Pramāṇa, Vol. 20, No. 2, February 1983, pp. 105–124. © Printed in India.

Generalized pencils of rays in statistical wave optics

R SIMON*
Department of Physics, Indian Institute of Science, Bangalore 560 012
*Permanent address: Department of Physics, The American College, Madurai 625 002, India

MS received 23 September 1982

Abstract. The recently introduced generalized pencil of Sudarshan which gives an exact ray picture of wave optics is analysed in some situations of interest to wave optics. A relationship between ray dispersion and statistical inhomogeneity of the field is obtained. A paraxial approximation which preserves the rectilinear propagation character of the generalized pencils is presented. Under this approximation the pencils can be computed directly from the field conditions on a plane, without the necessity to compute the cross-spectral density function in the entire space as an intermediate quantity. The paraxial results are illustrated with examples. The pencils are shown to exhibit an interesting scaling behaviour in the far-zone. This scaling leads to a natural generalization of the Fraunhofer range criterion and of the classical van Cittert-Zernike theorem to planar sources of arbitrary state of coherence. The recently derived results of radiometry with partially coherent sources are shown to be simple consequences of this scaling.

Keywords. Generalized pencils; paraxial approximation; partial coherence; radiometry; diffraction; interference; wave optics.

1. Introduction

Radiative transfer problems of interest in astrophysics have been conventionally treated by a phenomenological theory (Chandrasekhar 1950) which is built on the notion of *pencils of rays*. The strength of these pencils as a function of position and direction has been termed the specific intensity. In free space these pencils are assumed to travel in straight lines, and hence the specific intensity is constant along straight lines. In a scattering medium the specific intensity is assumed to obey a transport equation whose structure is similar to Boltzmann's transport equation. The implications of this phenomenological radiative transfer theory have been studied in great detail (Chandrasekhar 1950).

Since Maxwell's theory is the fundamental theory of electromagnetic phenomena at the classical level, to the extent that the phenomenological radiative transfer theory and the notion of pencils of rays are valid, they should be derivable as an approximation from Maxwell electrodynamics. We note that according to the second fundamental theorem of quantum optics due to Sudarshan (1969, 1979b), at the level of the two-point correlation function classical electrodynamics and quantum electrodynamics lead to indistinguishable results. Since our interest here is in two-point correlation functions, in view of the above mentioned theorem we may formally restrict our treatment to the classical level.

Following the pioneering work of Wolf (1976), there have been several attempts by Sudarshan (1979b, 1980, 1981) and others (Zubairy and Wolf 1977; Fante 1981; Zubairy 1981) to clarify the relationship between the phenomenological radiative transfer theory and electrodynamics. Clearly, the basic problem is to identify a linear functional of the two-point correlation function of the electromagnetic field which possesses all or almost all the properties attributed to the phenomenological pencils. In the phenomenological theory, the pencils due to different frequencies add without interference. Since absence of interference between different frequency components in the two-point correlation function corresponds to time stationarity (Born and Wolf 1970), it becomes clear that what is to be looked for is a linear functional of the cross-spectral density function of a time stationary ensemble. Two such functionals have been studied by Wolf and Sudarshan respectively. Each of these functionals, for a field of arbitrary state of coherence, satisfies only some of the properties of the phenomenological pencils. For instance, the linear functional studied by Sudarshan is not positive definite whereas the phenomenological specific intensity is. Now it is generally believed that there exists no linear functional of the cross-spectral density function which satisfies all the properties of the phenomenological pencils.

Sudarshan (1979a) has shown that by generalizing the notion of pencils of rays to include both *pradipa rays* (pencils with positive strength) and *tamasic rays* (pencils with negative strength), time stationary free space wave fields of arbitrary state of coherence can be described in an exact way as collection of *generalized pencils of rays* travelling in straight lines, thus obtaining an exact ray picture of wave optics. He has interpreted the well-known far-field solutions of typical diffraction and interference configurations in terms of these generalized pencils. However, this powerful geometric picture has so far not been exploited in optics research. A probable reason is that to compute the generalized pencils the cross-spectral density in the entire space must be known. We show that under a paraxial approximation the generalized pencils can be computed directly from the field conditions in the 'source plane' without the necessity to compute the cross-spectral density in the entire space as an intermediate quantity.

In many problems of interest in optics, the wave field (in the form of the complex field amplitude or the cross-spectral density function) is given in a plane, which one may take without loss of generality to be the plane $z=0$; and one is interested in calculating the wave field in the half space $z\geqslant 0$, satisfying the Sommerfeld radiation condition at infinity. We note that this kind of geometry includes the entire range of diffraction problems, beam propagation and radiometry with sources of arbitrary state of coherence. It should also be noted that for most of these problems the paraxial approximation yields adequate results.

In the present paper, we study a paraxial approximation which greatly simplifies the computation of the generalized pencils, and leads to an interesting scaling behaviour of the pencils in the far-zone. Section 2 starts with the definition of the generalized pencils. The pencils corresponding to the interference of a pair of plane waves are computed. A direct relationship between *ray dispersion* and statistical inhomogeneity of the field is derived. In § 3, we present a paraxial approximation to the generalized pencils. This approximation preserves the straight line propagation character of the exact pencils. Both pradipa and tamasic rays survive this approximation. Under this approximation, the generalized pencils in the half space $z \geqslant 0$ can be computed directly from the field conditions in the plane $z=0$, without the need to compute the wave field in the entire space as an intermediate quantity. To be specific, the strength of the generalized pencils through points in the plane $z=0$ is shown to be related to a two-dimensional Fourier transform of the cross-spectral density function in that plane; these pencils are shown to travel in straight lines into the half space $z > 0$. Thus the pencils through an arbitrary point in $z > 0$ can be constructed using simple geometry. And then, the quantities of interest in wave optics can be readily computed from the pencils so constructed; thus making the generalized pencils a practical tool for wave-optical computations. Conditions for the validity of the paraxial approximation are given. In § 4, these paraxial results are applied to some typical situations of interest in elementary wave optics.

In § 5, the pencils are shown to exhibit an interesting scaling behaviour in the far-zone. This scaling behaviour leads to a natural generalization of the Fraunhofer range criterion to sources of arbitrary state of coherence. A generalization of the classical van Cittert-Zernike theorem (van Cittert 1934; Zernike 1938; Klauder and Sudarshan 1968) to sources of arbitrary state of coherence is obtained as a direct consequence of this scaling. It is also shown that the well-known results of radiometry of partially coherent sources, which are usually derived using wave-optical methods, can be obtained in an elegant and easier way as simple consequences of this scaling behaviour.

A scalar treatment ignoring polarization is presented for simplicity.

2. The generalized pencils and ray dispersion in free space

In a time-stationary ensemble different frequency components do not interfere. Such an ensemble can be conveniently described through the hermitian cross-spectral density function (Born and Wolf 1970) $\overline{\Gamma}(\mathbf{r}_1, \mathbf{r}, w)$ defined by

$$\langle \phi_w^* (\mathbf{r}_1) \phi_w (\mathbf{r}_2) \rangle = \overline{\Gamma} (\mathbf{r}_1, \mathbf{r}_2; w)\, \delta (w - w') \tag{1}$$

where $\phi_w(\mathbf{r})$ is the analytic signal at $\mathbf{r}$ due to radiation of frequency w. The angular brackets denote ensemble average. The parametric dependence on w will be suppressed henceforth. The cross-spectral density in a plane $z =$ constant will be denoted by $\overline{\Gamma}_z (\boldsymbol{\rho}_1, \boldsymbol{\rho}_2)$, where

$$\boldsymbol{\rho}_i = (x_i, y_i), \quad i = 1, 2, \tag{2}$$

is the transverse two-vector part of the three-vector

$$\mathbf{r}_i = (\boldsymbol{\rho}_i, z_i).$$

It is convenient to work with sum and difference variables defined through

$$\mathbf{r} = \tfrac{1}{2} (\mathbf{r}_1 + \mathbf{r}_2), \quad \Delta\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2. \tag{3}$$

The Jacobian of this transformation is unity. We denote

$$\bar{\Gamma}(\mathbf{r}_1, \mathbf{r}_2) = \Gamma(\mathbf{r}, \Delta\,\mathbf{r}). \tag{4}$$

The generalized pencils corresponding to $\bar{\Gamma}(\mathbf{r}_1, \mathbf{r}_2)$ can now be defined as (Sudarshan 1979a)

$$W(\mathbf{r}, \mathbf{p}) = \frac{1}{(2\pi)^3}\int \Gamma(\mathbf{r}, \Delta\,\mathbf{r}) \exp(-i\mathbf{p}\cdot\Delta\,\mathbf{r})\, \mathrm{d}^3\,\Delta\,\mathbf{r}. \tag{5}$$

$W(\mathbf{r}, \mathbf{p})$ is called the Wolf function. From (1) and (5), $W(\mathbf{r}, \mathbf{p})$ is easily seen to be real; but in general it is not non-negative. It is interpreted to represent the strength of the pencil at the point $\mathbf{r}$ going in the direction of $\mathbf{p}$. $|\mathbf{p}|$ is related to dispersion [cf. (17)]. Pencils with positive strength have been called pradipa (shining) rays and pencils with negative strength have been called tamasic (dark) rays [Sudarshan 1979a, b]. From the free space wave equations for $\bar{\Gamma}(\mathbf{r}_1, \mathbf{r}_2)$ it can be shown that (Sudarshan 1981)

$$\mathbf{p}\cdot\nabla\, W(\mathbf{r}, \mathbf{p}) = 0, \tag{6}$$

and

$$(\tfrac{1}{4}\,\nabla^2 + k_0^2 - p^2)\, W(\mathbf{r}, \mathbf{p}) = 0, \tag{7}$$

where the differential operator acts on the $\mathbf{r}$ dependence. Equation (6) means that in free space the generalized pencils travel in straight lines. That is, for a given $\mathbf{p}$, the density W is constant in ordinary $\mathbf{r}$-space along lines parallel to $\mathbf{p}$. Equation (5) can be inverted to obtain the cross-spectral density function from a knowledge of the generalized pencils:

$$\Gamma(\mathbf{r}, \Delta\,\mathbf{r}) = \int W(\mathbf{r}, \mathbf{p}) \exp(i\,\mathbf{p}\cdot\Delta\,\mathbf{r})\, \mathrm{d}^3\, p. \tag{8}$$

Note that in both (5) and (8) one has to perform a three-dimensional integration. If we now set $\Delta z = 0$, so that both $\mathbf{r}_1$ and $\mathbf{r}_2$ lie on the same plane for some given z, we get

$$\Gamma_z(\rho, \Delta\,\rho) = \int W(\mathbf{r}, \mathbf{p}) \exp(i\,\mathbf{p}_\perp\cdot\Delta\,\rho)\, \mathrm{d}^2\, \mathbf{p}_\perp\, \mathrm{d}\,p_z, \tag{9}$$

where

$$\mathbf{p}_\perp = (p_x, p_y), \quad \mathbf{p} = (\mathbf{p}_\perp, p_z). \tag{10}$$

Equation (8) with $\Delta\,\mathbf{r} = 0$ gives the intensity at any point $\mathbf{r}$ as the integrated strength of all the pencils through that point:

$$I(\mathbf{r}) = \int W(\mathbf{r}, \mathbf{p})\, \mathrm{d}^3\, \mathbf{p}. \tag{11}$$

As an elementary illustration, we compute the pencils corresponding to the superposition of two plane waves $a \exp(-i\mathbf{K}_1, \mathbf{r})$ and $b \exp(-i\mathbf{K}_2\cdot\mathbf{r})$ of common frequency w. Clearly,

$$|\mathbf{K}_1|^2 = |\mathbf{K}_2|^2 = k_0^2 = w^2/c^2. \tag{12}$$

One obtains from (1)

$$\bar{\Gamma}(\mathbf{r}_1, \mathbf{r}_2) = \langle |a|^2 \rangle \exp [i \mathbf{K}_1 \cdot (\mathbf{r}_1 - \mathbf{r}_2)]$$
$$+ \langle |b|^2 \rangle \exp [i \mathbf{K}_2 \cdot (\mathbf{r}_1 - \mathbf{r}_2)]$$
$$+ \langle a^* b \rangle \exp [i (\mathbf{K}_1 \cdot \mathbf{r}_1 - \mathbf{K}_2 \cdot \mathbf{r}_2)]$$
$$+ \langle a b^* \rangle \exp [i (\mathbf{K}_2 \cdot \mathbf{r}_1 - \mathbf{K}_1 \cdot \mathbf{r}_2)], \quad (13)$$

where we have assumed that the ensemble average refers only to the amplitudes a, b and not to the vectors $\mathbf{K}_1, \mathbf{K}_2$. Now the pencils can be computed from eq. (5);

$$W(\mathbf{r}, \mathbf{p}) = \langle |a|^2 \rangle \, \delta (\mathbf{p} - \mathbf{K}_1) + \langle |b|^2 \rangle \, \delta (\mathbf{p} - \mathbf{K}_2)$$
$$+ 2 | \langle a^* b \rangle | \cos [(\mathbf{K}_1 - \mathbf{K}_2) \cdot \mathbf{r} + \epsilon] \, \delta [\mathbf{p} - \tfrac{1}{2} (\mathbf{K}_1 + \mathbf{K}_2)] \quad (14)$$

where ϵ is the phase of $\langle a^* b \rangle$. We can now interpret this expression as follows: Through every point $\mathbf{r}$ there is one pencil of strength $\langle |a|^2 \rangle$ going in the direction of $\mathbf{K}_1$, and another of strength $\langle |b|^2 \rangle$ going in the direction of $\mathbf{K}_2$. The strength and direction of each of these two pencils equal the intensity and direction of the respective plane wave. These pencils consist of only pradipa rays, and have $|\mathbf{p}| = k_0$. There is an additional pencil whose strength depends on the correlation between the two plane waves, as shown by the factor $\langle a^* b \rangle$. At every point this pencil is in the direction of the bisector of the angle between $\mathbf{K}_1$ and $\mathbf{K}_2$, as can be seen from figure 1. The strength of this pencil is not positive definite, and it has a sinusoidal variation in space in the direction of $(\mathbf{K}_1 - \mathbf{K}_2)$, *i.e.* perpendicular to $(\mathbf{K}_1 + \mathbf{K}_2)$. The special frequency, f_{spatial}, of this sinusoidal variation is $|\mathbf{K}_1 - \mathbf{K}_2|$. We note that dark rays are absent if and only if the two plane waves are mutually incoherent, *i.e.*, $\langle a^* b \rangle = 0$.

From figure 1, it is seen that

$$f_{\text{spatial}} = |\mathbf{K}_1 - \mathbf{K}_2| = 2 \, k_0 \sin \theta, \quad (15)$$

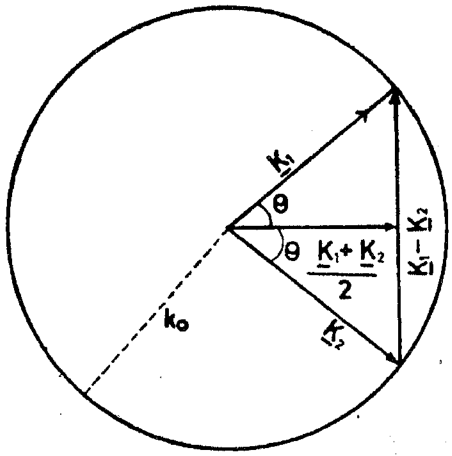

Figure 1. Interference of a pair of plane waves.

where θ is the half angle between $\mathbf{K}_1$ and $\mathbf{K}_2$. We also note that for the last pencil

$$|\mathbf{p}| = \left|\frac{\mathbf{K}_1 + \mathbf{K}_2}{2}\right| = k_0 \cos\theta, \tag{16}$$

so that

$$\frac{f^2_{\text{spatial}}}{4} + |\mathbf{p}|^2 = k_0^2. \tag{17}$$

The last equation is trivially satisfied by the pencils in the first two terms of (12). Further, note that for the third pencil the intensity integrated over the entire space is zero, showing that it consists of pradipa and tamasic rays in equal amounts.

Given an arbitrary free space field, we can Fourier analyse and express it as a superposition of plane waves. The diagonal elements of the cross-spectral density in the plane wave representation will correspond to the strength of the various plane wave components (similar to the first two terms on the right side of (13)), and the off-diagonal elements will correspond to correlation between pairs of plane waves (similar to the last two terms in (13)). The diagonal elements will give rise to pencils of the type in the first two terms on the right side of (14), and the off-diagonal ones will give pencils of the type in the last term of (14). Our analysis following (14) remains valid for these pencils. In particular, pencils of both types respect (17). Thus, (17) becomes an identity for free space pencils. This can also be seen by Fourier transforming (7) with respect to $\mathbf{r}$.

Equation (17) is an important result of our analysis. It fixes the range of values for $|\mathbf{p}|$ and f_{spatial} to be

$$0 \leqslant f^2_{\text{spatial}} \leqslant 4\,k_0^2, \tag{18}$$

$$0 \leqslant |\mathbf{p}|^2 \leqslant k_0^2. \tag{19}$$

Deviation of $|\mathbf{p}|$ from k_0 has been called 'ray dispersion' (Sudarshan 1980). Equation (17) relates ray dispersion directly to the spatial variation of the pencils, or, from (8), to the statistical inhomogeneity of the field. From (17) it is clear that absence of dispersion implies and is implied by statistical homogeneity of the field. This is the content of the well-known (Fante 1981; Zubairy 1981) counter argument of Collett *et al* (1977) to Tatarskii. Equation (17) goes beyond the argument presented by Collett *et al* in that it relates ray dispersion to the spatial spectrum of the field inhomogeneity in a quantitative manner. If $\boldsymbol{\Gamma}(\mathbf{r}, \Delta\,\mathbf{r})$ as a function of $\mathbf{r}$ is slowly varying (the scale being set by $\mathbf{k}_0^{-1}$), then $f_{\text{spatial}} \ll k_0$, and from (17) $|\mathbf{p}|^2 \approx k_0^2$.

Returning to (14), we see that each plane wave in the field amplitude is mapped into a non-dispersive generalized pencil, while correlations between pairs of plane waves give rise to dispersive generalized pencils. It also follows from (14) that absence of dispersion is both the necessary and the sufficient condition for absence of dark rays.

The results of our analysis following (14) can be summarised as follows: Absence of dispersion or absence of dark rays on the one hand, and absence of statistical inhomogeneity or absence of correlation between plane wave pairs on the other, are mutually equivalent.

Finally, we see that the pencils in (14) when used in (11) give the familiar intensity distribution corresponding to interference of a pair of plane waves:

$$I(\mathbf{r}) = \langle |a|^2\rangle + \langle |b|^2\rangle$$
$$+ 2\,|\langle a^* b\rangle|\cos[(\mathbf{K}_1 - \mathbf{K}_2)\cdot\mathbf{r} + \epsilon]. \tag{20}$$

This, of course, is non-negative.

3. The paraxial approximation

From the general theory, given the cross-spectral density function in the entire space, the exact generalized pencils can be computed from (5) and vice versa:

cross-spectral density function in the entire space $\Big\} \Longleftrightarrow$ exact pencils.

However, with a view to make this picture applicable to practical situations, in this section we show that under a paraxial approximation which preserves the straight line propagation character of the exact pencils, namely (6), the pencils can be computed in a simple way from a knowledge of the cross-spectral density function in the source plane alone.

Consider the geometry of interest in diffraction problems, beam propagation and physical radiometry: The cross-spectral density function (or the complex field amplitude) of a time stationary field is given on a plane, which we can take without loss of generality to be the $z = 0$ plane, as

$$\overline{\Gamma}_0(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2) = \langle \phi^*(\boldsymbol{\rho}_1, z_1 = 0)\,\phi(\boldsymbol{\rho}_2, z_2 = 0)\rangle, \tag{21}$$

and one is interested in computing the generalized pencils assuming that the field propagates into the region $z > 0$ (obeying the Sommerfeld radiation condition).

It is useful to construct the double angular spectrum of plane waves $\overline{A}(\mathbf{q}_1, \mathbf{q}_2)$ defined through

$$\overline{A}(\mathbf{q}_1, \mathbf{q}_2) = \frac{1}{(2\pi)^4}\int \overline{\Gamma}_0(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2)\exp[-i(\mathbf{q}_1\cdot\boldsymbol{\rho}_1 - \mathbf{q}_2\cdot\boldsymbol{\rho}_2)]\,\mathrm{d}^2\boldsymbol{\rho}_1\,\mathrm{d}^2\boldsymbol{\rho}_2, \tag{22}$$

where

$$\mathbf{q}_i = (q_{ix}, q_{iy}), \quad i = 1, 2. \tag{23}$$

Equation (22) can be inverted to obtain

$$\overline{\Gamma}_0(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2) = \int \overline{A}(\mathbf{q}_1, \mathbf{q}_2)\exp[i(\mathbf{q}_1\cdot\boldsymbol{\rho}_1 - \mathbf{q}_2\cdot\boldsymbol{\rho}_2)\,\mathrm{d}^2\mathbf{q}_1\,\mathrm{d}^2\mathbf{q}_2. \tag{24}$$

$\overline{A}(\mathbf{q}_1, \mathbf{q}_2)$ can be interpreted to represent the correlation between a pair of plane waves with propagation vectors $(\mathbf{q}_1, q_{1z})$ and $(\mathbf{q}_2, q_{2z})$ respectively, where q_{1z} and q_{2z} are fixed by the fact that these plane waves obey the free space Helmholtz equation and the Sommerfeld radiation condition:

$$q_{iz} = [k_0^2 - |\mathbf{q}_i|^2]^{1/2}, \quad |\mathbf{q}_i|^2 \leqslant k_0^2, \quad \text{(homogeneous plane waves)}$$

$$q_{iz} = -i\,[|\mathbf{q}_i|^2 - k_0^2]^{1/2}, \quad |\mathbf{q}_i|^2 > k_0^2, \quad \text{(inhomogeneous plane waves).} \tag{25}$$

Hence from (24)

$$\bar{\Gamma}(\mathbf{r}_1, \mathbf{r}_2) = \int \bar{A}(\mathbf{q}_1, \mathbf{q}_2) \exp[i(\mathbf{q}_1 \cdot \boldsymbol{\rho}_1 - \mathbf{q}_2 \cdot \boldsymbol{\rho}_2)] \exp[i(q_{1z}^* z_1 - q_{2z} z_2)]\, d^2\mathbf{q}_1\, d^2\mathbf{q}_2. \tag{26}$$

Transforming to sum and difference variables $\mathbf{q}$, q_z and $\Delta\mathbf{q}$, Δq_z one obtains,

$$\begin{aligned}\Gamma(\mathbf{r}, \Delta\mathbf{r}) &= \Gamma(\boldsymbol{\rho}, z; \Delta\boldsymbol{\rho}, \Delta z) \\ &= \int A(\mathbf{q}, \Delta\mathbf{q}) \exp[i(\mathbf{q}\cdot\Delta\boldsymbol{\rho} + \Delta\mathbf{q}\cdot\boldsymbol{\rho}) \\ &\quad \exp[i(q_z\,\Delta z + \Delta q_z\, z)\, d^2\mathbf{q}\, d^2\Delta\mathbf{q},\end{aligned} \tag{27}$$

and

$$\Gamma_0(\boldsymbol{\rho}, \Delta\boldsymbol{\rho}) = \int A(\mathbf{q}, \Delta\mathbf{q}) \exp[i(\mathbf{q}\cdot\Delta\boldsymbol{\rho} + \Delta\mathbf{q}\cdot\boldsymbol{\rho})]\, d^2\mathbf{q}\, d^2\Delta\mathbf{q}. \tag{28}$$

Substitution of (27) in (5) yields an expression for the generalised pencils:

$$\begin{aligned}W(\mathbf{r}, \mathbf{p}) &= W(\boldsymbol{\rho}, z; \mathbf{p}_\perp, p_z) \\ &= \int A(\mathbf{q}, \Delta\mathbf{q}) \exp[i(\Delta\mathbf{q}\cdot\boldsymbol{\rho} + \Delta q_z z)]. \\ &\quad \delta(\mathbf{q} - \mathbf{p}_\perp)\,\delta(q_z - p_z)\, d^2\mathbf{q}\, d^2\Delta\mathbf{q}.\end{aligned} \tag{29}$$

The last equation is an exact statement of (5) in the angular spectrum representation, and yields the exact pencils.

If the angular spectrum is narrow, *i.e.* if $\bar{A}(\mathbf{q}_1, \mathbf{q}_2)$ is appreciable only when

$$|\mathbf{q}_1|^2 \ll k_0^2; \quad |q_2|^2 \ll k_0^2, \tag{30}$$

then $|\mathbf{q}|^2 \ll k_0^2$ and $|\Delta\mathbf{q}|^2 \ll k_0^2$, and we can simplify (29) by making the paraxial approximation to q_z and Δq_z:

$$q_z = \tfrac{1}{2}\left\{\left[k_0^2 - \left(\mathbf{q} + \frac{\Delta\mathbf{q}}{2}\right)^2\right]^{1/2} + \left[k_0^2 - \left(\mathbf{q} - \frac{\Delta\mathbf{q}}{2}\right)^2\right]^{1/2}\right\} \approx k_0 \tag{31a}$$

$$\Delta q_z = \left[k_0^2 - \left(\mathbf{q} + \frac{\Delta\mathbf{q}}{2}\right)^2\right]^{1/2} - \left[k_0^2 - \left(\mathbf{q} - \frac{\Delta\mathbf{q}}{2}\right)^2\right]^{1/2} \approx -\frac{\mathbf{q}\cdot\Delta\mathbf{q}}{k_0}. \tag{31b}$$

Under this approximation (29) becomes

$$W(\boldsymbol{\rho}, z; \mathbf{p}_{\perp}, p_z) = \delta (k_0 - p_z) \int A (\mathbf{p}_{\perp}, \Delta \mathbf{q})$$

$$\exp [i \Delta \mathbf{q} \cdot \left(\boldsymbol{\rho} - \frac{\mathbf{p}_{\perp} z}{k_0}\right) d^2 \Delta \mathbf{q}$$

$$= \delta (k_0 - p_z) W_0\left(\boldsymbol{\rho} - \frac{\mathbf{p}_{\perp} z}{k_0}, \mathbf{p}_{\perp}\right), \tag{32}$$

where we have defined

$$W_0 (\boldsymbol{\rho}, \mathbf{p}_{\perp}) = \int A (\mathbf{p}_{\perp}, \Delta \mathbf{q}) \exp (i \boldsymbol{\rho} \cdot \Delta \mathbf{q}) d^2\Delta \mathbf{q}. \tag{33}$$

When (28) is used in the last equation one obtains

$$W_0 (\boldsymbol{\rho}, \mathbf{p}_{\perp}) = \frac{1}{(2\pi)^2} \int \Gamma_0 (\boldsymbol{\rho}, \Delta \boldsymbol{\rho}) \exp (- i \mathbf{p}_{\perp} \cdot \Delta \boldsymbol{\rho} \, d^2\Delta \boldsymbol{\rho}). \tag{34}$$

The pair (32) and (34) forms a very useful result of our analysis as will be shown in the subsequent sections. We can give an elegant geometrical interpretation of this result: The strength of the pencils through point $\boldsymbol{\rho}$ in the $z = 0$ plane going in the direction $(\mathbf{p}_{\perp}, k_0)$ is given by $W_0(\boldsymbol{\rho}, \mathbf{p}_{\perp})$, the Fourier transform of $\Gamma_0 (\boldsymbol{\rho}, \Delta \boldsymbol{\rho})$ with respect to $\Delta \boldsymbol{\rho}$. These pencils then travel in straight lines as shown by (32). Thus given the boundary condition, namely $\Gamma_0 (\boldsymbol{\rho}, \Delta \boldsymbol{\rho})$ in the $z = 0$ plane, one computes $W_0 (\boldsymbol{\rho}, \mathbf{p}_{\perp})$ using (34), and then the generalized pencil through an arbitrary point in the region $z > 0$ is constructed 'by drawing straight lines' as shown by (32), without the necessity to compute $\overline{\Gamma} (\mathbf{r}_1, \mathbf{r}_2)$ in the entire space as an intermediate quantity. We note that the only approximation made in deriving this result is the paraxial approximation stated in (31). It is interesting to observe that the straight line propagation character of the exact pencils is preserved under this approximation. This can be seen either from the

$$\left(\boldsymbol{\rho} - \frac{\mathbf{p}_{\perp} z}{k_0}\right)$$

factor in (32), or through substitution of (32) into (6).

Once the pencils are constructed using (34) and (32), the intensity, the cross-spectral density and other field quantities of interest could be readily computed from the pencils, thus making the generalized pencils a practical tool in wave-optic computations. Naturally, the field quantities so computed will reflect the paraxial approximation.

Equation (34) is essentially the definition of a Wigner distribution. Wigner distribution in the context of optics has been extensively studied by Bastiaans (1978, 1979). We also note that (34) has a close resemblance to Walther's first definition of radiance (Walther 1968).

The paraxial approximation is stated in (31). Clearly,

$$|\mathbf{q}|^2 \ll k_0^2 \text{ and } |\Delta \mathbf{q}|^2 \ll k_0^2 \tag{35}$$

form sufficient conditions for its validity. If the angular spectrum is narrow, then (35) is satisfied and the paraxial approximation will yield accurate results. Then it can be seen from the $\delta(\mathbf{q}-\mathbf{p}_\perp)$ factor in (29) that $W(\mathbf{r}, \mathbf{p})$ will be appreciable only when $|\mathbf{p}_\perp|^2 \ll k_0^2$, which means only paraxial rays will be present. Even if the angular spectrum is not narrow, we can still satisfy (35) provided we restrict to paraxial rays and if the angular correlation is small, *i.e.* if $A(\mathbf{q}, \Delta\mathbf{q})$ is appreciable only when $|\Delta\mathbf{q}|^2 \ll k_0^2$.

For later reference it is useful to define

$$W_z(\boldsymbol{\rho}, \mathbf{p}_\perp) = W_0\left(\boldsymbol{\rho} - \frac{\mathbf{p}_\perp z}{k_0}, \mathbf{p}_\perp\right). \tag{36}$$

From (32) and (9) one obtains

$$\begin{aligned} W_z(\boldsymbol{\rho}, \mathbf{p}_\perp) &= \int W(\boldsymbol{\rho}, z; \mathbf{p}_\perp, p_z)\, \mathrm{d}p_z \\ &= \frac{1}{(2\pi)^2} \int \boldsymbol{\Gamma}_z(\boldsymbol{\rho}, \Delta\boldsymbol{\rho}) \exp(-i\,\mathbf{p}_\perp \cdot \Delta\boldsymbol{\rho})\, \mathrm{d}^2\Delta\boldsymbol{\rho}, \end{aligned} \tag{37}$$

which correctly reduces to (34) when $z=0$. The last equation can be inverted to read

$$\boldsymbol{\Gamma}_z(\boldsymbol{\rho}, \Delta\boldsymbol{\rho}) = \int W_z(\boldsymbol{\rho}, \mathbf{p}_\perp) \exp(i\,\mathbf{p}_\perp \cdot \Delta\boldsymbol{\rho})\, \mathrm{d}^2\mathbf{p}_\perp. \tag{38}$$

The angular distribution of the pencils at a point will be termed the *ray pattern* at that point. Some simple features of this pattern can be seen as follows. Let us assume a planar incoherent source of linear dimension d. From the straight line propagation character of the pencils it is clear that the angular width of the ray pattern at a point $(\boldsymbol{\rho}, z)$ will be of the order of d/z, which means the width of $W_z(\boldsymbol{\rho}, \mathbf{p}_\perp)$ as a function of $\mathbf{p}_\perp$ will be of the order of $k_0 d/z$. Then from the Fourier transform relationship in (38), it follows that the width of $\boldsymbol{\Gamma}_z(\boldsymbol{\rho}, \Delta\boldsymbol{\rho})$ as a function of $\Delta\boldsymbol{\rho}$ (the transverse coherance length) will be of the order of $z/k_0 d$. This means that as the field propagates farther and farther away from the source the transverse coherence length increases or in other words coherance builds up through the process of propagation! This result, which is in essence the content of the well-known van Cittert-Zernike theorem, is shown here to be a direct consequence of the straight line propagation character of the generalized pencils. The generalized pencils have more light to throw on the van Cittert-Zernike theorem as will be shown in the subsequent sections.

4. Examples

In this section we illustrate the paraxial results of the last section by applying them to some typical configurations of interest in wave optics. Four configurations are considered, the first three corresponding to configurations studied earlier by Sudarshan (1979a). Our treatment is different from Sudarshan's in that we compute the generalized pencils in the entire domain in the paraxial approximation, in contrast to computing them in the Fraunhofer region. The fourth configuration corresponds to a model source which has recently attracted much interest in the theoretical study of radiometry with partially coherent sources. The planar source can be either primary or secondary, and is assumed to be located in the plane $z = 0$.

4.1 *Incoherent planar source*

We assume the cross-spectral density function to be of the form (Sudarshan 1979 a)

$$\Gamma_0 (\boldsymbol{\rho}, \Delta \boldsymbol{\rho}) = (2\pi)^2 I_0 (\boldsymbol{\rho})\, \delta (\Delta \boldsymbol{\rho}). \tag{39}$$

This gives vanishing transverse coherence length in the source plane and does not correspond to physical incoherent source. (The physical incoherent source should have a transverse coherence length of the order of k_0^{-1} (Beran and Parrent 1964)). We also note that the model source in (39) does not satisfy the conditions for the validity of the paraxial approximation namely (35). However, it serves to give a qualitative picture of the nature of the pencils generated by an incoherent planar source. We get results in agreement with wave optics if we restrict observation to paraxial points, so that only pencils with $|\mathbf{p}_\perp|^2 \ll k_2^0$ contribute. Using (39) in (34) one obtains

$$W_0 (\boldsymbol{\rho}, \mathbf{p}_\perp) = I_0 (\boldsymbol{\rho}), \tag{40a}$$

and from (32)

$$W (\boldsymbol{\rho}, z = 0, \mathbf{p}) = \delta (\mathbf{p}_z - k_0)\, I_0 (\boldsymbol{\rho}). \tag{40b}$$

This means that, for directions pointing to the region $z > 0$, the strength of the pencil at every point in the source plane is independent of direction; we have an isotropic ray pattern. The strength of the ray pattern at $\boldsymbol{\rho}$ equals $I_0 (\boldsymbol{\rho})$. Since $I_0 (\boldsymbol{\rho})$ is non-negative, we note that all the pencils in this case consist of pradipa rays only. From (32) the pencil at any point in the region $z \geqslant 0$ in direction $\mathbf{p}$ is

$$W (\mathbf{r}, \mathbf{p}) = \delta (p_z - k_0)\, I_0 \left(\boldsymbol{\rho} - \frac{\mathbf{p}_\perp z}{k_0}\right). \tag{41}$$

It is instructive to compute the cross-spectral density in the paraxial region in a z-plane sufficiently far away from the source plane, so that only the paraxial pencils contribute. From (36), (38) and (40) one obtains

$$\begin{aligned}\Gamma_z (\boldsymbol{\rho}, \Delta\boldsymbol{\rho}) &= \int I_0 \left(\boldsymbol{\rho} - \frac{\mathbf{p}_\perp z}{k_0}\right) \exp (i\, \mathbf{p}_\perp \cdot \Delta \boldsymbol{\rho})\, \mathrm{d}^2 \mathbf{p}_\perp \\ &= \frac{k_0^2}{z^2} \exp \left(i \frac{k_0}{z} \boldsymbol{\rho} \cdot \Delta \boldsymbol{\rho}\right) \int I_0 (\boldsymbol{\rho}') \exp \left(- i \frac{k_0}{z} \boldsymbol{\rho}' \cdot \Delta \boldsymbol{\rho}\right) \mathrm{d}^2 \boldsymbol{\rho}' \end{aligned} \tag{42}$$

Transforming $\boldsymbol{\rho}$ and $\Delta\boldsymbol{\rho}$ back to $\boldsymbol{\rho}_1$ and $\boldsymbol{\rho}_2$ this becomes

$$\bar{\Gamma}_z(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2) = \frac{k_0^2}{z^2} \exp\left(i\frac{k_0}{2z}\right)(\boldsymbol{\rho}_1^2 - \boldsymbol{\rho}_2^2)$$

$$\int I_0(\boldsymbol{\rho}') \exp\left(-i\frac{k_0}{z}(\boldsymbol{\rho}_1 - \boldsymbol{\rho}_2)\cdot\boldsymbol{\rho}'\right) \mathrm{d}^2\boldsymbol{\rho}'. \tag{43}$$

This is the celebrated van Cittert-Zernike theorem for planar incoherent sources. It relates the far-zone cross-spectral density to the Fourier transform of the source plane intensity. The usual derivation of this theorem employs wave-optic methods, while we have here obtained this result as a direct consequence of the straight line propagation character of the generalized pencils and of the isotropic nature of the ray pattern in the source plane. It is amusing to note that Sudarshan (1979b) in his pioneering work conjectured that the van Cittert-Zernike theorem was a consequence of the straight line propagation character of the generalized pencils. In § 5, we will generalize this theorem to planar sources of arbitrary state of coherence.

4.2 *Coherently illuminated double slit*

We assume that the two slits are located parallel to the y-axis at $x = +a$ and $x = -a$ respectively. Here again the conditions for the validity of the paraxial approximation are not met, hence observation must be restricted to paraxial regions. In the source plane

$$\Gamma_0(\boldsymbol{\rho}, \Delta\boldsymbol{\rho}) = \Gamma_0(x, y; \Delta x, \Delta y)$$

$$= [\delta(x-a) + \delta(x+a)]\,\delta(\Delta x)$$

$$+ \delta(x)\,[\delta(\Delta x - 2a) + \delta(\Delta x + 2a)]. \tag{44}$$

From (34) and the last equation

$$W_0(x, y;\ p_x, p_y) = \frac{1}{2\pi}[\delta(x-a)\,\delta(p_y) + \delta(x+a)\,\delta(p_y)$$

$$+ \delta(x)\,2\cos 2ap_x\,\delta(p_y)]. \tag{45}$$

The first two terms on the right side correspond to isotropic cylindrical pencils from either slit. These two pencils consist of pradipa rays only. There is a third pencil originating from the line midway between the two slits. The strength of this pencil is sinusoidal as a function of direction in the x-z plane. This pencil is due to the correlation between the illumination at the two slits, and is responsible for the formation of the familiar two slit interference pattern. It is easy to see that the strength of this third pencil decreases as the correlation between the two slits is reduced and that it disappears when the two slits become uncorrelated. Substitution of (45) in (32) gives the pencil through an arbitrary point in the region $z \geqslant 0$:

$$W(\mathbf{r}, \mathbf{p}) = \delta(p_z - k_0)\,\delta(p_y)\left[\delta\left(x - \frac{p_x z}{k_0} - a\right)\right.$$

$$\left. + \delta\left(x - \frac{p_x z}{k_0} + a\right) + \delta\left(x - \frac{p_x z}{k_0}\right) 2\cos 2ap_x\right]. \tag{46}$$

There is some similarity between this expression and the pencil corresponding to superposition of a pair of plane waves, namely (14). However, there are some essential differences: In the present case, the direction of the pencil at a point is a function of that point and the ray pattern has an inverse dependence on z through the factor z/k_0.

The intensity distribution corresponding to the pencil in (46) can be computed using (11):

$$I(\mathbf{r}) = \frac{k_0}{z}\left[2 + 2\cos\left(\frac{2k_0 ax}{z}\right)\right]$$

$$= 4\frac{k_0}{z}\cos^2\left(\frac{k_0 ax}{z}\right). \tag{47}$$

This is the two-slit interference pattern well-known in elementary wave optics.

4.3 *Coherently illuminated single slit*

We assume that the slit is of width $2a$ and occupies the region $-a \leqslant x \leqslant a$ parallel to the y-axis, and that the slit is uniformly illuminated with coherent light. Thus

$$\Gamma_0(x, y; \Delta x, \Delta y) = 2\pi.\ \mathrm{rect}\left[\frac{\Delta x}{2(a - |x|)}\right]\mathrm{rect}\,(x/a), \tag{48}$$

where

$$\mathrm{rect}\,(x/a) = 1 \qquad \text{if } |x| \leqslant a$$

$$= 0 \qquad \text{otherwise.} \tag{49}$$

If the slit width a is large compared to k_0^{-1}, then the angular spectrum is narrow and the condition for the paraxial approximation is satisfied. Substitution of (48) in (34) results in

$$W_0(x, y; p_x, p_y) = \frac{2\sin[2(a - |x|)p_x]}{p_x}\,\mathrm{rect}\left(\frac{x}{a}\right)\delta(p_y). \tag{50}$$

The pencil pattern at each point x in the slit has a sine function dependence on p_x with a characteristic width $1/(a - |x|)$ in p_x, hence the ray pattern at x has a characteristic angular width $1/k_0[a - |x|]$. When x is well inside the slit several wave lengths away from the edge, the ray pattern becomes narrow and needle-like about the z-direction. As x moves towards the edge, the ray pattern broadens. The pencil through every point consists of both pradipa rays and tamasic rays depending on p_x.

The pencil through an arbitrary point in the region $z>0$ can be constructed using (50) in (32). It will be shown in the next section (cf. (73)) that the pencils so constructed produce an intensity distribution in the far-zone which agrees with the well-known Fraunhofer diffraction pattern of a slit.

4.4 *Quasihomogeneous source*

Quasihomogeneous sources are model sources of great interest in radiometry with partially coherent sources (Carter and Wolf 1977; Collett and Wolf 1980). For these sources, the cross-spectral density in the source plane is separable in the form

$$\Gamma_0(\boldsymbol{\rho}, \Delta \boldsymbol{\rho}) = I_0(\boldsymbol{\rho})\, g_0(\Delta \boldsymbol{\rho}), \tag{51}$$

where $I_0(\boldsymbol{\rho})$ is a slowly varying function compared to $g_0(\Delta \boldsymbol{\rho})$. For instance, if both I_0 and g_0 are Gaussian, we have a Gaussian-correlated Gaussian quasihomogeneous source. Clearly, the condition for the validity of the paraxial approximation is satisfied if the effective width of g_0 is large compared to k_0^{-1}. From (34) we obtain for the present case

$$W_0(\boldsymbol{\rho}, \mathbf{p}_\perp) = I_0(\boldsymbol{\rho})\, \tilde{g}_0(\mathbf{p}_\perp), \tag{52}$$

where $\tilde{g}_0$ is the Fourier transform of g_0. It is seen that the ray pattern is the same at every point in the source plane and is given by $\tilde{g}_0$. The strength of the ray pattern at $\boldsymbol{\rho}$ is given by $I_0(\boldsymbol{\rho})$ and is a slowly varying function of $\boldsymbol{\rho}$. If g_0 is such that its Fourier transform is a non-negative function, then the pencils will consist of pradipa rays only. This is the case for example when we have Gaussian correlation.

The pencils through an arbitrary point can be constructed using (52) in (32):

$$W(r, p) = \delta(p_z - k_0)\, I_0\left(\boldsymbol{\rho} - \frac{\mathbf{p}_\perp z}{k_0}\right) \tilde{g}_0(\mathbf{p}_\perp). \tag{53}$$

The right side of (53) ceases to be separable in the form of (52) and hence the field in an arbitrary z-plane, generated by a quasihomogeneous source in the plane $z=0$, ceases to be quasihomogeneous, except in the case when $I_0(\boldsymbol{\rho})=\text{constant}$.

5. Far zone and scaling

Equation (34) expresses $W_0(\boldsymbol{\rho}, \mathbf{p}_\perp)$ as Fourier transform of $\Gamma_0(\boldsymbol{\rho}, \Delta\boldsymbol{\rho})$ with respect to $\Delta\boldsymbol{\rho}$. Once $W_0(\boldsymbol{\rho}, \mathbf{p}_\perp)$ is known, substitution into (32) constructs the pencil through an arbitrary point. We examine the nature of the pencil in the far-zone and show that, as a consequence of the straight line propagation character of the pencils, the ray pattern in the far-zone exhibits an interesting radial scaling property. This scaling property leads to a natural generalization of the Fraunhofer range criterian to sources of arbitrary state of coherence. It also gives a natural generalization of the van Cittert-Zernike theorem. The well-known results of radiometry with partially coherent sources are shown to follow from this scaling behaviour.

The effective width of $\Gamma_0(\boldsymbol{\rho}, \Delta\boldsymbol{\rho})$ as a function of $\Delta\boldsymbol{\rho}$ cannot be greater than the source dimension or the transverse coherence length in the source plane, whichever is smaller. The smaller of these two lengths will be denoted by a characteristic length. γ. Then from the Fourier transform relationship in (34) it follows that the effective width of $W_0(\boldsymbol{\rho}, \mathbf{p}_\perp)$ as a function of $\mathbf{p}_\perp$ cannot be less than $1/\gamma$. Equivalently, the effective angular width of the ray pattern at $\boldsymbol{\rho}$ in the source plane cannot be less than $1/k_0\gamma$.

From the straight line propagation character of the pencil as shown by (32), it is clear that the angular width of the ray pattern at the point $(\boldsymbol{\rho}, z)$ cannot be greater than d/z, where d is the source dimension. The geometry is shown in figure 2. If at no point in the source plane $W_0(\boldsymbol{\rho}, \mathbf{p}_\perp)$ as a function of $\mathbf{p}_\perp$ varies appreciably over this range, then we can replace $\mathbf{p}_\perp$ in (32) by a typical value, namely $\mathbf{p}_\perp^0 = \boldsymbol{\rho}\, k_0/z$ corresponding to the radial direction of $(\boldsymbol{\rho}, z)$ as shown in figure 2. Equation (32) now becomes

$$W(\boldsymbol{\rho}, z; \mathbf{p}_\perp, p_z) \approx \delta\,(p_z - k_0)\, W_0\left(\boldsymbol{\rho} - \frac{\mathbf{p}_\perp z}{k_0}, \mathbf{p}_\perp^0\right). \tag{54}$$

For this equation to be valid, the sufficient condition on z is easily seen to be

$$d/z \ll 1/k_0\gamma,$$

or

$$d\gamma \ll z/k_0 \tag{55}$$

The right side of (54) has an interesting radial scaling property. To see this, we rewrite it as

$$\begin{aligned} W(\boldsymbol{\rho}, z; \mathbf{p}_\perp, p_z) &= \delta(p_z - k_0). \\ &\quad W_0\left[\frac{z}{k_0}\left(\frac{k_0\boldsymbol{\rho}}{z} - \mathbf{p}_\perp\right), \mathbf{p}_\perp^0\right] \\ &= \delta\,(p_z - k_0)\, W_0\left[\frac{z}{k_0}\left(\frac{k_0\boldsymbol{\rho}}{z} - \mathbf{p}_\perp\right), \frac{\boldsymbol{\rho}\, k_0}{z}\right]. \end{aligned} \tag{56}$$

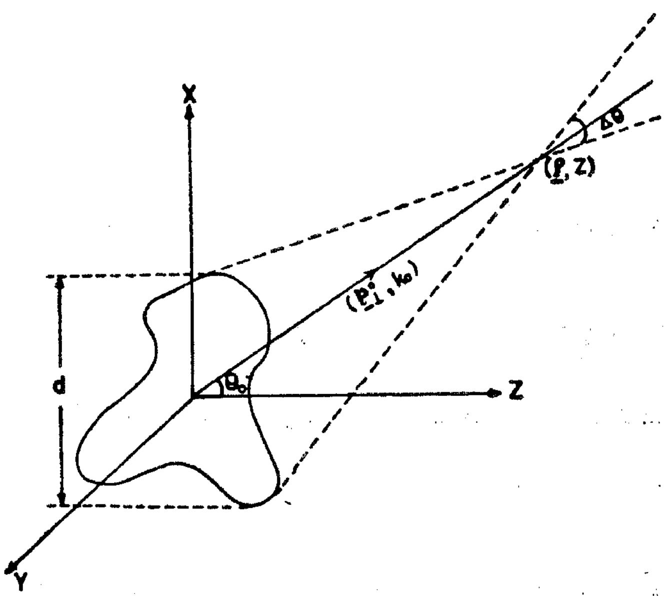

Figure 2. Geometry for the scaling behaviour.

This means that in the far-zone as the point of observation is moved along a radial line $\boldsymbol{\rho}/z =$ constant the ray pattern scales retaining its functional form, the scale factor being z/k_0. It should be noted that this scaling behaviour is again a direct consequence of the rectilinear propagation of the pencils. This property of the ray pattern in the far-zone has several interesting consequences.

We can use this scaling to identify the far-zone. Then (55) gives the far-zone range criterion. For fully coherent sources $\gamma = d$, and (55) becomes

$$d^2 \ll z/k_0 \tag{57}$$

which is the usual Fraunhofer range criterion for coherent sources. Thus (55) becomes the natural generalization of this criterion to sources of arbitrary state of coherence. Equation (55) implies that as the source becomes more and more incoherant, the far-zone comes closer and closer to the source. For incoherent sources (thermal source) $\gamma \approx k_0^{-1}$, and the far-zone range criterion becomes

$$d \ll z, \tag{58}$$

Using wave optic methods Leader (1978) has derived the far-zone range criterion for quasi homogeneous sources, and our result, namely (55), agrees with his result for such sources. However, we have obtained this result as a general condition for the scaling behaviour, and we have not placed any quasi-homogeneity restriction on the source.

As another consequence of the scaling of the ray pattern in the far-zone, we derive a generalization of the van Cittert-Zernike theorem to planar sources of arbitrary state of coherence. We start with substitution of (36) in (38):

$$\Gamma_z(\boldsymbol{\rho}, \Delta\boldsymbol{\rho}) = \int W_0\left(\boldsymbol{\rho} - \frac{\mathbf{p}_\perp z}{k_0}, \mathbf{p}_\perp^0\right) \exp\left(i\,\mathbf{p}_\perp \cdot \Delta\boldsymbol{\rho}\right) \mathrm{d}^2\mathbf{p}_\perp, \tag{59}$$

where we have assumed that we are in the scaling region. Introducing the change of integration variable

$$\boldsymbol{\rho}' = \boldsymbol{\rho} - \frac{\mathbf{p}_\perp z}{k_0}, \tag{60}$$

(59) becomes

$$\Gamma_z(\boldsymbol{\rho}, \Delta\boldsymbol{\rho}) = \frac{k_0^2}{z^2} \exp\left(i\frac{k_0}{z}\boldsymbol{\rho}\cdot\Delta\boldsymbol{\rho}\right) \int W_0(\boldsymbol{\rho}', \mathbf{p}_\perp^0) \cdot \exp\left(-i\frac{k_0}{z}\boldsymbol{\rho}'\cdot\Delta\boldsymbol{\rho}\right) \mathrm{d}^2\boldsymbol{\rho}'. \tag{61}$$

After a transformation from $\boldsymbol{\rho}$ and $\Delta\boldsymbol{\rho}$ to $\boldsymbol{\rho}_1$ and $\boldsymbol{\rho}_2$, the last equation becomes

$$\bar{\Gamma}_z(\boldsymbol{\rho}_1, \boldsymbol{\rho}_2) = \frac{k_0^2}{z^2} \exp\left[i\frac{k_0}{2z}(\boldsymbol{\rho}_1^2 - \boldsymbol{\rho}_2^2)\right] \int W_0\left(\boldsymbol{\rho}', \frac{\mathbf{p}_{\perp 1}^0 + \mathbf{p}_{\perp 2}^0}{2}\right) \exp\left[-i(\mathbf{p}_{\perp 1}^0 - \mathbf{p}_{\perp 2}^0)\cdot\boldsymbol{\rho}'\right] \mathrm{d}^2\boldsymbol{\rho}', \tag{62}$$

where we have defined

$$\mathbf{p}^0_{\perp i} = \boldsymbol{\rho}_i \frac{k_0}{z}, \qquad i = 1, 2. \tag{63}$$

Equation (62) is a generalization of the van Cittert—Zernike theorem and applies to every planar source. It agrees with the result of Wolf and Carter (1976) derived by a different procedure. Here we have obtained this generalized van Cittert-Zerniek theorem as a direct consequence of the far-zone scaling of the ray pattern which in turn is a consequence of the rectilinear propagation of the pencils.

Classical radiometry deals with far-zone properties of fields generated by incoherent sources. Recently there has been enormous interest in the far-zone fields produced by sources of arbitrary state of coherence (Wolf 1978, Baltes *et al* 1978), and many useful results have been established. Since the far-zone properties of the field are contained in the scaling behaviour, these now well-known results can be more easily deduced from the scaling property, thus giving a geometric picture of these results. We will consider only a few of these results for illustration.

We start by computing the intensity distribution in the scaling region. Substituting $\Delta\boldsymbol{\rho} = 0$ in (61) one obtains

$$I(\boldsymbol{\rho}, z) = \frac{k_0^2}{z^2} W_0(\boldsymbol{\rho}', \mathbf{p}^0_\perp)\, \mathrm{d}^2 \boldsymbol{\rho}', \tag{64}$$

where $$\mathbf{p}^0_\perp = \boldsymbol{\rho}\, \frac{k_0}{z}. \tag{65}$$

Equation (64) is a useful result; it explains the inverse square-law radial scaling of the intensity distribution in the far-zone. The far-zone angular distribution of the Intensity is called the radiant intensity and is defined as

$$J(\boldsymbol{\rho}/z) = \underset{z \to \infty}{\mathrm{Lt}}\; z^2 I(\boldsymbol{\rho}, z). \tag{66}$$

From (64) we obtain for the radiant intensity

$$J(\boldsymbol{\rho}/z) = k_0^2 \int W_0(\boldsymbol{\rho}', (\mathbf{p}^0_\perp)\, \mathrm{d}^2 \boldsymbol{\rho}'. \tag{67}$$

The right side is positive definite for it is the diagonal element of $\overline{A}(q_1, q_2)$. It is interesting to note that each point in source emits either pradipa or tamasic ray in direction $\boldsymbol{\rho}/z$, but they will all definitely add up to a positive radiant intensity!

From (34) we have

$$\int W_0(\boldsymbol{\rho}', \mathbf{p}^0_\perp)\, d^2 \boldsymbol{\rho}' = \frac{1}{4\pi^2} \int \left[\int \overline{\Gamma}\left(\boldsymbol{\rho}' + \frac{\boldsymbol{\sigma}}{2}, \boldsymbol{\rho}' - \frac{\boldsymbol{\sigma}}{2}\right) \mathrm{d}^2 \boldsymbol{\rho}' \right]$$

$$\exp(-i\, \mathbf{p}^0_\perp \cdot \boldsymbol{\sigma})\, \mathrm{d}^2 \boldsymbol{\sigma}. \tag{68}$$

If we define

$$F(\sigma) = \int \bar{\Gamma}\left(\boldsymbol{\rho}' + \frac{\sigma}{2},\ \boldsymbol{\rho}' - \frac{\sigma}{2}\right) \mathrm{d}^2 \boldsymbol{\rho}', \tag{69}$$

it follows from (67) and (68) that all sources having the same $F(\sigma)$ have the same far-zone intensity distribution. This is the well-known equivalence theorem of Collett and Wolf (1979).

If $\bar{\Gamma}_0$ is of the special form

$$\bar{\Gamma}_0 (\boldsymbol{\rho}_1, \boldsymbol{\rho}_2) = [I_0 (\boldsymbol{\rho}_1)]^{1/2} [I_0 (\boldsymbol{\rho}_2)]^{1/2} g_0 (\boldsymbol{\rho}_1 - \boldsymbol{\rho}_2), \tag{70}$$

then we have a Schell-model source (Schell 1961), which has recently attracted much interest. Defining the source amplitude auto-correlation function as

$$C_I (\sigma) = \int \left[I_0\left(\boldsymbol{\rho} + \frac{\sigma}{2}\right)\right]^{1/2} \left[I_0\left(\boldsymbol{\rho} - \frac{\sigma}{2}\right)\right]^{1/2} \mathrm{d}^2 \boldsymbol{\rho}, \tag{71}$$

and using (68) and (67) one obtains for Schell-model sources

$$I (\boldsymbol{\rho}, z) = \frac{k_0^2}{4\pi^2 z^2} \int C_I (\sigma) g_0 (\sigma) \exp\left(- i \frac{k_0}{z} \boldsymbol{\rho} \cdot \sigma\right) \mathrm{d}^2 \sigma. \tag{72}$$

This is Schell's Fourier transform theorem.

The results contained in (64) through (72) are well-known results of radiometry with partially coherant sources. Here they are shown to be consequences of the scaling property.

If the source dimension is finite in the x-direction and infinite in the y-direction, the far-zone range criterion can be satisfied for the x-dimension only, and consequently, the scaling property of the pencils can be realised for the p_x dependence only. In such cases $\mathbf{p}_\perp^0$ in (54) and the subsequent equations should be replaced by (p_x^0, p_y). Consequently, the change of integration variable from $\mathbf{p}_\perp$ to $\boldsymbol{\rho}'$ can be effected for the x-component only. For instance (64) will become

$$I (\boldsymbol{\rho}, z) = \frac{k_0}{z} \int W_0\left(x', y - \frac{p_y z}{k_0}; p_x^0, p_y\right) \mathrm{d}x' \, \mathrm{d}p_y, \tag{64'}$$

where

$$p_x^0 = x\, k_0/z. \tag{65'}$$

As an interesting application of (64′), we compute the far-zone intensity due to the pencils of the single slit case. Substituting (50) in (64′) one obtains

$$I (\boldsymbol{\rho}, z) = \frac{k_0}{z} \int 2 \sin\left[\frac{2\,(a - |x_i|)\, p_x^0}{p_x^0}\right] \cdot \mathrm{rect}\left(\frac{x_i}{a}\right) \delta (p_y) \, \mathrm{d}x_i \, \mathrm{d}p_y$$

$$= \frac{k_0}{z} 4 \left(\frac{\sin a p_x^0}{p_x^0}\right)^2. \tag{73}$$

This is the well-known Fraunhofer diffraction pattern due to a coherently-illuminated slit.

6. Conclusions

The present formalism gives an elegant geometrical picture of wave optics through the generalized pencils. The paraxial approximation used greatly simplifies the computation of the pencils, while retaining the rectilinear propagation character of the exact pencils. It should be emphasised that the results derived in the present paper depend on the rectilinear propagation character of the pencils in an essential way.

The pencils under the paraxial approximation do not respect the dispersion restriction, namely (19). This, however, does not pose any serious problem (Sudarshan 1980).

The description in terms of the generalized pencils has the advantage of possessing the simplicities of geometrical optics, while staying exact in principle. It treats coherent, incoherent and partially coherent wave fields on the same footing. In this paper we have studied many familiar situations in this new language so as to gain experience with the formalism and illuminate earlier results from a new point of view. It is further hoped that the present approach has a pedagogic advantage.

We have also used the present formalism to study the focussing of partially coherent fields and polarization in scattering. We intend to report these results in a subsequent paper.

Acknowledgements

It is a pleasure to thank Prof. N Mukunda for many useful discussions during the entire course of the work reported in this paper. The author thanks Prof. E C G Sudarshan for benefit of discussions on the generalized pencils of rays. A helpful communication from Prof. E Wolf is gratefully acknowledged.

References

Baltes H P, Geist J and Walther A 1978 in *Inverse source problems in optics* (ed.) H P Baltes (Berlin: Springer-Verlag) Ch. 5
Bastiaans M J 1978 *Opt. Commun.* **25** 26
Bastiaans M J 1979 *J. Opt. Soc. Am.* **69** 1710
Beran M J and Parrent G B 1964 *Theory of partial coherence* (New Jersey: Prentice Hall).
Born M and Wolf E 1970 *Principles of optics*, 4th ed. (London: Pergamon Press) Ch. 10
Carter W H and Wolf E 1977 *J. Opt. Soc. Am.* **67** 785
Chandrasekhar S 1950 *Radiative transfer* (New York: Dover)
Collett E, Foley J T and Wolf E 1977 *J. Opt. Soc. Am.* **67** 465
Collett E and Wolf E 1979 *J. Opt. Soc. Am.* **69** 942
Collett E and Wolf E 1980 *Opt. Commun.* **32** 27
Fante R L 1981 *J. Opt. Soc. Am.* **71** 460
Klauder J R and Sudarshan E C G 1968 *Fundamentals of quantum optics* (New York: W. A. Benjamin), Ch. 1

Leader J C 1978 *J. Opt. Soc. Am.* **68** 1332
Schell A C 1961 *The Multiple plate antennas*, Ph.D. thesis, Massachusetts Institute of Technology.
Sudarshan E C G 1969 *J. Math. Phys. Sci.* **3** 121
Sudarshan E C G 1979a *Phys. Lett.* **73A** 269
Sudarshan E C G 1979b *Physica* **96A** 315
Sudarshan E C G 1980 ICO-Conference, *Optics in four dimensions*, Enseneda, Mexico.
Sudarshan E C G 1981 *Phys. Rev.* **A23** 2802
van Cittert P H 1934 *Physica* **1** 201
Walther A 1968 *J. Opt. Soc. Am.* **58** 1256
Wolf E 1976 *Phys. Rev.* **D13** 869
Wolf E 1978 *J. Opt. Soc. Am.* **68** 6
Wolf E and Carter W H 1976 *Opt. Commun.* **16** 297
Zernike F 1938 *Physica* **5** 785
Zubairy M S 1981 *Opt. Commun.* **37** 315
Zubairy M S and Wolf E 1977 *Opt. Commun.* **20** 321