Dynamical properties of linear and projective transformations and their applications

S.G. Dani

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Let V be a finite-dimensional vector space over the field $\mathbb{R}$ of real numbers, namely $\mathbb{R}^d$ for some $d \geq 0$. We denote by $P(V)$ the corresponding projective space; thus $P(V) = (V\backslash\{0\})/\sim$, where $\sim$ is the equivalence relation identifying every $v \in V\backslash\{0\}$ with tv for all $t \in \mathbb{R}^*$ (nonzero real numbers). We consider V equipped with the usual Euclidean topology, and the projective space equipped with the quotient topology as the space of equivalence classes. Let $T : V \to V$ be a nonsingular linear transformation of V. Then T carries every equivalence class in $V\backslash\{0\}$, with respect to $\sim$ as above, to an equivalence class, and hence induces a homeomorphism of $P(V)$; we call this the projective transformation associated with T.

Here we consider the nonsingular linear transformations and projective transformations as dynamical systems. We describe their dynamical properties, and their applications to certain topics in Lie groups and ergodic theory: Borel's density theorem, Halmos's question on existence of ergodic automorphisms, invariant measures of automorphisms of locally compact groups etc.. This is primarily an expository article, but some new points have been brought out, correction to some arguments in [2] and [14] are noted, and some questions are raised. While for the most part we shall consider only real vector spaces, and their analogues over complex numbers, in § 7 we shall briefly describe also analogues of the results in the case of vector spaces over p-adic fields.

1 Preliminaries on general dynamical systems

We recall some definitions for a general dynamical system (X, τ), where X is a locally compact second countable space and τ is a homeomorphism of X. A point $x \in X$ is said to be *wandering*, if there exists a neighbourhood U of x in X such that $U \cap \tau^i(U) = \emptyset$ for all $i \in \mathbb{N}$ (natural numbers). A point $x \in X$ is said to be *nonwandering* if it is not wandering. The set of all nonwandering points is called the *nonwandering set* of (X, τ), and is denoted by $\Omega(X, \tau)$, or $\Omega(\tau)$ (or even just Ω, when there is no likelihood of confusion). It is easy to see that $\Omega(\tau)$ is closed and τ-invariant, namely $\tau(\Omega) = \Omega$. We note also the following.

Lemma 1.1. *Let (X, τ) be as above. A point x in X is nonwandering if and only if there exist sequences $\{x_i\}$ in X and $\{n_i\}$ in $\mathbb{N}$ such that, as $i \to \infty$, $x_i \to x$, $n_i \to \infty$, and $\tau^{n_i}(x_i) \to x$.*

Proof. Let $x \in X$. If there exist sequences $\{x_i\}$ in X and $\{n_i\}$ in $\mathbb{N}$ as above, then for every neighbourhood U of x there exists i such that $x_i \in U$ and $\tau^{n_i}(x_i) \in U$, which means that $U \cap \tau^{n_i}(U)$ contains $\tau^{n_i}(x_i)$, and hence is nonempty, and therefore x is nonwandering. This proves the 'if' part of the assertion. Now let x be a nonwandering point. If x is a periodic point of τ, namely if $\tau^n(x) = x$ for some $n \in \mathbb{N}$ then the desired converse is obvious. Suppose therefore that x is not periodic. Let $\{U_i\}$ be a fundamental system of neighbourhoods of x. Then for each i there exists $n_i \in \mathbb{N}$ such that $U_i \cap \tau^{n_i}(U_i) \neq \emptyset$ and hence $\tau^{-n_i}(U_i) \cap U_i \neq \emptyset$. For all i choose $x_i \in \tau^{-n_i}(U_i) \cap U_i$. Then clearly $x_i \to x$ and $\tau^{n_i}(x_i) \to x$. Also, since x is not a periodic point $\tau^{n_i}(x_i)$ are different from x for all large i, and since $\tau^{n_i}(x_i) \to x$ it follows that $n_i \to \infty$. This proves the lemma. $\square$

For every $x \in X$ we denote by $L^+(x, \tau)$ (respectively $L^-(x, \tau)$) the set of all $y \in X$ for which there exists a sequence $\{n_i\}$ of natural numbers such that $n_i \to \infty$ and $\tau^{n_i}(x) \to y$ (respectively $\tau^{-n_i}(x) \to y$) as $i \to \infty$. It is easy to see that for every $x \in X$, $L^+(x, \tau)$ and $L^-(x, \tau)$ are closed τ-invariant subsets; they are respectively called the *(positive) limit set* and the *negative limit set* of the τ-orbit of x. When τ is clear from the context we shall denote $L^+(x, \tau)$ and $L^-(x, \tau)$ by $L^+(x)$ and $L^-(x)$ respectively.

A point $x \in X$ is said to be *recurrent* for τ, if there exists a sequence $\{n_i\}$ in $\mathbb{N}$ such that $n_i \to \infty$ and $\tau^{n_i}(x) \to x$, namely if $x \in L^+(x)$.

Remark 1.2. We note that for all $x \in X$ the sets $L^+(x, \tau)$ and $L^-(x, \tau)$ are contained in $\Omega(\tau)$: If $\tau^{n_i}(x) \to y$ for a sequence $\{n_i\}$ in $\mathbb{N}$ such that $n_i \to \infty$, we can find a sequence m_i in $\mathbb{N}$ such that $m_i \to \infty$ and $\{m_i + n_i\}$ is a subsequence of $\{n_i\}$, and setting $y_i = \tau^{n_i}(x)$ we see that $y_i \to y$ and $\tau^{m_i}(y_i) = \tau^{m_i+n_i}(x) \to y$, and hence by Lemma 1.1 it follows that y is nonwandering. This shows that $L^+(x, \tau)$ is contained in $\Omega(\tau)$, and similarly $L^-(x, \tau)$ is contained in $\Omega(\tau)$. We note that in particular all recurrent points of (X, τ) are contained in $\Omega(\tau)$.

In the context of the systems of linear and projective transformations that we would be discussing in the following sections, it would also be of interest to define the following auxiliary notions related to the nonwandering set, in a general framework.

Let (X, τ) be a dynamical system as above. We define $\Psi(X, \tau)$ to be the subset of X consisting of all points y in X for which there exist a convergent sequence $\{x_i\}$ in X and a sequence $\{n_i\}$ in $\mathbb{N}$ with $n_i \to \infty$, such that $\tau^{n_i}(x_i) \to y$. We shall call $\Psi(X, \tau)$ the *core* of (X, τ); we shall sometimes denote $\Psi(X, \tau)$ by $\Psi(\tau)$ when the underlying space is clear from the context. We note that $\Psi(X, \tau)$ is a τ-invariant subset of X, and that, in view of Lemma 1.1, $\Omega(X, \tau)$ is contained in $\Psi(X, \tau)$.

We shall say that an open τ-invariant subset Y of X is *stably wandering* if the following holds: for any convergent sequence $\{y_i\}$ in Y, and any sequence $\{n_i\}$ in $\mathbb{N}$ such that $n_i \to \infty$, the sequence $\{\tau^{n_i}(y_i)\}$ has no limit point in Y. By Lemma 1.1, all points of a stably wandering set are wandering; thus every stably wandering set is contained in $X \backslash \Omega(\tau)$. On the other hand it may be seen that $X \backslash \Psi(\tau)$ is stably wandering. We note that the union of finitely many stably wandering sets is stably wandering, so in particular we may adjoin $X \backslash \Psi(\tau)$ to any stably wandering set and still get a stably wandering set.

A (Borel) measure μ on X is said to be τ-invariant if $\mu(\tau^{-1}(E)) = \mu(E)$ for all Borel subsets E of X. By the *support* of a measure μ on X we mean the complement of the largest open subset of X with μ measure 0; such a subset exists, as X is assumed to be second countable. For any measure μ on X we denote the support of μ by $\operatorname{supp} \mu$. We note that if μ is a τ-invariant measure then $\operatorname{supp} \mu$ is a τ-invariant subset of X. A measure is said to be *nonatomic* if the measure of every point is 0.

A measure μ on X is said to be *quasi-invariant* under the action of τ if for a Borel subset E of X we can have $\mu(\tau^{-1}(E)) = 0$ if and only if $\mu(E) = 0$; that is, the class of sets of μ-measure 0 is invariant under the action induced by τ on the class of all subsets of X.

A measure μ on X is said to be *ergodic* with respect to τ if for every τ-invariant Borel subset E of X either $\mu(E) = 0$ or $\mu(X \backslash E) = 0$. (Ergodicity is usually considered only for quasi-invariant - or invariant - measures, though the definition is meaningful even without such a condition. To each measure μ we can associate a quasi-invariant measure defined by $\nu(E) = \Sigma_{-\infty}^{\infty} 2^{-|j|}\mu'(\tau^{-j}(E))$ for all Borel subsets E, where μ' is a finite measure equivalent to (having the same sets of measure zero as) μ. Then μ is ergodic (in the generalised sense) if and only if ν is ergodic. In view of this, in the overall context of ergodic theory, not assuming quasi-invariance does not constitute an essential generalisation.)

The following simple property is useful in many contexts; it is a variation of an observation going back to G.A. Hedlund's papers in the 1930's.

Proposition 1.3. *Let μ be a nonatomic measure which is ergodic with respect to τ. Then for μ-almost all x in X, $L^+(x) \cup L^-(x)$ contains* $\operatorname{supp} \mu$. *In particular we have the following:*

i) μ*-almost all* x *are recurrent either for* τ *or for* τ^{-1}.

ii) if $\operatorname{supp}\mu = X$ *then there exists* $x \in X$ *such that the orbit* $\{\tau^n(x) \mid n \in \mathbb{Z}\}$ *is dense in* X.

Proof. Let $\{U_j\}$ be a sequence of open subsets of X such that $(\operatorname{supp}\mu) \cap U_j$ is nonempty for each j, and $\{(\operatorname{supp}\mu) \cap U_j\}$ forms a basis for the induced topology on $\operatorname{supp}\mu$. For each $j \geq 1$, let $E_j = \cup_{-\infty < i < \infty} \tau^{-i}(U_j)$, and let $E = \cap_{j=1}^{\infty} E_j$. Then each E_j is a τ-invariant open subset of positive μ-measure. Hence by ergodicity we have $\mu(X \backslash E_j) = 0$ for all j, and therefore $\mu(X \backslash E) = 0$. We show that for all $x \in E$, $\operatorname{supp}\mu$ is contained in $L^+(x) \cup L^-(x)$. Let $x \in E$ be given and let $O(x) = \{\tau^k(x) \mid k \in \mathbb{Z}\}$, the orbit of x. Let $S = \operatorname{supp}\mu \backslash O(x)$. Since μ is nonatomic, S is dense in $\operatorname{supp}\mu$, and hence it suffices to show that S is contained in $L^+(x) \cup L^-(x)$. Let $y \in S$ be arbitrary. Consider any $j \in \mathbb{N}$ such that $y \in U_j$. Then, as $x \in E \subset E_j$, there exists an integer k such that $\tau^k(x) \in U_j$. Since $\{(\operatorname{supp}\mu) \cap U_j\}$ is a basis of the topology on $\operatorname{supp}\mu$, it follows that y is contained in the closure of $O(x)$. Hence there exists a sequence $\{k_i\}$ in $\mathbb{Z}$ such that $\tau^{k_i}(x) \to y$ as $i \to \infty$. Since y is not contained in $O(x)$ it follows that $\{k_i\}$ has a subsequence tending either to ∞ or to $-\infty$. This shows that $y \in L^+(x) \cup L^-(x)$; this proves the proposition. □

By considering the translation map $n \mapsto n+1$ on $\mathbb{Z}$, the integers, it is easy to see that the nonatomicity condition in the proposition is in fact necessary.

When there is a finite invariant measure, we have the following analogous (and in some ways stronger) assertion, without the ergodicity condition; it is a version of what is called the Poincaré recurrence lemma; while assertion (i) below may be found in many books on ergodic theory, assertion (ii) is not readily accessible in literature; for the readers' convenience we include a proof of both the statements.

Proposition 1.4. *(Poincaré recurrence lemma) Let* μ *be a finite* τ*-invariant measure on* X. *Then the following statements hold:*

i) if E *is a measurable subset of* X, *then for* μ*-almost all* x *in* E, $\tau^i(x) \in E$ *for infinitely many positive integers* i;

ii) if R *is the set of all recurrent points of* (X, τ) *then* $\mu(X \backslash R) = 0$.

Proof. For any measurable subset E of X let $r(E)$ denote the subset of E consisting all x in E such that $\tau^i(x) \in E$ for some $i \geq 1$; namely $r(E)$ is the set of points of E which 'return' to E. We note that for any measurable subset E the sets $\{\tau^j(E \backslash r(E))\}$ are pairwise disjoint. As μ is τ-invariant the μ-measures of these subsets are all equal, and since the measure of X is finite it follows that $\mu(E \backslash r(E)) = 0$; thus $\mu(r(E)) = \mu(E)$. Now let E be a measurable subset of X and consider the sequence of sets $\{r^i(E)\}$ defined inductively by setting $r^i(E) = r(r^{i-1}(E))$ for all

$i \geq 1$, with $r^0(E) = E$. Then $\mu(E \backslash \cap_i r^i(E)) \leq \Sigma_i \, \mu(r^{i-1}(E) \backslash r^i(E)) = 0$, and clearly for $x \in \cap_i r^i(E)$ we have $\tau^i(x) \in E$ for infinitely many i. This proves (i).

Since X is a locally compact second countable space, it can be realised as a metric space; let d be a metric on X. Let $\delta > 0$ be arbitrary. Let $\{B_j\}$ be a sequence of balls of diameter at most δ such that $X \subset \cup B_j$, and let $X(\delta) = \cup \, r(B_j)$. Then $\mu(X \backslash X(\delta)) \leq \Sigma \, \mu(B_j \backslash r(B_j)) = 0$. We note that on the other hand if $x \in X(\delta)$, then $x \in r(B_j)$ for some j, so there exists $i \geq 1$ such that $\tau^i(x) \in B_j$ and hence $d(x, \tau^i(x)) \leq \delta$. Now let $Y = \cap_{k=1}^{\infty} X(1/k)$. Then clearly $\mu(X \backslash Y) = 0$. Furthermore, $x \in L^+(x)$ for every $x \in Y$, namely $Y \subset R$, and hence $\mu(X \backslash R) = 0$. This proves (ii). $\square$

In the following sections we will be studying the limit sets and nonwandering sets of linear and projective transformations, and applying the results to finite invariant measures and ergodic quasi-invariant measures of these transformations. In fact we shall consider a more general class of measures, defined as follows.

Definition 1.5. Let (X, τ) be a dynamical system. A measure μ on X is said to be *conservative* if for every open subset O of X with $\mu(O) > 0$ there exists $k \in \mathbb{N}$ such that $\mu(O \cap \tau^k(O)) > 0$.

Remark 1.6. Let (X, τ) be a dynamical system. Then we have the following (the proofs are easy and will be omitted):

i) If μ is either a finite τ-invariant measure, or a measure quasi-invariant and ergodic under the τ-action, then it is conservative.

ii) The support of any conservative measure for (X, τ) is contained in the non-wandering set $\Omega(X, \tau)$.

iii) If μ is a conservative measure on (X, τ) and Y is a closed τ-invariant subset of X containing $\Omega(\tau)$ then μ (viewed as a measure on Y) is conservative for (Y, τ).

2 Unipotent linear transformations

In this and the following sections we discuss the limit sets and nonwandering sets of linear and projective transformations.

Let $K = \mathbb{R}$ or $\mathbb{C}$, the field of real or complex numbers respectively. Let V be a finite-dimensional vector space over K of dimension $d \geq 1$, and $P(V) = (V \backslash \{0\})/K^*$, where K^* is the group of nonzero elements in K acting on $V \backslash \{0\}$ by scalar multiplication, be the projective space corresponding to V. We denote by $\pi : V \backslash \{0\} \to P(V)$ the canonical quotient map. For convenience, for any subset S of V we define $\pi(S)$ to be $\pi(S \backslash \{0\})$. Any nonsingular linear transformation T of V induces a homeomorphism of $P(V)$, the projective transformation corresponding to

T, which we denote by $\overline{T} : P(V) \to P(V)$. We denote by I the identity transformation of V, and the zero transformation will be denoted by 0. All subspaces and linear transformations will be meant to be over K, and we shall not indicate this specifically.

We begin with the special case of unipotent linear transformations. It may be recalled that a linear transformation T is said to be *unipotent* if $(T-I)^r = 0$ for some $r \in \mathbb{N}$ (namely $T-I$ is nilpotent); we note that if the condition holds for some $r \in \mathbb{N}$ then the Jordan canonical form shows that it holds for $r = d$. We note that a linear transformation is unipotent if and only if T has 1 as the only eigenvalue (even if $K = \mathbb{R}$, all eigenvalues over complex numbers are considered).

Let U be a unipotent linear transformation of V. Then V can be decomposed, using Jordan canonical form of U, as $V = V_1 \oplus \cdots \oplus V_l$, for some $l \geq 1$, such that each V_i, $1 \leq i \leq l$, is a U-invariant subspace, and admits a basis with respect to which the matrix of the restriction of U to V_i has the form

$$\begin{pmatrix} 1 & 1 & & & & & \\ & 1 & 1 & & & & \\ & & \cdot & \cdot & & & \\ & & & \cdot & \cdot & & \\ & & & & \cdot & \cdot & \\ & & & & & 1 & 1 \\ & & & & & & 1 \end{pmatrix} \qquad \ldots\ldots \qquad (1)$$

where all the diagonal entries and the entries next to them on the right hand side (except in the last row) are 1, and the other entries are understood to be 0; the 1×1 matrix (1) should also be viewed as a special case of the above. It can be seen that the number l as above is precisely $d - \mathrm{Rank}\,(U-I)$.

Let us first consider a unipotent linear transformation which has a matrix form as in (1) above; namely the case with $l = 1$, or equivalently such that $\mathrm{Rank}\,(U-I) = d-1$; we shall call such a unipotent matrix an *indecomposable* unipotent matrix. Let $\{e_1, \ldots, e_d\}$ be a basis of V such that $U(e_1) = e_1$ and $U(e_j) = e_j + e_{j-1}$ for all $j = 2, \ldots, d$, namely such that the matrix of U with respect to $\{e_j\}$ is as above. As usual we identify any vector $v = \Sigma_{j=1}^d v_j e_j$ with the column vector with v_j in the jth row for all $j = 1, \ldots, d$. It can be seen that for $v = \Sigma_{j=1}^d v_j e_j$ and $k \geq d$ the vector

$U^k(v)$ corresponds to the column vector

$$\begin{pmatrix} v_1 + kv_2 + {}^kC_2v_3 + \ldots + {}^kC_{d-1}v_d \\ v_2 + kv_3 + \ldots + {}^kC_{d-2}v_d \\ . \\ . \\ . \\ v_{d-1} + kv_d \\ v_d \end{pmatrix} \qquad \ldots\ldots \quad (2)$$

where ${}^kC_r$ denotes, for $1 \leq r \leq k$, the number of ways of choosing r symbols from k symbols.

We note that in the above expression all the entries are polynomials in k, and the entry in the first row has the highest degree among them. It follows that in this case $L^+(v, U)$ is empty for all $v \in V$ except for v in the span of e_1; the latter are fixed under the action of U, and for them of course $L^+(v, U) = \{v\}$. It follows also that for the corresponding projective transformations $\overline{U}$, $L^+(\pi(v), \overline{U}) = \pi(e_1)$ for all $v \in V \backslash \{0\}$, since if any of the coordinates $v_2, \ldots, v_d$ of v are not zero then the first entry in the above column dominates all other coordinates, as k tends to infinity.

It is tempting to conclude, with similar reasoning about the dominance of the first row in mind, that the nonwandering sets, and the cores, of U and $\overline{U}$ also consist of just the span of e_1, and the point $\pi(e_1)$, respectively. Such an assertion was indeed made for $\overline{U}$ in [2], and also in [14]; (we note that the term "irreducible" in [14] corresponds to indecomposible in the sense as above, and not to irreducible in the usual sense of representation theory). It turns out however that this is not true, as was pointed out to the author by Gopal Prasad, with the following counterexample. Let $d = 3$ and consider the sequence of vectors

$$v^{(k)} = \begin{pmatrix} 1/k \\ 1 \\ -2/(k-1) \end{pmatrix}.$$

It is then straightforward to verify that as $k \to \infty$, $v^{(k)} \to e_2$ and $\overline{U}^k(\pi(v^{(k)})) \to \pi(e_2)$, which shows, by Lemma 1.1, that $\pi(e_2)$ is also a nonwandering point of $\overline{U}$. Since $\Omega(\overline{U})$ is $\overline{U}$-invariant and closed, it follows furthermore, that $\Omega(\overline{U})$ contains $\pi(v)$ for all nonzero v in the subspace spanned by e_1 and e_2.

It turns out that for any indecomposable unipotent U, as above, it is indeed true that the nonwandering set consists of the span of e_1. Also, $\Omega(\overline{U})$ coincides with $\pi(W')$, where W' is the unique $d - [d/2]$-dimensional U-invariant subspace of V, where $[d/2]$ denotes the largest integer not exceeding $d/2$. However, the arguments for proving these statements (that the author knows at present) are rather cumbersome. On the other hand, the precise results are of limited interest in our present context. We shall therefore not go into the proofs here; they may be taken up elsewhere, on another occasion. For the present we content ourselves observing the following weaker statements:

Proposition 2.1. *Let the notation be as above, and suppose that* $d \geq 2$. *Let* $<e_1, \ldots, e_{d-1}>$ *be the subsapce spanned by* $e_1, \ldots, e_{d-1}$. *Then we have the following:*

i) $\Omega(U) \subset \Psi(U) \subset \ <e_1, \ldots, e_{d-1}>$, *and*

ii) $\Omega(\overline{U}) \subset \pi(<e_1, \ldots, e_{d-1}>)$; *moreover* $P(V) \backslash \pi(<e_1, \ldots, e_{d-1}>)$ *is a stably wandering subset for the* $\overline{U}$*-action on* $P(V)$.

Proof. Suppose $\{v^{(k)} = \Sigma_{j=1}^d v_j^{(k)} e_j\}$ is a convergent sequence in V and $\{n_k\}$ is a sequence in $\mathbb{N}$ such that $n_k \to \infty$ and $U^{n_k}(v^{(k)}) \to w = \Sigma_{j=1}^d w_j e_j$ as $k \to \infty$, with $w_d \neq 0$. Since the e_d-component of $U^{n_k}(v^{(k)})$ is $v_d^{(k)}$ we get that $v_d^{(k)} \to w_d \neq 0$, and hence the expression in (2) shows that the e_1 components of $U^{n_k}(v^{(k)})$ can not converge; this is a contradiction, showing that $\Psi(U) \subset <e_1, \ldots, e_{d-1}>$. This proves (i).

Now let $\{v^{(k)} = \Sigma_{j=1}^d v_j^{(k)} e_j\}$ be a sequence in $V \backslash \{0\}$ such that $\{\pi(v^{(k)})\}$ is converges to $\pi(v)$ where $v = \Sigma_{j=1}^d v_j e_j$ with $v_d \neq 0$, and $\{n_k\}$ be a sequence in $\mathbb{N}$ such that $n_k \to \infty$ and $\overline{U}^{n_k}(\pi(v^{(k)})) \to \pi(w)$, for some $w = \Sigma_{j=1}^d w_j e_j$ with $w_d \neq 0$. Replacing $\{v^{(k)}\}$ by suitable multiples we may assume that $v_j^{(k)} \to v_j$ for all $j = 1, \ldots, d$. Since $v_d \neq 0$ by (2) we have $\overline{U}^{n_k}(\pi(v^{(k)})) \to \pi(e_1)$. As $d \geq 2$ this shows that $P(V) \backslash \pi(<e_1, \ldots, e_{d-1}>)$ is a stably wandering set. In particular $\Omega(\overline{U}) \subset \pi(<e_1, \ldots, e_{d-1}>)$. This proves (ii). □

Remark 2.2. Let the notation be as above, with $d \geq 2$. As $\Psi(U) \subset <e_1, \ldots, e_{d-1}>$ it follows that $V \backslash <e_1, \ldots, e_{d-1}>$ is a stably wandering subset for the U-action. However, in contrast with the assertion in (i) for $\Psi(U)$, $\Psi(\overline{U})$ is not contained in $\pi(<e_1, \ldots, e_{d-1}>)$. This may be seen as follows. For all $k \in \mathbb{N}$ let $v^{(k)} = -ke_{d-1} + e_d$. Then $\overline{U}^k(\pi(v^{(k)})) = \pi(e_d)$ for all k, and since $\pi(v^{(k)}) \to \pi(e_{d-1})$, this shows that $\pi(e_d) \in \Psi(\overline{U})$.

Remark 2.3. Let the notation be as above. Further, let $F = <e_1>$, and W be the subspace defined to be $<e_1, \ldots, e_{d-1}>$ if $d \geq 2$ and $W = <e_1>$ for $d = 1$. Then, together with the above, clearly $\Psi(U) \subset W$ and $\Psi(\overline{U}) \subset \pi(W)$ for all $d \geq 1$. We note that for all $d \geq 1$, F is the space of all points fixed by U, and W is the subspace spanned by $F \cup \{U(v) - v \mid v \in V\}$. This characterisation turns out to be of significance in the sequel. We note that if U is a unipotent linear transformation of V which is not necessarily indecomposable, F is the subspace consisting of all points fixed by U, and W is the subspace spanned by $F \cup \{U(v) - v \mid v \in V\}$. then F and W are the sums of the corresponding subspaces, defined as above, in the indecomposable components.

We now deduce the following.

Proposition 2.4. *Let U be a unipotent linear transformation of V. Let F be the subspace of V consisting of all points fixed by U, and W be the subspace spanned by $F \cup \{U(v) - v \mid v \in V\}$. Then we have the following:*

i) for $v \in V$, $L^+(v, U)$ and $L^-(v, U)$ are contained in F;

ii) for $v \in V \backslash \{0\}$, $L^+(\pi(v), \overline{U})$ and $L^-(\pi(v), \overline{U})$ are contained in $\pi(F)$;

iii) $\Omega(V, U) \subset \Psi(V, U) \subset W$;

iv) $P(V) \backslash \pi(W)$ is a stably wandering set of the $\overline{U}$-action on $P(V)$; in particular, the nonwandering set $\Omega(P(V), \overline{U})$ is contained in $\pi(W)$.

Proof. For an indecomposable unipotent transformation the assertions are immediate from the observations above.

Now consider the general case. We decompose V by Jordan canonical form, as $V_1 \oplus \cdots \oplus V_l$, for some $l \geq 1$, such that for each $i = 1, \ldots, l$, V_i is U-invariant and the restriction of U to V_i is indecomposable as a unipotent transformation. Since the convergence involved is componentwise, assertions (i) and (iii) as in the proposition follow immediately from the special case of indecomposable transformations, and Remark 2.3.

To prove the assertions for the projective transformation $\overline{U}$ we proceed as follows. For each $i = 1, \ldots, l$ let $\Phi_i = V \backslash \oplus_{j \neq i} V_j$; namely Φ_i is the set of points for which the i-component is nonzero. Then $\pi(\Phi_i)$ is an open subset of $P(V)$ and we have a natural projection map $\eta_i : \pi(\Phi_i) \to P(V_i)$. Now let $v \in V \backslash \{0\}$, and $w \in V \backslash \{0\}$ be such that $\pi(w) \in L^+(\pi(v), \overline{U})$. Then we see that for all i such that $w \in \Phi_i$, necessarily $v \in \Phi_i$, and $\eta_i(\pi(w)) \in L^+(\eta_i(\pi(v)), \overline{U})$. From the special case of indecomposable transformations and Remark 2.3 it follows that $\eta_i(w)$ is contained in $\pi(F)$. Since this holds for all i such that $w \in \Phi_i$ it follows that $w \in F$. This shows that $L^+(\pi(v), \overline{U})$ is contained in $\pi(F)$; similarly we get that $L^-(\pi(v), \overline{U})$ is contained in $\pi(F)$. This proves assertion (ii).

For all $i = 1, \ldots, l$ let W_i be the subspace which is the image of W (as in the hypothesis) in V_i, under the natural projection of V onto V_i with respect to the decomposition as above. We note that by the special case of indecomposable transformations and Remark 2.3 $P(V_i) \backslash \pi(W_i)$ is a stably wandering set for $(P(V_i), \overline{U})$, for all $i = 1, \ldots, l$. For every $v \in V \backslash \{0\}$ such that $\pi(v) \in P(V) \backslash \pi(W)$ there exists i, $1 \leq i \leq l$, such that $v \in \Phi_i$ and $\eta_i(\pi(v)) \in P(V_i) \backslash \pi(W_i)$. Together with the preceding conclusion this shows that $P(V)\backslash\pi(W)$ is a stably nonwandering set for the $\overline{U}$-action on $P(V)$. In particular the nonwandering set $\Omega(P(V),\overline{U})$ is contained in $\pi(W)$. This proves (iv), and completes the proof of the proposition. $\square$

3 General linear and projective transformations

We shall next consider the nonwandering sets and limit sets of general linear and projective transformations.

As before let V be a finite-dimensional K-vector space where $K=\mathbb{R}$ or $\mathbb{C}$, and let $P(V)$ be the corresponding projective space. Let T be a nonsingular linear transformation of V. For every positive real number λ let V_λ denote the largest T-invariant subspace of V such that all (possibly complex) eigenvalues of the restriction of T to V_λ are of absolute value λ. Clearly V_λ is nontrivial only for finitely many λ, and letting Λ to be the set of all λ for which V_λ is nonzero we have

$$V=\oplus_{\lambda>0}V_\lambda=\oplus_{\lambda\in\Lambda}V_\lambda,$$

direct sums; we note that V_λ is the sum of all generalised eigenspaces with respect to T corresponding to all eigenvalues with absolute value λ.

We equip V with a norm, denoted by $\|\cdot\|$. Using Jordan decomposition it can be seen that for any $\theta>1$ there exists a $c>1$ such that for all $v\in V_\lambda$ and all $i=1,2,\ldots$

$$c^{-1}\theta^{-i}\lambda^i\|v\|\leq\|T^i(v)\|\leq c\theta^i\lambda^i\|v\|. \tag{3}$$

For $\lambda\in\Lambda$ we denote by T_λ the restriction of T to V_λ, and by $\overline{T}_\lambda$ the restriction of $\overline{T}$ to $\pi(V_\lambda)$.

Proposition 3.1. *Let the notation be as above. Then the following statements hold:*
i) for any $v=\Sigma_{\lambda>0}v_\lambda$, $L^+(v,T)=L^+(v_1,T_1)$ (including when $V_1=\{0\}$);
ii) for any $v\in V\backslash\{0\}$, $L^+(\pi(v),\overline{T})=\cup_{\lambda\in\Lambda}L^+(\pi(v_\lambda),\overline{T}_\lambda)$;
iii) $\Omega(V,T)=\Omega(V_1,T_1)$.
iv) $\Omega(P(V),\overline{T})=\cup_{\lambda\in\Lambda}\Omega(\pi(V_\lambda),\overline{T}_\lambda)$.

Proof. Assertions (i) and (ii) follow from straightforward computation of limits, using (3) above; we omit the details.

Now, for $v=\Sigma_{\lambda>0}v_\lambda\in V$, we see from (3) that if $v_\lambda\neq0$ for some $\lambda\neq1$ then v is wandering. Also if $1\in\Lambda$ and v_1 is wandering with respect to T_1 then v is wandering. Thus we get that $\Omega(V,T)\subset\Omega(V_1,T_1)$. On the other hand if $v\in V_1$ is nonwandering for T_1 then clearly it is nonwandering for T. This proves (iii).

Finally, let $p = \pi(v)$, where $v = \Sigma_{\lambda \in \Lambda} v_\lambda \in V$. If there are $\lambda_1, \lambda_2 \in \Lambda$ such that $\lambda_1 \neq \lambda_2$ and v_{λ_1} and v_{λ_2} are nonzero then using (3) we see that p has a neighbourhood U such that $\{\overline{T}^i(U)\}$ are pairwise disjoint, so p is wandering in this case. Therefore $\Omega(P(V), \overline{T}) \subset \cup_{\lambda \in \Lambda} \Omega(\pi(V_\lambda), \overline{T}_\lambda)$. The other way inclusion is obvious, and thus we get (iv). $\square$

We note that for any $\lambda \in \Lambda$, $\Omega(\pi(V_\lambda), \overline{T}_\lambda)$ is the same as $\Omega(\pi(V_\lambda), \overline{\lambda^{-1} T_\lambda})$, and that all eigenvalues of $\lambda^{-1} T_\lambda$ over V_λ are of absolute value 1. Therefore, in view of Proposition 3.1 to determine the nonwandering sets, and similarly the limit sets, of linear as well as projective transformations it suffices to consider T for which all eigenvalues are of absolute value 1.

To analyse the sets further we use Jordan decomposition of linear transformations. As usual we denote by $GL(V)$ the group of nonsingular linear transformations of V (over K), equipped with the usual topology. We recall that $S \in GL(V)$ is said to be *semisimple* if it is diagonalisable over $\mathbb{C}$ (even if $K = \mathbb{R}$). By Jordan decomposition any $T \in GL(V)$ can be written uniquely as $T = SU$, where $S, U \in GL(V)$, S is semisimple, U is unipotent and S and U commute with each other; S and U are called the semisimple and unipotent Jordan components of T. Now let T be such that all eigenvalues of T are of absolute value 1. Then the eigenvalues of the semisimple Jordan component S of T are also of absolute value 1, and since it is diagonalisable this implies that S is contained in a compact subgroup of $GL(V)$. We now prove the following:

Proposition 3.2. *Let $T : V \to V$ be a linear transformation such that all eigenvalues of T are of absolute value 1. Let $T = SU$ be the Jordan decomposition of T in $GL(V)$, where S and U are the semisimple and unipotent components respectively. Let F be the subspace consisting of points fixed by U, and W the subspace spanned by $F \cup \{U(v) - v \mid v \in V\}$. Then the following conditions hold:*

i) for $v \in V$, $L^+(v, T)$ and $L^-(v, T)$ are contained in F;

ii) for $v \in V \backslash \{0\}$, $L^+(\pi(v), \overline{T})$ and $L^-(\pi(v), \overline{T})$ are contained in $\pi(F)$;

iii) $\Omega(V, T) \subset \Psi(V, T) \subset W$;

iv) $\Omega(P(V), \overline{T}) \subset \pi(W)$.

Proof. Recall that S is contained in a compact subgroup, and let $\mathcal{C}$ be the smallest compact subgroup of $GL(V)$ containing S, namely the closure of the subgroup generated by S. Since U commutes with S, it follows that the subspaces F and W are invariant under the action of S, and hence also under the action of all elements of $\mathcal{C}$.

Now let $v \in V$ and suppose that $T^{n_i}(v) \to w \in V$ for a sequence $\{n_i\}$ in $\mathbb{N}$, with $n_i \to \infty$. Passing to a subsequence of $\{n_i\}$ we may assume that S^{n_i} is convergent, say $S^{n_i} \to C \in \mathcal{C}$. As S and U commute with each other, it follows that $U^{n_i}(v) = S^{-n_i}T^{n_i}(v) \to C^{-1}(w)$. Therefore by Proposition 2.4 $C^{-1}(w) \in L^+(v, U) \subset F$. Since F is invariant under the action of $\mathcal{C}$ it follows that $w \in F$. This shows that $L^+(v, T) \subset F$ for all $v \in V$. Similarly we see that $L^-(v, T) \subset F$ for all $v \in V$. This proves (i). A similar argument with the projective action, together with Proposition 2.4 shows that (ii) holds.

Now let $w \in \Psi(V, T)$. Then we have a convergent sequence $\{v_i\}$ in V and a sequence $\{n_i\}$ in $\mathbb{N}$ with $n_i \to \infty$ such that $T^{n_i}(v_i) \to w$. Arguing as in the proof of (i), using Proposition 2.4, we see that there exists a $C \in \mathcal{C}$ such that $C^{-1}(w) \in W$. Since W is $\mathcal{C}$-invariant this shows that $w \in W$. Thus we get that $\Psi(V, T) \subset W$, which proves (iii).

Now let $v \in V\backslash\{0\}$ be such that $\pi(v) \in \Omega(P(V), \overline{T})$. Then by Lemma 1.1 there exist sequences $\{v_i\}$ in $V\backslash\{0\}$ and $\{n_i\}$ in $\mathbb{N}$ such that $\pi(v_i) \to \pi(v)$, $n_i \to \infty$, and $\overline{T}^{n_i}(\pi(v_i)) \to \pi(v)$. We may assume the S^{n_i} converges, say $S^{n_i} \to C \in \mathcal{C}$, as $i \to \infty$. Then $\overline{U}^{n_i}(\pi(v_i)) = \overline{S}^{-n_i}\overline{T}^{n_i}(\pi(v_i)) \to \pi(C^{-1}(v))$, as $i \to \infty$. Suppose that $\pi(v) \notin \pi(W)$. Since $\pi(v_i) \to \pi(v)$ and $P(V)\backslash\pi(W)$ is a stably wandering set for the action of $\overline{U}$ on $P(V)$ (see Proposition 2.4) the preceding conclusion implies that $\pi(C^{-1}(v)) \in \pi(W)$, or equivalently $C^{-1}(v) \in W$. Since W is invariant under the action of C we get that $v \in W$. But this contradicts the assumption that $\pi(v) \notin \pi(W)$, and hence shows that $\pi(v) \in \pi(W)$. Therefore $\pi(v) \in \Omega(P(V), \overline{T})$. This proves (iv) and completes the proof of the proposition. □

We note that if $T = SU$ is the Jordan decomposition of T, where S and U are the semisimple and unipotent components respectively, then the subspace F consisting of all vectors fixed by U is the largest T-invariant subspace of V such that the restriction of T to F is a semisimple linear transformation, and the subspace W as above, namely the subspace spanned by $F \cup \{U(v) - v \mid v \in V\}$, is the smallest T-invariant subspace such that $F \subset W$ and the factor of T on the quotient vector space V/W is a semisimple linear transformation. Propositions 2.4 and 3.2 together therefore yield the following:

Theorem 3.3. *Let $T : V \to V$ be a nonsingular linear transformation of a K-vector space V where $K = \mathbb{R}$ or $\mathbb{C}$. Let F be the largest T-invariant subspace of V such that the restriction of T to F is semisimple (as a linear transformation) and W be the smallest T-invariant subspace of V such that $F \subset W$ and the factor of T on V/W is semisimple. For each $\lambda \in \mathbb{R}^+$ let V_λ be the maximal T-invariant subspace of V such that all eigenvalues of T over V_λ are of absolute value λ, and Λ be the (finite) subset of $\mathbb{R}^+$ consisting of all λ for which V_λ is nonzero. Then the following conditions are satisfied:*

i) for any $v \in V$, $L^+(v, T)$ and $L^-(v, T)$ are contained in $V_1 \cap F$;

ii) for any $v \in V\backslash\{0\}$, the limit sets $L^+(\pi(v), \overline{T})$ and $L^-(\pi(v), \overline{T})$ are contained in $\cup_{\lambda\in\Lambda}(\pi(V_\lambda) \cap \pi(F))$;

iii) $\Omega(V, T)$ is contained in $V_1 \cap W$;

iv) $\Omega(P(V), \overline{T})$ is contained in $\cup_{\lambda\in\Lambda}(\pi(V_\lambda) \cap \pi(W))$.

Remark 3.4. Let the notation be as in Theorem 3.3. Then $W = V$ if and only if $F = V$; this can be seen from the characterisation of the two subspaces in terms of the unipotent Jordan component of T, and Remark 2.3. We note also that F is the largest T-invariant subspace of W on which the restriction of T is semisimple.

We now deduce the following corollary about the supports of conservative measures of linear and projective transformations. We recall that the result applies in particular to supports of finite invariant measures, and ergodic quasi-invariant measures (see Remark 1.6).

Corollary 3.5. *Let the notation be as in Theorem 3.3. Then the following conditions are satisfied:*

i) for any conservative measure μ of (V, T), $\operatorname{supp}\mu$ *is contained in $V_1 \cap F$;*

ii) for any conservative measure μ of $(P(V), \overline{T})$, $\operatorname{supp}\mu$ *is contained in the set $\cup_{\lambda\in\Lambda}(\pi(V_\lambda) \cap \pi(F))$.*

Proof. To begin with we note that in the case of finite invariant measures (in the place of the general conservative measures) the assertions in the corollary follow immediately from Theorem 3.3 and the Poincaré recurrence lemma, namely assertion (ii) in Proposition 1.4.

We now consider the general case of conservative measures, for the case of projective transformations. The proof will be by induction on the dimension of V. The assertion being clear for dimension 1, we shall proceed with the general case, assuming that the statement holds in all dimensions lower than the one under consideration. Now let μ be a conservative measure of $(P(V), \overline{T})$. Then $\operatorname{supp}\mu$ is contained in the nonwandering set $\Omega(\overline{T})$, and hence by Theorem 3.3 it is contained in $\pi(W)$, where, as before, W is the smallest T-invariant subspace of V such that $F \subset W$ and the factor of T on V/W is semisimple. We can identify $\pi(W)$ canonically with $P(W)$, and view μ as a measure on $P(W)$ invariant under the action of the transformation induced by the restriction of T to W, say T'. If $W = V$ then by Remark 3.4 $F = V$, and in that case we are through. Therefore in proving (ii) we may assume that W is a proper subspace of V. Recall also that F as above is also the largest subspace of W such that T' is semisimple, as a linear transformation of W. Now, since μ is conservative for the system $(P(W), \overline{T}')$, by the induction hypothesis it follows that $\operatorname{supp}\mu$ is contained in $\pi(F)$. Since we know already that it is also contained in $\cup_{\lambda\in\Lambda}\pi(V_\lambda)$, this proves statement (ii) in the corollary. Assertion (i) can be proved in the same way, and is even simpler, with appropriate substitutions in the argument. □

Corollary 3.6. *Let $T : V \to V$ be a nonsingular linear transformation of a K-vector space V where $K = \mathbb{R}$ or $\mathbb{C}$, and suppose that all eigenvalues of T are real and positive. Then we have the following:*

i) $L^+(\pi(v), \overline{T})$ and $L^-(\pi(v), \overline{T})$ consist of fixed points of $\overline{T}$;

ii) for any conservative measure μ of $\overline{T}$ on $P(V)$, $\operatorname{supp} \mu$ is contained in the set of fixed points of $\overline{T}$.

Proof. Since all eigenvalues of T are real and positive, for any $\lambda > 0$ the restriction of T to the subspace V_λ (notation as before) has λ as the only eigenvalue. Furthermore, the restriction of T to $V_\lambda \cap F$, being semisimple, consists of scalar multiplication by λ. Thus for T as in the hypothesis the induced action of $\overline{T}$ on $\pi(V_\lambda) \cap \pi(F)$ is trivial for every $\lambda \in \Lambda$. The assertions in the corollary now follow immediately from Theorem 3.3(ii) and Corollary 3.5 respectively. □

Theorem 3.3 and Corollary 3.5 in turn imply the following, about the dynamics of the systems arising from linear and projective transformations.

Corollary 3.7. *Let $T : V \to V$ be a nonsingular linear transformation of a finite-dimensional real vector space V. Let $X = V$ or $P(V)$, and τ be the homeomorphism T or $\overline{T}$ respectively. Let Y be a closed τ-invariant subset of X such that one of the following conditions holds:*

i) the set of points of Y that are recurrent for either τ or τ^{-1} is a dense subset of Y, or

ii) there exists a conservative measure μ with $\operatorname{supp} \mu = Y$.

Then Y is a disjoint union of minimal closed τ-invariant subsets. Furthermore, if Y is a minimal closed τ-invariant subset then there exists a homeomorphism ψ of Y onto a compact abelian subgroup C of $GL(V)$ such that $\psi\tau\psi^{-1} : C \to C$ is a translation of C by an element of C.

Proof. We shall consider the case of projective transformations, namely $X = P(V)$ and $\tau = \overline{T}$; the case of linear transformations can be dealt with in a similar way. Suppose condition (i) holds and let R be the set of points of Y which are recurrent for either τ or τ^{-1}. Then by Theorem 3.3 every point of R is contained in $\cup_{\lambda \in \Lambda} \pi(V_\lambda)$, in the notation as before. Since R is dense in Y this implies that Y is contained in $\cup_{\lambda \in \Lambda} \pi(V_\lambda)$. Also, if condition (ii) holds then by Corollary 3.5 Y is contained in $\cup_{\lambda \in \Lambda} \pi(V_\lambda)$. As Λ is finite and each V_λ is $\overline{T}$-invariant, this shows that it suffices to prove the statement in the corollary for $Y \cap \pi(V_\lambda)$ (separately) for each λ. In other words, we may assume that all eigenvalues of T are of the same absolute value. Furthermore, as the scalars act trivially on $\pi(V_\lambda)$ we may assume that all the eigenvalues are of absolute value 1. By Theorem 3.3 and Corollary 3.5 Y is also contained in $\pi(F)$, where F is a T-invariant subspace on which the restriction of T is semisimple. Thus get that it suffices to prove the corollary in the case when T is a semisimple linear transformation all whose eigenvalues are of absolute value 1. Such a T is contained in a compact subgroup C of $GL(V)$, which we may assume furthermore to be such that the cyclic subgroup generated by T is dense in C. It is now straightforward to deduce the assertions as in the corollary, considering the orbits of the C-action on $P(V)$. $\square$

We note also the following; it can also be proved directly, but we note it here as an observation from Theorem 3.3.

Corollary 3.8. *Let $T : V \to V$ be a linear transformation of a finite-dimensional vector space V over $\mathbb{R}$. If the closure of an orbit of T has only finitely many connected components, then it is compact.*

Proof. Let $v \in V$ and $C = \overline{\{T^j(v) \mid j \in \mathbb{Z}\}}$, the closure of the T-orbit of v. Suppose C has only finitely many connected components. Then every point of C is a limit point of the orbit $\{T^j(v) \mid j \in \mathbb{Z}\}$. Thus $C \subset L^+(v,T) \cup L^-(v,T)$ and hence by Theorem 3.3 it is contained in $V_1 \cap F$, in the notation of the theorem. The restriction of T to $V_1 \cap F$ is a semisimple transformation and all its eigenvalues are of absolute value 1. Therefore all its orbits have compact closure. Since C contains a dense orbit, namely that of v, this implies that C is compact. $\square$

4 Borel's density theorem

Given a subgroup Γ of $GL(V)$, where V a finite-dimensional vector space over $\mathbb{R}$ (for convenience we shall now restrict to the case of the field of real numbers, the analogous case for complex number being subsumed by this in the present context), it is of interest in many contexts to know the smallest algebraic subgroup of $GL(V)$ containing Γ. We recall that a subgroup H of $GL(V)$ is said to be *algebraic* if there exist polynomials $P_1, \dots, P_k$, for some $k \geq 1$, in the d^2 variables corresponding to the entries of the matrices, such that when $GL(V)$ is identified with $GL(d, \mathbb{R})$ via a suitable (fixed) basis, H consists precisely of the common zeros of $P_1, \dots, P_k$, namely, $\{g = (g_{ij}) \in GL(n, \mathbb{R}) \mid P(g_{ij}) = 0\}$; (over the field of real numbers it suffices to take $k = 1$, since the set of common zeros as above coincides with the set of zeros of $P_1^2 + \cdots + P_k^2$). The intersection of two algebraic subgroups is an algebraic subgroup, and hence given Γ as above there exists a smallest algebraic subgroup of $GL(V)$ containing Γ; it is called the Zariski closure of Γ. When Γ and

G are subgroups of $GL(V)$ such that $\Gamma \subset G$, we say that Γ is Zariski dense in G if G is contained in the Zariski closure of Γ.

For $r \in \mathbb{N}$, let $V^{(r)}$ be the r-th tensor power of V and $\rho^{(r)} : GL(V) \to GL(V^{(r)})$ denote the r-th tensor power representation of $GL(V)$. We note that given Γ and G as above, Γ is Zariski dense in G if and only if for every $r \in \mathbb{N}$ any one-dimensional subspace L of $V^{(r)}$ which is invariant under the action of Γ is also invariant under the action of G; this is a standard fact from the theory of algebraic groups, going back to the work of Chevelley, and can be proved along the lines of the proof of Theorem 5.1 in [1] (or may be deduced from it); for the reader not familiar with algebraic groups, it may be recommended that the above condition be taken as the definition of Γ being 'Zariski dense' in G, it being a natural condition by itself.

A theorem due to A. Borel, known as Borel's density theorem, asserts that if G is a connected semisimple Lie subgroup of $GL(V)$ with no compact normal subgroup of positive dimension, and Γ is a lattice in G (namely a discrete subgroup of G such that G/Γ admits a finite measure invariant under the action of G by left translations) then Γ is Zariski dense in G. (The original theorem is somewhat more general - see Remark 4.6 below.) This means for example that the subgroup $SL(n, \mathbb{Z})$ consisting of integral unimodular $n \times n$ matrices is Zariski dense in $SL(n, \mathbb{R})$, since $SL(n, \mathbb{Z})$ is a lattice in $SL(n, \mathbb{R})$ (see [17], Ch. X).

Furstenberg [11] gave a simple proof of Borel's density theorem, based on considerations of invariant measures under actions of groups on projective spaces. Motivated by Furstenberg's note and some variants of his argument obtained in [16], the question was considered in [2] in terms of nonwandering sets of projective transformations. Apart from the case of finite invariant measures as in Borel's density theorem the ideas were applied to ergodic quasi-invariant measures; actually conservative measures as we consider here, is now seen to be the 'right' setting for the purpose. While the argument in [2] essentially does yield the results stated there, as it stands it is incomplete in view of the error in an assertion about nonwandering sets mentioned in Section 2 above. How to rectify the argument is seen from the proof of Corollary 3.5. Also, we present below, with proof, somewhat more general results than in [2].

Theorem 4.1. *Let Γ and G be closed subgroups of $GL(V)$, where V is a finite-dimensional vector space over $\mathbb{R}$, with $\Gamma \subset G$. Let μ be a measure on G/Γ such that* $\operatorname{supp} \mu$ *contains the coset Γ. Let R be the set of all elements g in G such that all the eigenvalues of g (as an element of $GL(n, \mathbb{R})$) are real and positive, and μ is a conservative measure for the translation action of g on G/Γ. Let G' be the subgroup generated by $R \cup \Gamma$. Then Γ is Zariski dense in G'. In particular if R generates a dense subgroup of G then Γ is Zariski dense in G.*

Proof. To prove the first statement in the theorem it suffices to show that if, for some $r \in \mathbb{N}$, $p \in P(V^{(r)})$ is fixed under the action of $\rho^{(r)}(\Gamma)$ then it is also fixed under the action of $\rho^{(r)}(R)$. Let $r \in \mathbb{N}$ and a $p \in P(V^{(r)})$ fixed by $\rho^{(r)}(\Gamma)$ be given. Then we have an orbit map $\eta : G/\Gamma \to P(V^{(r)})$ defined by $\eta(g\Gamma) = \rho^{(r)}(g)(p)$ for all $g \in G$. The map η is equivariant under the actions of G on the two spaces, and hence it follows that the measure $\eta(\mu)$ is conservative under the action of $\rho^{(r)}(g)$, for all $g \in R$. All eigenvalues of $\rho^{(r)}(g)$, $g \in R$, are real and positive, and hence by Corollary 3.6 it follows that the support of $\eta(\mu)$ is fixed pointwise by each $\rho^{(r)}(g)$, $g \in R$. Since by the condition in the hypothesis $\operatorname{supp}\mu$ contains the coset Γ, the point p as above is contained in the support of $\eta(\mu)$, and hence we get in particular that p is fixed under the action of $\rho^{(r)}(R)$, as sought to be proved. Therefore Γ is Zariski dense in G'. The second statement now follows from the fact that algebraic subgroups are closed in the usual topology on $GL(V)$. □

Remark 4.2. Let G and Γ be closed subgroups of $GL(V)$, with V as above, and suppose that $\Gamma \subset G$ and G/Γ admits a finite measure which is invariant under the G-action. Then the action of every element of G is conservative, and hence the theorem implies that the subgroup G' generated by the set of elements of G whose eigenvalues are all real and positive is contained in the Zariski closure of Γ. We note that G' contains in particular all unipotent elements contained in G. Now suppose that G is a connected Lie group. If, furthermore, G is a semisimple Lie group with no nontrivial compact factor group, then G is generated by the unipotent elements contained in G, and hence $G' = G$. If G is a more general connected Lie subgroup of $GL(V)$ then the subgroup H generated by the unipotent elements in it is a closed normal subgroup such that G/H is a direct product of a compact group with a vector group (i.e. $\mathbb{R}^d$ for some d). The subgroup G' evidently contains H, and G'/H is a vector group. In general it need not be the case that G/G' is compact; e.g. we can have one-parameter subgroups (topologically isomorphic to $\mathbb{R}$) of $GL(n, \mathbb{R})$ in which no nontrivial element has only real eigenvalues. However, if G is an algebraic subgroup, or an open subgroup of an algebraic subgroup (such a subgroup is called almost algebraic) then it turns out that G/G' is compact. The condition that G/G' is compact means, in a sense, that G' is large, and by Theorem 4.1 it is contained in the Zariski closure of Γ.

Along the lines of the above set of ideas the following is proved in [3].

Theorem 4.3. *Let V be a finite-dimensional vector space over $\mathbb{R}$ and G be an algebraic subgroup of $GL(V)$. Let μ be a probability measure on $P(V)$ and let $I(\mu)$ and $J(\mu)$ be the closed subgroups defined by*

$$I(\mu) = \{g \in G \mid \mu \text{ is preserved under the action of } g\}$$

and

$$J(\mu) = \{g \in G \mid g(v) = v \text{ for all } v \in \operatorname{supp} \mu\}.$$

Then $I(\mu)/J(\mu)$ is compact.

The reader is also referred to [5] for similar results for automorphisms groups of Lie groups, in the place of $GL(V)$ (the latter being the automorphism group of V).

Theorem 4.1 can also be applied in situations where μ may not be a finite invariant measure. To this end we note the following corollary from [2].

Corollary 4.4. *Let G and Γ be as in Theorem 4.1. Let μ be a measure on G/Γ which is quasi-invariant under the G-action and ergodic with respect to the action of an element $g_0 \in G$ which has only positive real eigenvalues. Then Γ is Zariski dense in G.*

Proof. Since μ is quasi-invariant for the G-action and conservative for the action of g_0 it follows that it is conservative for the action of any conjugate of g_0, namely gg_0g^{-1} with $g \in G$. Hence from the condition in the hypothesis and Theorem 4.1 we get that gg_0g^{-1} is contained in the Zariski closure of Γ, for all $g \in G$. Let H be the subgroup generated by gg_0g^{-1}, $g \in G$. Then H is a normal subgroup of G, and $\overline{H\Gamma}$ is a subgroup of G. Furthermore, for every $g \in G$, $g\overline{H\Gamma}/\Gamma$ is a closed subset of G/Γ invariant under the action of g_0. Since the action of g_0 is ergodic, and $\operatorname{supp} \mu$ is the whole of G/Γ (as μ is quasi-invariant under the G-action), it follows the g_0-action has a dense orbit; (see Proposition 1.3). Hence $g\overline{H\Gamma}/\Gamma = G/\Gamma$ for some g. That in turn means that $G = \overline{H\Gamma}$. As H is contained in the Zariski closure of Γ, this shows that Γ is Zariski dense in G, thus proving the corollary. □

As an application of Corollary 4.4 it was deduced in [2] that if M is a Riemannian manifold of constant negative curvature, such that the associated geodesic flow (defined on the unit tangent bundle of M) is ergodic (with respect to the Riemannian measure), then the fundamental group of M is Zariski dense in the group of isometries of M (which is a group isomorphic to $SO(n, 1)$, where n is the dimension of M).

We next prove a variation of Theorem 4.1, not involving considerations of the eigenvalues being real and positive.

Theorem 4.5. *Let G and Γ be as in Theorem 4.1, and let μ be a measure on G/Γ such that* $\operatorname{supp} \mu = G/\Gamma$. *Let $g_0 \in G$ be such that μ is conservative for the translation action of g_0 on G/Γ. Then there exists a closed subgroup G' of finite index in G, and a representation $\rho : G' \to GL(W)$ over a finite-dimensional vector space W, such that the following conditions are satisfied:*

i) $\ker \rho$ is contained in the Zariski closure of Γ, and *ii) if S denotes the subgroup of $GL(W)$ consisting of scalar transformations and $PGL(W) = GL(W)/S$, then for any $k \in \mathbb{N}$ such that $g_0^k \in G'$, $\rho(g_0^k)S$ is contained in a compact subgroup of $PGL(W)$.*

Proof. Let H be the Zariski closure of Γ and suppose that it is a proper subgroup of $GL(V)$. Then there exist $r \in \mathbb{N}$ and $p \in P(V^{(r)})$ (notation as before) such that H is precisely the subgroup consisting of elements g in $GL(V)$ for which $\rho^{(r)}(g)(p) = p$. Then we get a map $\eta : G/\Gamma \to P(V^{(r)})$ defined by $\eta(g) = \rho^{(r)}(g)(p)$ for all $g \in G$. Also, η is equivariant and hence the measure $\eta(\mu)$ is conservative under the action of $\rho^{(r)}(g_0)$.

We denote by π the canonical projection of $V^{(r)}\backslash\{0\}$ onto $P(V^{(r)})$. Let $\mathcal{E}$ be the family of all subsets E of $P(V^{(r)})$ of the form $\pi(L)$ where L is a finite union of vector subspaces of $V^{(r)}$ (as before, while taking the image in $P(V^{(r)})$ under π, 0 is to be omitted from consideration). It can be seen that the intersection of any collection of subsets from $\mathcal{E}$ is again an element of $\mathcal{E}$, and hence for any subset of $P(V^{(r)})$ there exists a smallest element of $\mathcal{E}$ containing it. Let E be the smallest element of $\mathcal{E}$ containing $\operatorname{supp}\eta(\mu)$. Let $W_1, \dots, W_s$ be the subspaces of $V^{(r)}$ such that $E = \cup_{j=1}^s \pi(W_j)$. Since $\operatorname{supp}\mu = G/\Gamma$ it follows that $\cup_{j=1}^s W_j$ is invariant under the action of $\rho^{(r)}(G)$. Therefore there exists a closed subgroup G' of finite index in G such that each W_j is $\rho^{(r)}(G')$-invariant. Now let j be such that $p \in \pi(W_j)$, and choose $W = W_j$. Let $\rho : G' \to GL(W)$ be the representation defined by $\rho(g)(w) = \rho^{(r)}(g)(w)$ for all $g \in G'$ and $w \in W$.

We now show that the assertions as in the statement of the theorem hold for the choices as above. If $g \in \ker\rho$ then $\rho^{(r)}(g)$ fixes W pointwise, and since $p \in \pi(W)$, it is fixed under the projective action of $\rho^{(r)}(g)$. By the choice of r and $p \in P(V^{(r)})$ this implies that g is contained in the Zariski closure of Γ. This proves (i). Now let $k \in \mathbb{N}$ and S be as in the statement of (ii). For any $\lambda > 0$ let $V_\lambda^{(r)}$ be the largest $\rho(g_0^k)$-invariant subspace on which all its eigenvalues are of absolute value λ, and Λ be the finite set of all λ's such that $V_\lambda^{(r)}$ is nonzero. Also let F be the largest $\rho(g_0^k)$-invariant subspace on which the restriction of $\rho(g_0^k)$ is semisimple. The measure $\eta(\mu)$ is conservative for the projective action of $\rho(g_0^k)$ and hence by Theorem 3.3 its support is contained in $\cup_{\lambda\in\Lambda}(\pi(V_\lambda^{(r)}) \cap \pi(F))$. As E is the smallest element of $\mathcal{E}$ containing the support of $\eta(\mu)$, this implies that W as above is contained in $V_\lambda^{(r)} \cap F$ for some $\lambda \in \Lambda$. Since the restriction of $\lambda^{-1}\rho^{(r)}(g_0^k)$ to $V_\lambda^{(r)} \cap F$ is contained in a compact subgroup of $GL(V_\lambda^{(r)} \cap F)$, the preceding conclusion implies that $\rho(g_0^k)S$ is contained in a compact subgroup of $PGL(W)$. This proves the theorem. $\square$

Remark 4.6. Consideration of Zariski density involves, a priori, the groups in question being realised as matrix groups. In general, given a connected Lie group

G and a closed subgroup Γ, we can consider the Zariski density of $\rho(\Gamma)$ in $\rho(G)$, under various finite-dimensional representations ρ. The adjoint representation of G arises naturally in this respect, in many contexts. Thus the original form of Borel's density theorem states that if G is a connected semisimple Lie group with no nontrivial compact factors (or equivalently no compact normal subgroup of positive dimension) and Γ is a lattice in G then $\mathrm{Ad}\,(\Gamma)$ is Zariski dense in $\mathrm{Ad}\,(G)$, where Ad denotes the adjoint representation of G over its Lie algebra, say $\mathcal{G}$. This can be deduced from Theorem 4.1 or Theorem 4.5, by considering the closures of $\mathrm{Ad}\,(\Gamma)$ and $\mathrm{Ad}\,(G)$ as subgroups of $GL(\mathcal{G})$, in the place of the subgroups Γ and G of $GL(V)$ in the theorem.

Zariski density results for subgroups of Lie groups have various interesting consequences. We shall not go into them here. The reader is referred to [17] for some of the basic applications.

5 Automorphisms and affine automorphisms of locally compact groups

Some of the basic examples of ergodic dynamical systems, especially in early literature, consist of (continuous) automorphisms and, more generally, affine transformations (composites of automorphisms with translations) of compact groups (see for example [18]). With this in background Halmos asked in his book [12] (see page 29) whether an automorphism of a noncompact locally compact group can be ergodic. The question, and various extensions of it have been the subject of many papers. The question has been answered in the negative and also considerably stronger versions of this have been proved; see [3], [4], [7], [8], [9], and various references cited there. We shall not go into the general theory around the question here, but content ourselves recalling some of the results along the way, obtained via the study of dynamics of linear and projective transformations; this involves in particular focussing on (connected) Lie groups.

In [3] the following theorem was proved, for quasi-invariant measures of automorphisms, via considerations of projective transformations.

Theorem 5.1. *Let G be a connected locally compact noncompact topological group. Then there exists a closed subgroup N of G, such that N is invariant under all continuous automorphisms of G, G/N is noncompact, and the following holds: if T is a continuous automorphism of G, Ω the nonwandering set of T, and $Y = \overline{\Omega N/N} \subset G/N$ then there exists an action of a compact group C on Y, such that* *the action induced by the factor of T on Y coincides with the action of an element of C.*

This may be compared with assertion (ii) in Corollary 3.7 in the particular case of $G = V$, a vector space. The theorem signifies that a similar assertion as in the special case holds in general, for a noncompact factor group of the given group G; perhaps the statement as in the theorem holds for a compact subgroup N of G, invariant under all continuous automorphisms of G, but this has not been ascertained.

It may be noted that the theorem implies in particular that a connected noncompact locally compact group does not admit an ergodic automorphism (with respect to the Haar measure, or with respect to any quasi-invariant measure whose support is the whole of G). It shows also that a connected noncompact locally compact group does not admit a continuous automorphism with a dense orbit. The corresponding assertion, namely that G as above does not admit an affine automorphism with a dense orbit, was also upheld in [4] for all affine automorphisms. We now give a sketch of how the dynamics of linear transformations is applied for the purpose.

Firstly we note that by a theorem of Montgomery and Zippin ([15], Chapter IV) every connected locally compact group G contains a unique maximal compact normal subgroup K such that G/K is a Lie group. The subgroup K is invariant under all continuous automorphisms of G, and this reduces the study of the question as above to the case of Lie groups.

Now let G be a connected Lie group. Let $\mathrm{Aut}\,(G)$ denote the group of all (continuous) automorphisms of G. It has the structure of a Lie group, with respect to the topology of uniform convergence on compact subsets (see for example [13]). In general it is not a connected Lie group; e.g. when G is a torus group of dimension $d \geq 2$ then $\mathrm{Aut}\,(G)$ is an infinite discrete group. We note also that except in the case when G is a torus, $\mathrm{Aut}\,(G)$ is of positive dimension. We denote by $\mathrm{Aff}\,(G)$ the group of all affine automorphisms of G, namely homeomorphisms of the form $T_g \circ \alpha$ where α is an automorphism of G and T_g denotes the translation by $g \in G$. Clearly $\mathrm{Aff}\,(G)$ can be realised as the semidirect product of $\mathrm{Aut}\,(G)$ and G, and we equip it with the topology and Lie group structure as a semidirect product.

Now let G be a connected Lie group other than a torus. Let $\mathcal{L}$ denote the Lie algebra of $\mathrm{Aff}\,(G)$ and Ad: $\mathrm{Aff}\,(G) \to GL(\mathcal{L})$ be the adjoint representation of $\mathrm{Aff}\,(G)$. Let $\mathcal{A}$ be the Lie subalgebra of $\mathcal{L}$ corresponding to $\mathrm{Aut}\,(G)$, and let r be the dimension of $\mathcal{A}$; since G is not a torus, $r > 0$. Let $V = \wedge^r \mathcal{L}$, be the r-th exterior power of $\mathcal{L}$ and $\rho = \wedge^r \mathrm{Ad} : \mathrm{Aff}\,(G) \to GL(V)$ be the r-th exterior power of the adjoint representation as above. Let $v_0 \in V$ be a nonzero vector contained in the one-dimensional subspace $\wedge^r \mathcal{A}$ of V. Let $\eta : G \to V$ be the map defined by $\eta(g) = \rho(g)(v_0)$ for all $g \in G$. Consider any $g \in G$ such that $\rho(g)(v_0) = v_0$. Then

$\mathcal{A}$ is Ad (g)-invariant, and hence g normalises the subgroup Aut $(G)^0$, the connected component of the identity in Aut (G). Since Aff (G) is a semidirect product of Aut (G) and G, this implies that all elements of Aut $(G)^0$ commute with g, in Aff (G). This means that g as above is fixed by all automorphisms from Aut $(G)^0$. This is quite a strong condition in general, and means in particular that g is contained in the centre of G, since all inner automorphisms of G (automorphisms induced by conjugation by elements of G) are contained in Aut $(G)^0$. Suppose that there is no nontrivial element satisfying this condition; this holds for instance if the center of G is trivial. Then η is an injective map, and gives an embedding of G in V (as a subset).

Now let T be any affine automorphism of G, say $T = T_g \circ \alpha$, where $g \in G$ and $\alpha \in$ Aut (G). Clearly v_0 as above is an eigenvector of $\rho(\alpha)$. Let λ be the eigenvalue of $\rho(\alpha)$ corresponding to v_0. Then we have, for any $x \in G$,

$$\begin{aligned}\eta(T(x)) = \rho(g\alpha(x))(v_0) = \rho(T_g \circ \alpha \circ T_x \circ \alpha^{-1})(v_0) = \rho(T \circ T_x \circ \alpha^{-1})(v_0) \\ = \lambda^{-1}\rho(T)(\rho(x)(v_0)) = \lambda^{-1}\rho(T)(\eta(x)).\end{aligned}$$

This means that under the embedding η, application of T on G corresponds to restriction of the linear transformation τ of V, defined by $\tau(v) = \lambda^{-1}\rho(T)(v)$ for all $v \in V$.

Thus we have embedded G, under a suitable condition, in a vector space V in such a way that any affine automorphism of G is the restriction of a linear transformation τ of V. Now suppose that there exists an affine automorphisms T of G with a dense orbit in G. Let $x \in G$ be such that $\{T^j(x) \mid j \in \mathbb{Z}\}$ be dense in G. Then considering the images under η we conclude that, with $v = \eta(x)$, $\{\tau^j(v) \mid j \in \mathbb{Z}\}$ is dense in $\rho(G)(v_0)$. This implies in particular that the closure of the τ-orbit of v is connected. Then by Corollary 3.8 it is compact. Hence we get that $\rho(G)(v_0)$ is contained in a compact subset. This is a strong condition on G, irrespective of what $\rho : G \to GL(V)$ is, so long as it is faithful as in this case, and holds only if G is a product of a compact group with $\mathbb{R}^d$ for some d. In the situation when G has an affine automorphism with a dense orbit this further implies that G is compact, as desired.

The general case, where η as above may not be injective, is reduced to the above special case by successively going modulo the centre; we shall not go into the of this details here; (see [4]).

Remark 5.2. In analogy with the question of automorphisms of affine automorphisms acting ergodically or with a dense orbit, one may ask about groups of automorphisms or affine automorphisms acting ergodically or with a dense orbit. For the case of abelian groups of automorphisms this question has been studied in [7] and [9]. We shall however not go into the details here, the themes involved being rather different.

6 Intersections of orbits of linear transformations with subspaces

In this section I discuss some interesting questions encountered in the course of the work in [10], relating to certain dynamical aspects of linear or projective transformations.

Let V be a finite-dimensional vector space over $\mathbb{R}$ and let $T : V \to V$ be a nonsingular linear transformation. Let $v \in V$ and for any subspace W of V let $\nu(v, W) = \{n \in \mathbb{N} \mid T^n(v) \in W\}$, namely $\nu(v, W)$ consists of the 'times of visiting W', starting from the point v, under iteration by T. How are the sets $\nu(v, W)$, for various choices of $v \neq 0$ and proper subspaces W (depending on T)? If W contains a nonzero subspace invariant under T^r for some $r \in \mathbb{N}$ then it can be seen that for any $j \in \mathbb{N}$ there exist $v \in V$ such that $\nu(v, W)$ contains the infinite arithmetic progression $\{j + kr \mid k \in \mathbb{N}\}$. On the other hand it can be seen that if T is a unipotent transformation then for any v whose orbit is not wholly contained in W, $\nu(v, W)$ has at most $d - 1$ elements, where d is the dimension of V. Is the most general situation a combination of these two cases?

Conjecture 6.1. *Let $T : V \to V$ be a nonsingular linear transformation of a finite-dimensional vector space over $\mathbb{R}$. Then there exists $m \in \mathbb{N}$ such that for any $v \in V$ and any subspace W, if $\nu(v, W)$ contains m elements then it contains an infinite arithmetic progression.*

Furthermore, it seems plausible that m as above admits an upper bound depending only on the dimension of V, and that when $\nu(v, W)$ does contain an infinite arithmetic progression, there exists one such progression whose complement in $\nu(v, W)$ has at most m elements.

The following proposition shows that the conjecture is true under an additional condition. The result is noted in [10], Remark 5.2; here we give the argument in some detail. For the proof we need the following lemma.

Lemma 6.2. *Let f be a function on $\mathbb{R}$, of the form $f(t) = \Sigma_{i=1}^r e^{\alpha_i t} Q_i(t)$ for all $t \in \mathbb{R}$, where $\alpha_1, \ldots, \alpha_r$ are distinct real numbers, and $Q_1, \ldots, Q_r$ are polynomials. If f is not identically zero then the number of zeros of f, namely the cardinality of the set $\{t \in \mathbb{R} \mid f(t) = 0\}$, is at most $r - 1 + \Sigma_{i=1}^r \deg Q_i$.*

Proof. Let $\mathcal{C}$ denote the class of nonzero (not identically zero) functions on $\mathbb{R}$ of the form $\Sigma_{i=1}^{r} e^{\alpha_i t} Q_i(t)$, where $\alpha_1, \ldots, \alpha_r$ are real numbers, and $Q_1, \ldots, Q_r$ are polynomials; for any $f \in \mathcal{C}$ there exists a unique expression as above, for which $\alpha_1, \ldots, \alpha_r$ are distinct. For $f = \Sigma_{i=1}^{r} e^{\alpha_i t} Q_i(t)$, with α_i's distinct, let $m(f) = r - 1 + \Sigma_{i=1}^{r} \deg Q_i$. We proceed by induction on the number $m(f)$. We note that $m(f) \geq 0$ for all $f \in \mathcal{C}$. If $m(f) = 0$ then f has the form $ae^{\alpha t}$, with $a \in \mathbb{R}^*$, which has no zero, and hence the desired contention holds. Now consider any $f = \Sigma_{i=1}^{r} e^{\alpha_i t} Q_i(t) \in \mathcal{C}$ as above, with $m(f) \geq 1$, assuming that the contention of the lemma holds for all $g \in \mathcal{C}$ with $m(g) < m(f)$. Taking out the factor $e^{\alpha_1 t}$, which has no zeros, we may also assume that $\alpha_1 = 0$. Now consider the derivative f' of f. We have

$$f'(t) = \Sigma_{i=1}^{r} \alpha_i e^{\alpha_i t} Q_i(t) + e^{\alpha_i t} Q_i'(t) = Q_1'(t) + \Sigma_{i=2}^{r} e^{\alpha_i t}(\alpha_i Q_i(t) + Q_i'(t)).$$

Now if f' is identically zero then f is constant, and hence we are through in this case. We may therefore suppose that f' is not identically zero. Then $f' \in \mathcal{C}$. Also, $m(f') = r - 1 + (\deg Q_1 - 1) + \Sigma_{i=2}^{r} \deg Q_i = m(f) - 1$, if $\deg Q_1 \geq 1$, and if Q_1 is a constant polynomial then $f'(t) = \Sigma_{i=2}^{r} e^{\alpha_i t}(\alpha_i Q_i(t) + Q_i'(t))$, and hence $m(f') = (r-1) - 1 + \Sigma_{i=2}^{r} \deg Q_i = m(f) - 1$. Thus in either case we have $m(f') = m(f) - 1$. Therefore by the induction hypothesis f' has at most $m(f) - 1$ zeros. Therefore by the mean value theorem we get that f has at most $m(f)$ zeros. This proves the lemma. □

Proposition 6.3. *Let $T : V \to V$ be a nonsingular linear transformation of a finite-dimensional vector space V of dimension d over $\mathbb{R}$. Suppose that all eigenvalues of T are real. Let $v \in V$ and W be any subspace of V. Then $\nu(v, W)$ either contains all even or all odd natural numbers, or has at most $2(d-1)$ elements. If, furthermore, all eigenvalues of T are positive then either $\nu(v, W) = \mathbb{N}$ or it has at most $(d-1)$ elements.*

Proof. We note that $k \in \nu(v, W)$ if and only if either $k = 2l$ and $(T^2)^l(v) \in W$ or $k = 2l+1$ and $(T^2)^l(Tv) \in W$, for some $l \in \mathbb{N}$. As the eigenvalues of T^2 are positive, this shows that if we prove the second statement in the proposition, then the first one follows.

We now suppose that all eigenvalues of T are positive. Let V' be the subspace of V spanned by $\{T^k(v) \mid k \in \mathbb{N}\}$. We have to show that for all subspaces W which do not contain V', $\nu(v, W)$ has at most $d - 1$ elements. In proving this, without loss of generality we may assume, and we shall, that W is a hyperplane (subspace of codimension 1).

Now let φ be a linear functional on V with W as its kernel. Let $\lambda_1, \ldots, \lambda_r$ be the distinct eigenvalues of T. By the Jordan canonical form there exists a basis $\{e_{ij} \mid$

$i = 1, \ldots, r;\ j = 1, \ldots, d_i\}$, where d_i is the dimension of the generalised eigenspace corresponding to the eigenvalue λ_i, such that for $v = \Sigma_{i,j}\, v_{ij} e_{ij}$ we have $T^k(v) = \Sigma_{i,j}\, \lambda_i^k P_{ij}(k) e_{ij}$ where P_{ij} is a polynomial of degree at most $d_i - 1$ (with coefficients depending on v_{ij}'s). Then for $v = \Sigma_{i,j}\, v_{ij} e_{ij}$ we have $\varphi(T^k(v)) = \Sigma_{i,j}\, \lambda_i^k P_{ij}(k) \varphi(e_{ij})$ which may be written as $\Sigma_{i=1}^r \lambda_i^k Q_i(k)$, where $Q_i = \Sigma_{j=1}^{d_i} \varphi(e_{ij}) P_{ij}$ is a polynomial of degree at most $d_i - 1$. As λ_i are positive we may write them as e^{α_i}, where $\alpha_i = \log \lambda_i$, for all $i = 1, \ldots, r$, and applying Lemma 6.2 we then see that there are at most $r - 1 + \Sigma\, (d_i - 1) = d - 1$ integers k such that $\varphi(T^k(v)) = 0$. Since W is the kernel of φ this shows that $\nu(v, W)$ has at most $d - 1$ elements. This proves the proposition. $\square$

Conjecture 6.1 can also be verified, along lines of the proof of Proposition 6.3 under the slightly weaker condition that all eigenvalues of T be roots of real numbers, namely $z \in \mathbb{C}$ such that $z^n \in \mathbb{R}$ for some $n \in \mathbb{N}$. We do not know if a similar assertion holds under a still weaker condition. The following weaker statement holds however. It is a special case of Proposition 5.1 in [10], for vectors in the place of subspaces there. The proof depends on a version of the theorem of Szemeredi, due to Gowers, on finding arithmetic progressions in sets of integers. We shall however not go into the details of the proof.

Proposition 6.4. *Let V be a finite-dimensional vector space over $\mathbb{R}$ and $T : V \to V$ be a nonsingular linear transformation. Then for every $\delta > 0$ there exists $k \in \mathbb{N}$ such that the following holds: if $v \in V$ and there exists a proper subspace W of V such that*

$$|\{1 \le j \le k \mid T^j(v) \in W\}| > \delta k,$$

(where $|\cdot|$ denotes the cardinality of the set), then there exists $r \in \mathbb{N}$ such that v is contained in a proper T^r-invariant subspace of V.

7 The case of p-adic vector spaces

Theory analogous to what we discussed in the preceding sections for linear transformations of real vector spaces works also for (finite-dimensional) vector spaces over p-adic fields. There are however differences in certain respects. We briefly go over the results in this case. We will not go into the proofs, which typically are similar to the real case, with appropriate modifications.

Let K be a locally compact totally disconnected field (local field) of characteristic zero; we note that such a field is a finite extension of the field $\mathbb{Q}_p$ of p-adic numbers, for some prime number p. Let V be a finite-dimensional K-vector space, and T be a nonsingular K-linear transformation of V. Let $P(V)$ be the projective space corresponding to V, consisting of equivalence classes of nonzero points of V, the equivalence being given by $v \sim w$ if $w = kv$ for some $k \in K^*$ (nonzero elements of K). As before, we denote by $\pi : V - \backslash\{0\}$ the natural quotient map, and for any subset S of V we write $\pi(S)$ for the image of $S\backslash\{0\}$ in $P(V)$. The linear transformation T induces a map of $P(V)$ which we call the projective transformation corresponding to T, and denote by $\overline{T}$. We are interested in the dynamics of T and $\overline{T}$.

We mention, to begin with, that unlike in the case of linear transformations of real vector spaces, in the case of totally disconnected fields as above every unipotent linear transformation is contained in a compact subgroup of $GL(V)$. In view of this, in the asymptotic dynamical behaviour of T and $\overline{T}$ the role of the unipotent Jordan component is quite different than in the case of the reals.

As before, for every positive real number λ let V_λ denote the largest T-invariant subspace of V such that all eigenvalues (in the algebraic closure of K) of the restriction of T to V_λ are of (p-adic) absolute value λ. Then $V = \oplus_{\lambda>0} V_\lambda = \oplus_{\lambda\in\Lambda} V_\lambda$, direct sums, where Λ is the finite set of λ's for which V_λ is nonzero. On each V_λ there exists a norm, denoted by $\|\cdot\|$ (as a p-adic vector space) such that $\|T(v)\| = \lambda\|v\|$ for all $v \in V_\lambda$; this can be deduced using Jordan decomposition and the fact that the unipotent elements are contained in a compact subgroup of $GL(V)$. Then $\|T^i(v)\| = \lambda^i\|v\|$ for all $v \in V_\lambda$ and $i \in \mathbb{Z}$, which may be compared with (3) in §3 for the case of real vector spaces; here we get a stronger relation, for a specific norm, than in the case of the reals. Using this we can prove, along the lines of the proof of Proposition 3.1, the following result corresponding to Theorem 3.3 in the real case.

Theorem 7.1. *Let the notation be as above. Then the following statements hold:*
i) For any $v \in V$, $L^+(v,T) = L^+(v_1,T_1) \subset V_1$;
ii) for any $v \in V\backslash\{0\}$, $L^+(\pi(v),\overline{T}) \subset \cup_{\lambda\in\Lambda}\pi(V_\lambda)$;
iii) $\Omega(V,T) = \Omega(V_1,T_1) = V_1$;
iv) $\Omega(P(V),\overline{T}) = \cup_{\lambda\in\Lambda}\Omega(\pi(V_\lambda),\overline{T}_\lambda) = \cup_{\lambda\in\Lambda}\pi(V_\lambda)$.

As in the real case one can deduce from this the following corollary for supports of conservative measures.

Corollary 7.2. *Let the notation be as above. Then the following holds:*
i) for any conservative measure μ *of* (V,T), $\operatorname{supp}\mu$ *is contained in* V_1;
ii) for any conservative measure μ *of* $(P(V),\overline{T})$, $\operatorname{supp}\mu \subset \cup_{\lambda\in\Lambda}\pi(V_\lambda)$.

In a way, these results may seem 'weaker' than the corresponding results in the real case, since unlike in the earlier case the sets on the right hand side are not contained in the space of the fixed points of the unipotent Jordan component (F in the corresponding earlier results). On the other hand, unlike in the real case, the restriction of T to V_1 is contained in a compact subgroup of $GL(V_1)$. Similarly it can be seen that for each $\lambda \in \Lambda$ (notation as above) $\overline{T}_\lambda$, which can be viewed as an element of the group $PGL(V_\lambda) = GL(V_\lambda)/K^*$ (where K^* is the subgroup of $GL(V_\lambda)$ consisting of nonzero scalar transformations), is contained in a compact subgroup of $PGL(V_\lambda)$; we note that since λ may not be the absolute value of any element of K, the argument for this, which we shall not go into, involves considering a finite extension field of K.

In the light of the above observations, and Theorem 7.1, it is easy to see that the statement as in Corollary 3.7 holds also for finite-dimensional vector spaces over p-adic fields (verbatim statement with reals replaced by p-adic). This can be applied to obtain results on Zariski closures of subgroups (analogous to Borel's density theorem, along the lines of the proof of Theorem 4.5) and supports of conservative measures of automorphisms of groups. We shall however not go into the details of this here.

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Permanent Address:
School of Mathematics
Tata Institute of Fundamental Research
Homi Bhabha Road, Colaba
Mumbai 400 005, India
E-mail: dani@math.tifr.res.in