state of polarization. Thus, for a meridional ray a scalar theory is good enough.

The situation is different for skew rays. Since skewness is preserved, on the sphere of directions the ray will continue to traverse the circuit in the same sense all along and hence a non zero area will be enclosed. As a consequence left and right circular polarizations pick up different phases and hence the plane of polarization undergoes a rotation. As one consequence we see that scalar theory is much less adequate for skew rays than for meridional rays: even when the position and direction of a ray are restored by a lens system, there is no guarantee that its polarization will be restored, if the ray under consideration is a skew ray.

This brings in the following question: Is it possible to design a lens system which makes an identity mapping not only on the position and direction of all rays but also on their polarization? I think the answer will be in the negative. Assuming this, we have at hand a new optimization problem of designing an imaging system paying due respect to Maxwell by taking polarization into account.

Pancharatnam phase as a symplectic phase

As a final observation it may be noted that the Pancharatnam phase itself can be interpreted as a symplectic phase in the sense I now describe.

Consider light propagating along the z-axis. We can choose two orthogonal polarizations, x and y polarizations or right and left circular polarizations. With each one of these two polarization modes we can associate a boson operator. Optical systems which produce linear canonical transformations on these boson operators constitute the group Sp(4,R). Now the requirement that the intensity is preserved forces us to the maximal compact subgroup of Sp(4,R). But the maximal compact subgroup of Sp($2n$,R) is U(n) for every n. Thus, we recover the U(2), or SU(2), subgroup which forms the basis for the Pancharatnam geometry involving birefringent plates and optically active media.

To conclude, since the groups SU(2) and Sp(2,R) play a dominant role in the context of optical phases, it is likely that geometric constructions of the type which Prof. Mukunda described in his talk[15], namely Hamilton's theory of turns and its generalization to Sp(2,R), may be useful tools for these problems[26-28].

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REFERENCES

1. Pancharatnam, S., *Proc. Indian Acad. Sci.,* 1956, **A44,** 247, reprinted in *Collected Works of S. Pancharatnam* (Oxford University Press, 1975) p. 77.
2. Berry, M. V., *Proc. R. Soc. London,* 1984, **A392,** 45.
3. Simon, B., *Phys. Rev. Lett.,* 1983, **51,** 2167, Wilczek, F. and Zee, A., *Phys. Rev. Lett.,* 1984, **52,** 2111; Aharonov, Y., and Anandan, J., *Phys. Rev. Lett.,* 1987, **58,** 1593 ; Jackiw, R., *Int. Jour. Mod. Phys.,* 1988, **A3,** 285; Samuel, J., and Bhandari, R., *Phys. Rev. Lett.,* 1988, **60,** 2339; Agarwal, G. S., *Phys. Rev.,* 1988, **A38,** 5957.
4. Berry, M. V., *J. Phys.,* 1985, **A18,** 15, Hannay, J., *J. Phys.,* 1985, **A18,** 221; Gozzi, E. and Thacker, W., *Phys. Rev.,* 1987, **D35,** 2388, 2389.
5. Tycko, R., *Phys. Rev. Lett.,* 1987, **58,** 2281; Bitter, T. and Dubbers, D., *Phys. Rev. Lett.,* 1987, **59,** 251.
6. Suter, D., Chinges, G., Harris, R. A., and Pines, A., *Mol. Phys.,* 1987, **61,** 1327; Suter, D., Mueller, K. T., and Pines, A., *Phys. Rev. Lett.,* 1988, **60,** 1218.
7. Ramaseshan, S., and Nityananda, R., *Curr. Sci.,* 1986, **55,** 1225.
8. Berry, M. V., *J. Mod. Opt.,* 1987, **34,** 1401.
9. Bhandari, R., and Samuel, J., *Phys. Rev. Lett.,* 1988, **60,** 1211; Bhandari, R., *Phys. Lett.,* 1988, **A133,** 1.
10. Simon, R., Kimble, H. J., and Sudarshan, E. C. G., *Phys. Rev. Lett.,* 1988, **61,** 19; Maddox, J., *Nature,* 1988, **334,** 99.
11. Chyba, T. H., Wang, L. J., Mandel, L., and Simon, R., *Opt. Lett.,* 1988, **13,** 562; Tompkin W. R., Malcuit, M. S., Boyd, R. W. and Chiao, R. Y., preprint, 1988.
12. Berry, M. V., Proceedings of this Symposium.
13. Ramaseshan, S., Proceedings of this Symposium.
14. Bhandari, R., Proceedings of this Symposium.
15. Mukunda, N., Proceedings of this Symposium.
16. Wolf, K. B., "Elements of Euclidean Optics" in *Lie Methods in Optics-II Workshop,* to be published.
17. Born, M., and Wolf, E., *Principles of Optics* (Pergamon, London, 1959).
18. Tomita, A., and Chiao, R. Y., *Phys. Rev. Lett.,* 1986, **57,** 937; Chiao, R. Y., and Wu, Y. S., *Phys. Rev. Lett.,* 1988, **57,** 933.
19. Walls, D. F., *Nature,* 1983, **306,** 141.
20. Geometric phase associated with the Heisenberg-Weyl group has been studied by Chaturvedi, S., Sriram, M. S., and Srinivasan, V., *J. Phys.,* 1987, **A.L1071.** See also, Simon, R., and Kumar, N., *J. Phys.,* 1988, **A21,** 1725.
21. Jordon, T. F., and Chiao, R. Y., *Phys. Lett.,* 1988, **A132,** 77.
22. Perelomov, A. M., *Sov. Phys. Usp.,* 1977, **20,** 703.
23. See, for example, Siegman, A. E., *Lasers* (Oxford University Press, 1986), where further references can be found.
24. Stokes, G. G., *Trans. Cambridge Philos. Soc.,* 1952, **9,** 399; Ramachandran, G. N., and Ramaseshan, S., *Handb. Physik,* **XXV/1,** (Springer-Verlag, Berlin, 1961).
25. Kitano, M., and Yabuzaki, T., Preprint, 1988.
26. Simon, R., Mukunda, N., and Sudarshan, E. C. G., *Phys. Rev. Lett.,* 1989, **62,** 1331.
27. Simon, R., Mukunda, N., and Sudarshan, E. C. G., *J. Math. Phys.,* 1989, **30,** 1000–1006.
28. Simon, R., Mukunda, N., and Sudarshan, E. C. G., *Pramana,* 1989, **32,** 769–792.