On the Pythagorean triples in the Śulvasūtras

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The performance of the fire rituals (*yajna*) practised by the Vedic people involved construction of altars (*vedi*) and fire-places (*agni*) in specific geometrical shapes. The constructions and the geometric principles involved in them have been elaborately described in the *Śulvasūtras*; the root *śulv* means 'to measure', and the name signifies exposition on mensuration. The Śulvasūtras are part of the *Kalpasūtras* from the *Vedāngas* of the *Yajurveda*. Different Vedic *śākhās* (branches) had their versions of Śulvasūtras. There are nine known Śulvasūtras of which four, those of Baudhāyana, Āpastamba, Manu and Kātyāyana, are noted to be of mathematical significance. For general reference in this respect the reader is referred to ref. 1; (see also refs 2–7). It may be mentioned that while precise dates of the Śulvasūtras are uncertain, it is generally believed that they were composed sometime during the period 800–400 BCE, Baudhāyana Śulvasūtra being the most ancient of them, and Kātyāyana Śulvasūtra the latest. There is considerable mutual commonality between the Śulvasūtras which indicates that, despite possible differences in the period and location of their authors, the works represent selections from a common body of knowledge.

The purpose of this note is to discuss the role played in the Śulvasūtras by what are called 'Pythagorean triples'. We recall that a triple of natural numbers (x, y, z) is called a *Pythagorean triple* if $x^2 + y^2 = z^2$; the numbers in the triple represent the lengths (with respect to any fixed unit of length) of the two sides and the hypotenuse respectively, in a right-angled triangle. A Pythagorean triple is said to be *primitive* if the entries have no common factor greater than 1. The following primitive Pythagorean triples occur in the Śulvasūtras[1,2]:

$$(3, 4, 5), (5, 12, 13), (8, 15, 17), (7, 24, 25), (12, 35, 37). \quad (1)$$

Various multiples of these triples also occur in the Śulvasūtras in various contexts; other triples (a, b, c) with $a^2 + b^2 = c^2$, where a, b, c are magnitudes that are not mutually commensurable, are also involved in the sūtras but we shall not be concerned with them here.

For reference to the sūtras we follow the numbering as in ref. 1; in this respect Asl., Bsl., Ksl. and Msl. will stand for Āpastamba, Baudhāyana, Kātyāyana, and Mānava Śulvasūtras respectively.

1. The Nyancana procedure

The Pythagorean triples occur in the Śulvasūtras, principally in the context of constructions of squares, rectangles, and trapezia. For constructing these figures it is crucial to be able to draw perpendiculars to given lines, at given points; (in the case of trapezia there is no right angle to be seen in the figure, but it involves drawing parallel lines, which can be realized by drawing perpendiculars to a fixed line at two points at a desired distance).

It may be recalled that the standard method from Euclidean geometry for constructing a perpendicular to a given line L at a point P on it goes as follows: choose two points Q and R on L equidistant from P (one on either side of P), and draw circles with centres at Q and R, with radius exceeding the distance between P and Q (or R); the two circles intersect each other in two distinct points, and joining these two points yields a line perpendicular to the given line, passing through P; in practice one draws only segments of the circles near the anticipated position of the points of intersection. Similar construction is also found in the Śulvasūtras. However another procedure involving Pythagorean triples was more common. The method depends on the converse of what is now called the Pythagoras theorem, namely that if the three sides of a triangle are a, b and c, and $a^2 + b^2 = c^2$, then the angle between the sides of lengths a and b is a right angle. Incidentally, familiarity with the Pythagoras theorem is seen in the Śulvasūtras in various contexts and there has been speculation about whether a proof of the theorem was known, but we shall not concern ourselves with it here.

We now describe the method, following the overall style of the Śulvasūtras, but using modern notation. The steps in the construction are shown in Figure 1. Let a, b, c be three magnitudes (not necessarily integral) such that $a^2 + b^2 = c^2$. Let A and B be two given points at a distance a. We take a cord of length $b + c$, and mark the endpoints as (say) P and Q. The point on the cord at a distance b from Q is marked as N; this mark on the cord is called *Nyancana* (pronounced as Nyan-cha-na). (In some versions of Baudhāyana Śulvasūtra[1,2] the word is written as *Nyanchana* and in Kātyāyana and Mānava Śulvasūtras[1] it is referred to as *Niranchana*. The term does not appear in Āpastamba Śulvasūtra though the method is used.) The endpoints P and Q of the cord are to be tied at the points A and B respectively. The cord is now stretched away from the segment AB, to one side of the plane, holding it at the Nyancana mark. The Nyancana mark will then reach a specific point on the plane (ground), say C; (there is precisely one such point on either side of the line segment AB, these being the two points at a distance b from B and c from A). Now ABC is a triangle with sides a, b and c, and since by choice $a^2 + b^2 = c^2$, by the converse of the Pythagoras theorem, the angle between AB and BC is a right angle.

This provides a way to draw a line perpendicular to a given line segment, at a given point. The word Nyancana means[2] 'lying with face downwards'; the mark presumably got the name because the cord was laid on the ground holding it, stretched. The Śulvasūtras do not use any name for the process as a whole. However, in view of the crucial role played by the Nyancana mark, and the significance of the term as seen from the above meaning, we shall refer to the procedure as the *Nyancana method*.

Clearly the general procedure involved can also be used for construction of triangles with a given base of length a, and sides of lengths b and c, using a cord of length $b + c$, provided $b + c > a$, even for a, b, c not satisfying the relation $a^2 + b^2 = c^2$; the latter relation is used in the above construction to ensure the resulting triangle to be a right-angled triangle. There are some instances in the Śulva-

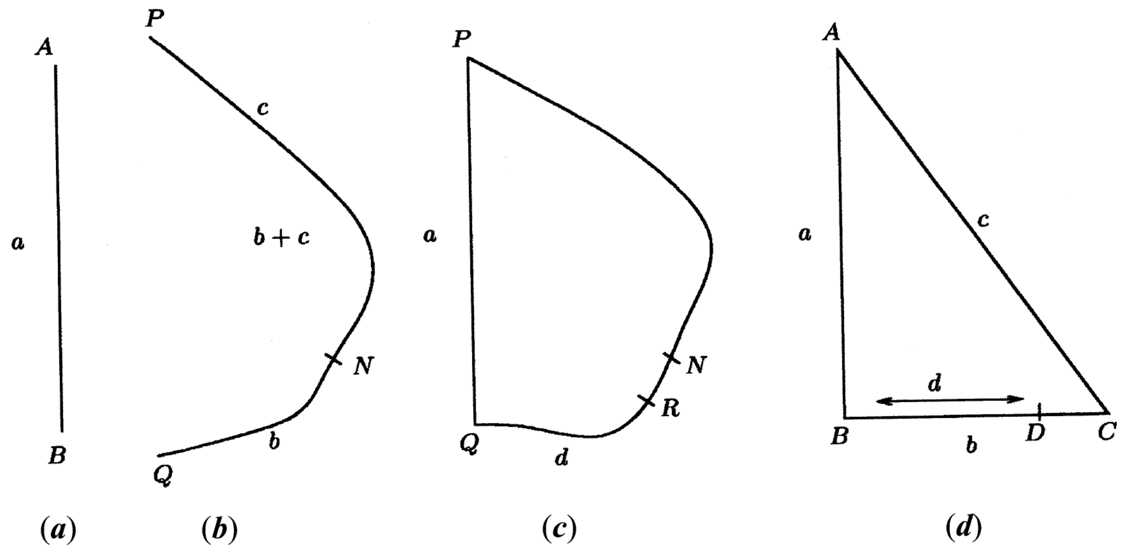

Figure 1. Nyancana construction.

sūtras where this possibility is used in its wider context (e.g. Bsl. 3.3–3.4), but the main application is in construction of right angles.

For the construction of right angles by this method it indeed suffices to have any magnitude a, b, c, such that $a^2 + b^2 = c^2$, with a, b, c not necessarily mutually commensurable. However, it is more convenient to use Pythagorean triples (namely with integral a, b, c), as the requisite marking points for these are more easily determinable in terms of any chosen unit. It is interesting to note that Āpastamba Śulvasūtra seems to make a specific mention of the relevance of using such triples. Āpastamba Śulvasūtra begins (after an opening remark in sūtra Asl. 1.1) with a description of the Nyancana procedure with respect to the Pythagorean triples (5, 12, 13) and (3, 4, 5), in Asl. 1.2 and Asl. 1.3 respectively. This is followed in Asl. 1.4 by statement of the Pythagoras theorem, and then a clause which reads[1,3] as *tābhirjneyābhiruktam viharaṉam.* While this crisp clause is open to interpretation to some extent, according to Bürk (who produced one of the standard editions of Āpastamba Śulvasūtra), it links the Nyancana construction with Pythagorean triples (see ref. 1, p. 234; see also Note 2 in the Appendix below for a related comment).

In many applications of the Nyancana procedure the following additional feature is involved. Let d be a magnitude less than b. Let the point on the cord PQ at a distance d from Q be marked as R. Now, after stretching the cord holding it at N we plot the point on the plane, say D, where the point R on the cord is located. This yields a point at a distance d from the point B, on the line perpendicular to AB at B.

2. Constructions of squares and rectangles

Many sūtras in the Śulvasūtras describe constructions of squares and rectangles. Some of them involve a method analogous to the compass construction as in Euclidean geometry, the role of the compass being played by a cord with one end fastened to a pole. In others, the Nyancana method is used. In practical contexts dealt with in the Śulvasūtras, the latter procedure has some advantages, as the construction can then be 'packaged' for convenient use as we shall see below.

To construct a square with a given line segment AB as the mid-line (the line of symmetry, joining the midpoints of one pair of opposite sides) the following construction is described in the Śulvasūtras (see Asl. 1.3, Bsl. 1.5, Ksl. 1.4 and Msl. 1.11). Take a cord of twice the length of AB. Let P and Q denote the endpoints of the cord, to be tied to the points A and B respectively. We now mark the following points on the cord. Let X be the midpoint of PQ, R the midpoint of XQ, and N the midpoint of XR. The point N is to be the Nyancana mark. Note that if the length of the segment AB is a, then the lengths of QN and PN are $3a/4$ and $5a/4$ respectively, and that $(3a/4)^2 + a^2 = (5a/4)^2$. Now, following the Nyancana procedure the cord is stretched holding by the mark at N, and a point D is marked on the ground at the location of the mark R. Then BD is perpendicular to the segment AB. Furthermore, as the length of the cord segment QR is $a/2$, by choice, the point D is at a distance $a/2$ from B, the right distance for it to be one of the corners of the desired square. Now carrying out the construction by stretching to the other side gives the other corner of the square, across B, and by similar construction, with the role of the points A and B interchanged, we get the other two corners of the desired square, on the two sides of A.

The above construction clearly makes use of the Pythagorean triple (3, 4, 5). Similarly the triple (5, 12, 13) is used by Baudhāyana, in Bsl. 1.8, for construction of rectangles; (see Note 3 in Appendix for a comment in this respect). The same construction is also given in Asl. 1.2; see also Ksl. 1.5. For this, the given measure is extended by half of itself, and the Nyancana mark N is given at $\frac{1}{6}$th of the added half, from the dividing point; in other words, if the length of the segment AB is a, we take a cord of length $\frac{3}{2}a$, with endpoints P and Q to be tied at A and B respectively, mark X at a distance a from P, and N at a distance $\frac{1}{6}$th of the added part XQ, from X. Thus QN and PN are of length $\frac{5}{12}a$ and $\frac{13}{12}a$ respectively. Let R be a point marked on the cord between Q and N, say at a distance d from Q. Now following the Nyancana method on the two sides of the segment AB, from the point R we get two points at a distance d on either side of B, on the line perpendicular to AB. Similarly, interchanging the role of A and B we get two points at a distance d on either side of A. The four points together form a rectangle with sides a and $2d$, with AB as the midline. Since the point R is to be marked between Q and N, the length d of QR is at most $\frac{5}{12}a$, and hence the width of the rectangle will be at most $\frac{5}{6}a$. Broader rectangles can be constructed by elongating the side by a suitable multiple, in a second step. While Bsl. 1.8 is silent on the last part about getting broader rectangles by elongation, Asl. 1.2 includes a remark stating that the rectangle can be lengthened or shortened as desired.

One significant point about this procedure, given that certain standard sizes were to be used for the rituals, is that one could have cords with requisite markings in store, tie them to the two ends of the desired mid-line and then each stretch of the cord will produce one corner each; the cord would thus serve as a handy device in this way. It may also be borne in mind in this respect that rigid equipment like protractors would not have been convenient in producing a reasonably accurate right angle, in the context of the magnitude of the distances involved (of the order of 30 to 40 feet) and the ground surface.

Notice also that the points to be marked on the cord have a simplicity, more than the triples that are used, in numerical terms; the subdivisions of the cord are only in halves and thirds, even though the triples in question involve 5 and 13. There is no explicit indication in the Śulvasūtras about how they subdivided line segments into equal parts, and in any case it would be more cumbersome to divide into more parts. In this context, the above mentioned point offers an advantage. The property of the triple that is involved in this is that the sum of the odd entries in the triple has only 2 and 3 as the prime factors. It is interesting to note in this respect that all the five primitive triples occurring in Śulvasūtras have this property; for (8, 15, 17) and (7, 24, 25) the sum is 32, while for (12, 35, 37) it is 72.

3. The Mahāvedi

The Mahāvedi (the grand altar) is an altar in the shape of a symmetric (isosceles) trapezium with the base (longer of the parallel sides) measuring 30 padas, height of 36 padas, and the face (the shorter of the parallel sides) measuring 24 padas (see Figure 2). (The term *pada* means 'foot', and the measure was about a foot (12 inches). Various units are used in the Śulvasūtras, depending on the context, but in the sequel we shall not go into the details of the units involved, as it is not relevant to our theme here.)

The *Samhitās* of the *Krsna Yajur Veda* and the *Śatapatha Brāhmana*, which are considerably older compositions than the Śulvasūtras, also mention the Mahāvedi, with the same dimensions as above. The *Śatapatha Brāhmana* also contains a method of construction (see ref. 8) for the passage describing the method); the method however is approximate, and does not involve geometrical constructions using cords.

In Baudhāyana Śulvasūtra, after stating the dimensions of the vedi, for the construction the user is referred to the methods described earlier in the sūtras for constructing trapezia (Bsl. 4.3); (this can be either by Nyancana or by a procedure similar to the compass construction in Euclidean geometry). In Āpastamba Śulvasūtra on the other hand, we find a detailed treatment specifically for construction of the Mahāvedi (in Āpastamba Śulvasūtra it is also called Saumiki vedi). Two constructions are described in this respect, one using a single cord (*ekarajjuvidhi*) and another with two cords (*dvirajjuvidhi*). The ekarajjuvidhi uses the Nyancana construction in the following form: We begin by fixing the segment forming the line of symmetry, along east-west, with length 36 padas. Following the Nyancana procedure with $a = 36$, $b = 15$ and $c = 39$ padas gives the corner points of the base. The cord is also equipped with a mark at a distance of 12 padas from the end. When the Nyancana construction is applied, with the same values as above, but with the role of the endpoints reversed, the mark at 12 gives the corner points of the face.

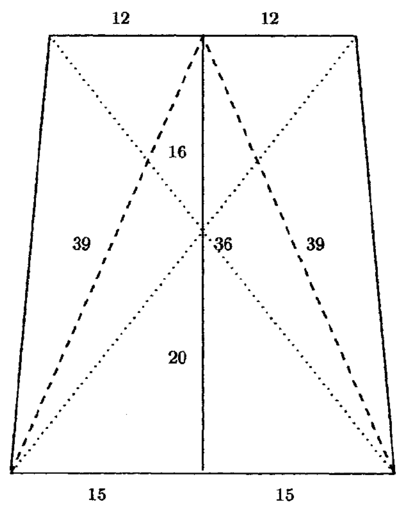

Figure 2. Mahāvedi.

The dvirajjuvidhi also employs the Nyancana method, but the corner points of the base and the face are determined using different Pythagorean triples. In this context, Āpastamba mentions the four Pythagorean triples (3, 4, 5), (5, 12, 13), (8, 15, 17) and (12, 35, 37). He also points out the multiples (12, 16, 20), (15, 20, 25) and (15, 36, 39); see Note 2 in the Appendix below for a comment relating to this). Notice that the corner points of the face can now be found by the Nyancana method, using any of the triples (12, 16, 20), (5, 12, 13) or (12, 35, 37), each of which involves 12, which is the distance of the corner point from the midpoint of the face. Similarly, the corner points of the base can be determined by Nyancana procedure using any of the triples (15, 20, 25), (15, 36, 39) or (8, 15, 17), each of which involves 15, which is the distance of the corner point from the midpoint of the base; in the sūtras the triples with 12 and with 15 have been paired, in the order as above, each pair making up a scheme for the dvirajjuvidhi.

4. The 'mark at fifteen'

Vedis were also constructed in the shape of trapezia of other sizes (than those of the Mahāvedi). We note here a rather interesting feature of the description of the constructions of certain vedis in Āpastamba Śulvasūtra (which seems to have gone unnoticed). The vedis described in the Asl. 6.3–6.4 (*nirudhapaśubandha vedi*), Asl. 6.6, Asl. 6.7 (*paitrki vedi*), Asl. 6.8 (*uttara vedi*), and Asl. 7.1 involve construction of trapezia, squares and a rectangle, whose dimensions are quite varied; (the first two are symmetric trapezia with face, base and altitude given respectively by 86, 104 and 188, and 3, 4 and 6, in certain units, the third is a square of a general size, the fourth is a square with a side of 10 units, and the last is a rectangle with sides 9 and 27 units). However, the description of the construction of each of them refers to 'Having stretched (the cord) by the mark at fifteen' (*pancadaśikenaivāpāyamya*). In the commentary in ref. 14 these figures are stated to be constructed using various multiples of the Pythagorean triple (5, 12, 13) (by the Nyancana method). The explanation is however unsatisfactory, for two reasons: first, the reference to 'Having stretched by the mark at fifteen' finds no place in the explanation given by the authors; secondly, the entries of the specific multiples of the above Pythagorean triple that the authors associate with the construction, in each of the cases, are not found in the sūtras (in fact, no numbers other than the 'fifteen' as in the above phrase occur in the sūtras, except in stating the dimensions of the vedis).

The following quite different explanation seems natural if we take into account the overall context. It may be recalled that the main task in constructing any of these figures is to draw perpendiculars to the line segment which is to serve as the line of symmetry, and to mark points at the desired lengths. The concise style of presentation in this part of Āpastamba Śulvasūtra shows that the target users are supposed to be familiar with the general procedures involved and that it would suffice to recall only the main steps.

In each of the instances mentioned above, the author first gives the dimensions of the vedi to be constructed, then states that it is to be constructed by the *ekarajjuvidhi* described (by him) earlier, this is followed by 'Having stretched by the mark at fifteen', and then the distance at which the corners are to fixed, the latter being simply half the given widths.

Once these points are borne in mind, one can see that 'Having stretched by the mark at fifteen' is used as a way of saying 'Having drawn a perpendicular'. The former expression thus serves as a 'colloquial' phrase to describe the latter action; it may be recalled here that in the Śulvasūtras there is no term for 'perpendicular' (the ideas concerning it occur essentially through figures such as squares and rectangles), and also that the term Nyancana – or a variation of it – does not appear in Āpastamba Śulvasūtra. Also, the motivation for using the phrase 'Having stretched by the mark at fifteen' to mean 'Having drawn a perpendicular' is not far to seek. We have seen that in the *ekarajjuvidhi* construction of the Mahāvedi, the Nyancana method was used with the Pythagorean triple (15, 36, 39), and that it involved stretching the cord by the mark at 15 for drawing a perpendicular. In view of the great significance of the Mahāvedi, the choice of this phraseology is therefore quite natural.

The above mentioned sūtras thus show that drawing perpendiculars by the Nyancana method was routine during this phase, and that the Pythagorean triple (15, 36, 39) had come to be intricately associated with the process, to the extent of lending a name to it.

5. Baudhāyana's list

Apart from the description of various constructions discussed in the preceding sections, there is one other, rather unique, occurrence of Pythagorean triples, in the Śulvasūtras. This is in Bsl. 1.13 of Baudhāyana Śulvasūtra. All the Pythagorean triples listed in the introduction (marked by (1)) are found in Bsl. 1.13. This follows a statement, in Bsl. 1.12, of (what is now called) Pythagoras theorem.

The sūtra Bsl. 1.13 translates as '*This is observed in rectangles having sides* 3 *and* 4, 12 *and* 5, 15 *and* 8, 7 and 24, 12 *and* 35, 15 *and* 36.'; see ref. 1; see also Note 4 in the Appendix below. Thus, the sūtra gives the pairs of the side-length (the first two entries) in the triples listed in the introduction; these pairs of course determine the last entry. The sūtra includes, apart from the pairs corresponding to the triples in (1), also the pair (15, 36), which completes to the triple (15, 36, 39). This last one is a multiple of the second triple from (1); the pair here may be seen to appear in the reverse order, but the choice of the order however is likely to be for convenience in composition of the sūtra and may not signify anything more.

The overall context indicates that the triples are described as examples related to the theorem. The inclusion of the triple (15, 36, 39) in the list is also significant in this respect. At first sight it seems puzzling, as it is the only one from the list that is not primitive. Baudhāyana was of course aware that taking multiples yielded new triples; then why is only one particular multiple included in the list following the statement of the theorem? A likely explanation for this is that the triple (15, 36, 39) enjoyed substantial familiarity among the target users of the manual, it being involved in the Mahāvedi. As was noted in §3, the Mahāvedi had a long tradition by the time of the Śulvasūtras, which lent special significance to the triple (15, 36, 39) to be included as an example.

With regard to Baudhāyana's list it may also be noted that it is the only place where the triple (7, 24, 25) occurs. We have seen earlier that the other four primitive triples from the list occur in Āpastamba's description of the construction of the Mahāvedi.

6. Geometric propositions

While the Pythagorean triples listed earlier and some of their multiples are the only ones occurring in the Śulvasūtras, explicitly in numerical terms, certain geometrical propositions in the Śulvasūtras point to more Pythagorean triples. In the Kātyāyana Śulvasūtra there is a sūtra, Ksl. 6.7, which states the following (the original is considered somewhat difficult to understand[1] but doubtless conveys the meaning as stated): when a square with a given area is desired, construct an (isosceles) triangle with base one less than the given number, and the other two (equal) sides adding to one more than the given number; the altitude of the triangle will produce the desired square. (Note that in the context of the rope-stretching construction procedures it is quite natural to prescribe the sum of the two (equal) sides, rather than the side itself.) Clearly, the Pythagoras theorem is assumed in this, and in modern notation the assertion means that

$$t = \left(\frac{t+1}{2}\right)^2 - \left(\frac{t-1}{2}\right)^2, \qquad (2)$$

(t being any magnitude). Taking $t = (2x + 1)^2$, x a natural number, leads to the Pythagorean triples

$$(2x + 1,\ 2x^2 + 2x,\ 2x^2 + 2x + 1),\quad x = 1, 2, \ldots, \qquad (3)$$

and taking $t = x^2$, x an even number, yields the triples

$$(2x,\ x^2 - 1,\ x^2 + 1),\quad x = 2, 4, \ldots \qquad (4)$$

The triples from both the series are primitive. In the Greek tradition the triples as in (3) and (4) are attributed to Pythagoras and Plato respectively (see ref. 9, p. 165, or ref. 10, p. 116). It is interesting to note that the five primitive triples in Baudhāyana's list contain the first three terms of each of these series (one of the triples being common). This brings to mind the convention of picking three terms of a sequence as illustrative examples which is followed even today. It is however unclear whether substitutions as above, as a square integer, would have occurred at that time naturally, especially in the absence of algebraic formalism and the fact that the main purpose of the geometrical proposition concerns quantities which are a priori not given as squares (in the alternative case, the side of the desired square being already known).

In obtaining the above series of Pythagorean triples one only had to choose a value for t, as a square integer, for the side of a square as in the proposition. A less direct, but still plausible way for producing Pythagorean triples from the geometrical ideas in the Śulvasūtras emerges from the procedure described for the construction of a square with area equal to that of a given rectangle. The procedure is given in Bsl. 2.5, Asl. 2.7 and Ksl. 3.2. For this it is noted (in geometrical formulation) that the area of a rectangle with sides, say s and t, with $s > t$, is the same as the difference of the areas of the squares of sides $\frac{1}{2}(s+t)$ and $\frac{1}{2}(s-t)$; for constructing the desired square one is then referred to the construction of a square with an area equal to the difference of two given squares. In numerical terms this (the first part) corresponds to the following equation

$$st = \left(\frac{s+t}{2}\right)^2 - \left(\frac{s-t}{2}\right)^2. \qquad (5)$$

This is a more general version of the relation (2) as above, and substituting suitable square integral values for s and t produces the Pythagorean triples

$$(2mn, m^2 - n^2, m^2 + n^2),\ m, n = 1, 2, \ldots, \text{ and } m > n. \qquad (6)$$

This suggests a possible route along which they could have arrived at the above triples. There is however no evidence that they did so, and the rather definitive statements found in this respect in literature (see e.g. ref. 11, Part II, §20) have therefore to be regarded with skepticism. (See Note 5 in the Appendix for some historical facts concerning the triples in (6).)

7. Related issues

How did the ancient Indians obtain the Pythagorean triples? It is pointed out by van der Waerden (see ref. 12, p. 9) that the triples could not have been found empirically by measuring triangles. He states that the author of Baudhāyana Śulvasūtra knew the equation $x^2 = z^2 - y^2$ and used it in altar constructions, and that it is easy to rewrite the equation as

$$x^2 = (z + y)\,(z - y). \qquad (7)$$

From this and other arguments he concludes that the triples would have been found using the above equation. This is quite plausible. In the course of the discussion however the following comment appears, pertaining to the triples in (1):

- In all these triples $z-y$ is small, which makes the calculation of the triples by means of the formula $(z+y)(z-y) = x^2$ very easy. The triple (20, 21, 29), which is not easy to obtain, is not mentioned by Baudhāyana.

It is indeed true that in the triples in question, $z-y$ is small, viz. 1 or 2. However the suggestion that only triples with small value for $z-y$ could be found easily using eq. (7) is incorrect. The following simple possibility seems to have been overlooked in this respect: choose x to be an odd number and factorize x^2 into two distinct factors p and q, with say $p > q$; then setting $y = (p-q)/2$ and $z = (p+q)/2$ yields a Pythagorean triple (x, y, z); a similar construction is also possible starting with even x, provided the factors p and q are chosen to be even. The numbers $(p+q)/2$ and $(p-q)/2$, namely the average and the discrepancy from the average, are recognizable mentally, or geometrically (without involving algebra), as the two numbers whose sum and difference would give the factors p and q.

The comment seems to suppose that to use eq. (7) one would have to start by choosing a value for $z-y$ first and then pick $z+y$ with the same parity and such that $(z+y)(z-y)$ is a square. This however need not be the only course for applying formula, to get examples of Pythagorean triples. Thus the fact that (20, 21, 29), or any other triple with large difference, not being mentioned in the Śulvasūtras is not attributable to it being 'difficult' to produce such triples.

As seen from the observations in the earlier sections, the Pythagorean triples that find direct mention had a limited context, either as examples or for use in applying the Nyancana method. The latter neither calls for a large variety of Pythagorean triples, nor is it facilitated by knowing more general or large Pythagorean triples. It may be noted that even among those triples that are applied, only (3, 4, 5) and (5, 12, 13) are common to all the four Śulvasūtras as above. Furthermore, other than in Baudhāyana's list, the only occurrence of (8, 15, 17) and (12, 35, 37) is in Āpastamba Śulvasūtra in connection with the construction of the Mahāvedi, and the triple (7, 24, 25) is found only in Baudhāyana's list. Thus one sees also a definite correlation between the simplicity of the triple and its being used in the Śulvasūtra constructions. Therefore the non-occurrence of other Pythagorean triples does not lend itself to any inference about whether more such triples were known or not.

It is well-known[13] that the Babylonian cuneiform tablet Plimpton 322 contains fifteen Pythagorean triples with quite large entries, including (13500, 12709, 18541) which is a primitive triple, indicating in particular that there was sophisticated understanding on the topic (see however Note 6 in the Appendix). Since these tablets predate the Śulvasūtra period by several centuries, taking into account the contextual appearance of some of the triples, it is reasonable to expect that similar understanding would have been there in India. Also the geometric propositions in §6 show that mathematicians from the Śulvasūtra period had the means to construct a variety of Pythagorean triples, as in (3), (4) and (6).

As the main objective of the Śulvasūtras was to describe the constructions of altars and the geometric principles involved in them, the subject of Pythagorean triples, even if it had been well-understood may still not have featured in the Śulvasūtras. The occurrence of the triples in the Śulvasūtras is comparable to mathematics that one may encounter in an introductory book on architecture or another similar applied area, and would not correspond directly to the overall knowledge on the topic at that time. Since, unfortunately, no other contemporaneous sources have been found it may never be possible to settle this issue satisfactorily.

Appendix: Auxiliary notes

1. It is often said that the ancient Egyptians also used the Pythagorean triple (3, 4, 5) for producing right angles, but this does not seem to be well founded (see ref. 14, Appendix 5).
2. Heath[15] states on p. 146 that Āpastamba knew only seven Pythagorean triples, four of them being primitive. While no specific reference is given, the seven triples he has in mind are evidently the ones that were recalled in §3 in connection with the Mahāvedi, and the remark seems to be based on an incorrect reading of some of Bürk's observations in respect of the clause *tābhirjneyābhiruktam viharaṇam* mentioned in §1. In Āpastamba Śulvasūtra that clause and the Pythagorean triples occur at quite different places, and not in a connected way as Heath seems to suppose. Various other aspects of the sūtras also show that the above statement from ref. 15 is unwarranted.
3. In Thibaut's translation (see ref. 2, p. 50) and also in the commentary in ref. 1; p. 150, the sūtra Bsl. 1.8 (which was discussed in §3) is interpreted as being for construction of squares (only), rather than rectangles. This however is not correct; that it is meant for rectangles is clear from the fact that the marker for the corner is treated as variable (*iṣṭe'msārtham*), even as the length of the square is already fixed. In this sūtra Baudhāyana is actually continuing from Bsl. 1.6 which is an elaborate description of construction of rectangles, after a brief digression in Bsl. 1.7 to note how to obtain trapezia, rather than rectangles, by following the overall procedure described in Bsl. 1.6. The sūtras concerning construction of squares appear consecutively in Bsl. 1.4 and 1.5, and the '*athāparam* (now another method)' in the beginning of Bsl. 1.8 could not be referring to that. It may also be observed that the mark on the cord cannot produce the corner point of the square with the given length (determined by the length of the mid-line, *prācī*). This is because the mark has to be between the Nyancana mark N and the point Q in the notation as in §3, and hence the width of the rectangle obtained using the mark is at most $\frac{5}{6}a$, a being the length of the mid-line; as noted in §3, to get a square one can elongate the side, in a second step; e.g. the mark may be made at $\frac{1}{4}a$ and then the segment may be doubled.

   In the context of the above observation, the parenthetic interpolations in the translations of the sūtra in refs 1 and 2, referring to squares as the figure in question, need to be either suitably modified, or deleted.
4. Thibaut's translation (given in ref. 2) of Bsl. 1.13 is essentially the same as the one from ref. 1 recalled in §5, with 'seen' in the place of 'observed', and 'oblongs' in the place of 'rectangles'. Datta[7] proffers also the alternative 'realized in', in place of 'observed in'. The verb-form involved is *upalabdhih* which would correspond also to 'obtained in'. It may also be mentioned that in many versions of the Baudhāyana Śulvasūtra the sūtra commences with the word *tāsām*, but it is missing from Thibaut's version. Since the word means something like 'from among these' it would be of interest to understand what it signifies in the context of the sūtra. The word has been included in the text given in ref. 1, but does not seem to have been taken into account in the translation, or the commentary.
5. Here, it may be worthwhile to recall some historical facts concerning the Pythagorean triples displayed in (6) to put the subject in its proper perspective (see ref. 9, for details). Euclid's *Elements* contains the triples. Diophantus was familiar with them and employed them systematically in solving various problems, using the parametrisation as above. In India, explicit mention of them is found in the work of Brahmagupta in the sixth century. In an anonymous Arabic manuscript of 972 it is stated that for every primitive right triangle the sides may be expressed in the form as in (6). Extant proofs of this are from eighteenth century and later (see ref. 9, p. 167 and ref. 10, p. 295).
6. A recent article[16] of Robson makes a persuasive case to the effect that the list in Plimpton 322 was not meant to describe a set of Pythogorean triples (that it happens to be so is incidental). If that is the case, our related remark in §7 would not be applicable. Thanks are due to Rahul Roy for bringing the article to the author's attention.

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