# APPROACHING NEW POINTS BY APPLICATION OF LINEAR TRANSFORMATIONS

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ABSTRACT

We discuss some applications of the study of flows on homogeneous spaces to describe the set of points in the euclidean spaces which can be approached, starting from an initial point, by application of linear transformations from various special classes.

## INTRODUCTION

Let $E^n$ be the usual $n$-dimensional euclidean space; the elements of $E^n$ will be written as $n$-tuples $\mathbf{x} = (x_1, \ldots, x_n)$, where $x_1, \ldots, x_n$ are real numbers, namely the coordinates of $\mathbf{x}$. Let $T$ be a set of linear transformations of $E^n$. Suppose we are given two points $\mathbf{v} = (v_1, \ldots, v_n)$ and $\mathbf{w} = (w_1, \ldots, w_n)$. Consider the following question: can we reach $\mathbf{w}$ from $\mathbf{v}$ by a transformation from $T$? That is, does there exist a transformation $\sigma$ from $T$ such that $\sigma(\mathbf{v}) = \mathbf{w}$? In the cases that interest us $T$ will be countable. Then for each $\mathbf{v}$ the question is doomed to have negative answer for all but countably many $\mathbf{w}$. Let us however relax the question and ask the following: Can we *approach* $\mathbf{w}$ starting from $\mathbf{v}$ by transformations from $T$? that is, are there points arbitrarily near $\mathbf{w}$ that can be reached? in yet other words, can we find a sequence of transformations $\sigma_1, \sigma_2, \ldots, \sigma_k, \ldots$ in $T$ such that $\sigma_k(\mathbf{v})$ converges to $\mathbf{w}$ as $k \to \infty$? Let $C_T(\mathbf{v})$ be the set of points $\mathbf{w}$ in $E^n$ for which the answer to these questions is affirmative. $C_T(\mathbf{v})$ of course includes the set of $\mathbf{w}$ such that $\sigma(\mathbf{v}) = \mathbf{w}$ for some $\sigma$ in $T$. But while the latter is countable whenever $T$ is countable, the former can be a large set and perhaps even the whole of $E^n$.

In general it is a difficult problem to describe $C_T(\mathbf{v})$ for a given set of transformations. It is the purpose of this article to describe some results achieving this in certain specific natural situations. In §1 we describe the results; these can be understood without much mathematical background. Section 2 is devoted to giving an outline of the underlying ideas from dynamics involved in the approach; in §3 we give a proof of the main result stated in §1. In the concluding section we discuss some related questions.

## §1. Results on approximation

For any $m \geq 1$ let $S_m$ denote the set of linear transformations $\sigma$ of $E^m$ such that for all $\mathbf{v} = (v_1, \ldots, v_m)$ in $E_m$,

$$\sigma((v_1, \ldots, v_m)) = \left(\sum_{i=1}^{m} a_{1i} v_i, \ldots, \sum_{i=1}^{m} a_{mi} v_i\right), \quad (1)$$

where $a_{ij}$ are all integers and the determinant of the matrix $(a_{ij})$ is 1. It would be worthwhile to note that every such $\sigma$ can be expressed as a composite of transformations of the form $\sigma_{kl}$ or $\sigma_{kl}^{-1}$, where $1 \leq k \leq n$, $1 \leq l \leq n$, $k \neq l$ and $\sigma_{kl}$ is defined, for all $(v_1, v_2, \ldots, v_m)$ in $E^m$, by

$$\sigma_{kl}[(v_1, \ldots, v_m)] = (v_1, \ldots, v_{l-1}, v_l + v_k, v_{l+1}, \ldots, v_m). \quad (2)$$

The transformation $\sigma_{kl}$ consists of fixing the hyperplane defined by the equation $v_k = 0$ and shearing the other points along the $l$th axis by an amount equal to the $k$th coordinate.

A. Now consider the set of transformations $T = S_n$ of $E^n$. It is easy to see that if $\mathbf{v} = (v_1, \ldots v_n)$ is an integral vector (that is, $v_1, \ldots v_n$ are integers) then any $\mathbf{w}$ of the form $\sigma(\mathbf{v})$, $\sigma \in T$, is also an integral vector. Since limits of integral vectors are again integral vectors, we see that for any integral vector $\mathbf{v}$, $C_T(\mathbf{v})$ consists of integral vectors. Similarly we see that if $\mathbf{v}$ is such that $t\mathbf{v}$ is an integral vector for some positive real number $t$ then $t\mathbf{w}$ is an integral vector for all $\mathbf{w}$ in $C_T(\mathbf{v})$; consequently in this case also $C_T(\mathbf{v})$ is a proper subset of $E^n$. It turns out however that for all other points $C_T(\mathbf{v})$ equals $E^n$. Namely we have the following:

**1.1 Theorem:** Let $T$ be as above and let $\mathbf{v} = (v_1, \ldots v_n)$ be such that for some $i$, $j$, $v_i \neq 0$ and $v_j/v_i$ is an irrational number. Then $C_T(\mathbf{v}) = E^n$. That is, any point $\mathbf{w}$ can be approached starting from $\mathbf{v}$ by applying transformations from $T$.

A proof of this may be found in Dani[1]. There is also another somewhat more elementary proof possible, which however is not found in literature.

B. Next let $n = mp$, where $m, p \geq 1$. We view $E^n$ as consisting of $p$-tuples $(\mathbf{x}_1, \ldots, \mathbf{x}_p)$ where $\mathbf{x}_1, \ldots, \mathbf{x}_p$ are vectors in $E^m$. Each $\sigma$ in $S_m$ then yields a transformation of $E^n$, denoted here by $\bar{\sigma}$, defined by

$$\bar{\sigma}[(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_p)] = [\sigma(\mathbf{x}_1), \sigma(\mathbf{x}_2), \ldots, \sigma(\mathbf{x}_p)] \quad (3)$$

for all $(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_p)$ in $E^n$. That is, $\bar{\sigma}$ consists of applying $\sigma$ simultaneously to all component vectors $\mathbf{x}_1, \ldots, \mathbf{x}_p$. Let $T$ be the set of all transformations $\bar{\sigma}$ of $E^n$ obtained from all $\sigma$ in $S_m$ as above. We note that if $\mathbf{v} = (\mathbf{x}_1, \ldots, \mathbf{x}_p)$ and $\mathbf{w} = (\mathbf{y}_1, \ldots, \mathbf{y}_p)$, where each $\mathbf{y}_i$ is approachable starting from $\mathbf{x}_i$ by transformations from $S_m$ it still does not mean that $\mathbf{w}$ is in $C_T(\mathbf{v})$ since we need to apply the transformations from $S_m$ simultaneously on all components. In fact we see that if there exist constants $c_1, \ldots, c_p$, not all zero, such that $c_1\mathbf{x}_1 + c_2\mathbf{x}_2 + \ldots + c_p\mathbf{x}_p$ is an integral vector (which incidentally would necessarily be the case if $p \geq m$ then for any $\mathbf{w} = (\mathbf{y}_1, \ldots, \mathbf{y}_p)$ in $C_T(\mathbf{v})$, $c_1\mathbf{y}_1 + c_2\mathbf{y}_2 + \ldots + c_p\mathbf{y}_p$ is an integral vector. If this happens, then an arbitrary vector in $E^n$ cannot be approached starting from $\mathbf{v}$, even though from each component it may be possible to approach any vector in $E^m$. This however turns out to be the only constraint to approachability of an arbitrary vector in $E^n$. Namely, the following holds:

**1.2. Theorem:** Let $m$, $p$, $n$ and $T$ be as above and $\mathbf{v} = (\mathbf{x}_1, \ldots, \mathbf{x}_p)$ where $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_p$ are in $E^m$. Suppose that there do not exist constants $c_1, c_2, \ldots, c_p$, not all zero, such that $c_1\mathbf{x}_1 + c_2\mathbf{x}_2 + \ldots + c_p\mathbf{x}_p$ is an integral vector (so $p \leq m-1$, in particular). Then $C_T(\mathbf{v}) = E^n$; that is, every vector is approachable starting from $\mathbf{v}$ by transformations from $T$.

**Remark.** If there exist constants $c_1, c_2, \ldots, c_p$, not all zero, such that $c_1\mathbf{x}_1 + c_2\mathbf{x}_2 + \ldots + c_p\mathbf{x}_p$ is an integral vector then $\mathbf{w} = (\mathbf{y}_1, \ldots, \mathbf{y}_p)$ is in $C_T(\mathbf{v})$ if and only if $c_1\mathbf{y}_1 + c_2\mathbf{y}_2 + \ldots + c_p\mathbf{y}_p$ is an integral vector for every $p$-tuple of constants $c_1, c_2, \ldots, c_p$ such that $c_1\mathbf{x}_1 + c_2\mathbf{x}_2 + \ldots + c_p\mathbf{x}_p$ is an integral vector.

This theorem was proved in Dani and Raghavan[2]. It may be noted that there we also proved a similar result for sympletic matrices. We shall however not go into the details of that.

C. While in the above result the transformations $\bar{\sigma}$ were obtained by applying the same transformation $\sigma$ to all components, one may also ask a similar question where to some of the components we apply $\sigma$ while to some others we apply another transformation depending on $\sigma$. We shall now consider such an instance.

Let $m$, $p$ and $q$ be such that $m \geq 2$, $0 \leq p \leq m-1$, $0 \leq q \leq m-1$ and $1 \leq p+q \leq m$. Let $n = m(p+q)$. If $\sigma$ in $S_m$ is given by the relation (1), for a suitable matrix $(a_{ij})$, let ${}^t\sigma$ be the transformation of $E^m$ defined by

$${}^t\sigma[(v_1, \ldots, v_m)] = \left( \sum_{j=1}^{m} a_{j1} v_j, \ldots, \sum_{j=1}^{m} a_{jm} v_j \right),$$

namely, ${}^t\sigma$ corresponds to the transpose of the matrix $(a_{ij})$. Evidently ${}^t\sigma$ is in $S_m$ for all $\sigma$

in $S_m$. Now for each $\sigma$ in $S_m$ let $\tilde{\sigma}$ be the transformation of $E^n$ defined by

$$\tilde{\sigma}[(\mathbf{x}_1, \ldots, \mathbf{x}_p, \mathbf{f}_1, \ldots, \mathbf{f}_q)] = [\sigma(\mathbf{x}_1), \ldots, \sigma(\mathbf{x}_p), {}^t\sigma^{-1}(\mathbf{f}_1), \ldots, {}^t\sigma^{-1}(\mathbf{f}_q)] \quad (4)$$

for all $(\mathbf{x}_1, \ldots, \mathbf{x}_p, \mathbf{f}_1, \ldots, \mathbf{f}_q)$ in $E^n$, $\mathbf{x}_1, \ldots, \mathbf{x}_p$ and $\mathbf{f}_1, \ldots, \mathbf{f}_q$ being the component vectors in $E^m$. Let $T$ be the set of all transformations $\tilde{\sigma}$ of $E^n$ obtained from all $\sigma$ in $S_m$ as above. As in the earlier case, reaching from $(\mathbf{x}_1, \ldots, \mathbf{x}_p, \mathbf{f}_1, \ldots, \mathbf{f}_q)$ to $(\mathbf{y}_1, \ldots, \mathbf{y}_p, \mathbf{g}_1, \ldots, \mathbf{g}_q)$ under $\tilde{\sigma}$ amounts to reaching simultaneously from $\mathbf{x}_k$ to $\mathbf{y}_k$ for all $k = 1, \ldots, p$ under $\sigma$ and from $\mathbf{f}_l$ to $\mathbf{g}_l$ for all $l = 1, \ldots, q$ under ${}^t\sigma^{-1}$

Recall that two vectors $\mathbf{a} = (a_1, \ldots, a_m)$ and $\mathbf{b} = (b_1, \ldots, b_m)$ in $E^m$ are perpendicular to each other if and only if $a_1b_1 + \ldots + a_mb_m = 0$. We write $\mathbf{a} \perp \mathbf{b}$ to say $\mathbf{a}$ is perpendicular to $\mathbf{b}$. Let

$$X = \{ (x_1, \ldots, x_p, \mathbf{f}_1, \ldots, \mathbf{f}_q) \; \mathbf{x}_1, \ldots, \mathbf{x}_p, \text{ and } \mathbf{f}_1, \ldots, \mathbf{f}_q \text{ are in } E^m \text{ and } x_k \perp f_l \text{ for all } k = 1, \ldots, p \text{ and } l = 1, \ldots, q\} \quad (5)$$

It is easy to verify that if $\mathbf{v} = (\mathbf{x}_1, \ldots, \mathbf{x}_p, \mathbf{f}_1, \ldots, \mathbf{f}_q)$ is in $X$ then $\tilde{\sigma}(\mathbf{v})$, defined as above, is also in $X$. This implies that for any $\mathbf{v}$ in $X$, $C_T(\mathbf{v})$ is contained in $X$. That is, if a vector $\mathbf{w}$ in $E^n$ can be approached, starting from $\mathbf{v}$ by application of transformations $\tilde{\sigma}$ as above, then it must belong to $X$; further, if $\mathbf{v} = (\mathbf{x}_1, \ldots, \mathbf{x}_p, \mathbf{f}_1, \ldots, \mathbf{f}_q)$ and $\mathbf{w} = (\mathbf{y}_1, \ldots, \mathbf{y}_p, \mathbf{g}_1, \ldots, \mathbf{g}_q)$ then $(\mathbf{y}_1, \ldots, \mathbf{y}_p)$ and $(\mathbf{g}_1, \ldots, \mathbf{g}_q)$ must be approachable starting from $(\mathbf{x}_1, \ldots, \mathbf{x}_p)$ and $(\mathbf{f}_1, \ldots, \mathbf{f}_p)$ respectively under the action as in B, for which the conditions are described by Theorem 1.2. Conversely, it turns out that these conditions suffice to ensure approachability of $\mathbf{w}$ starting from $\mathbf{v}$. That is, we have the following.

**1.3. Theorem.** Let $p$, $q$, $m$, $n$, $T$ and $X$ be as above. Let $\mathbf{v} = (\mathbf{x}_1, \ldots, \mathbf{x}_p, \mathbf{f}_1, \ldots, \mathbf{f}_q)$ be an element in $X$. Suppose that there do not exist constants $a_1, \ldots, a_p$, not all zero, such that $a_1\mathbf{x}_1 + \ldots + a_p\mathbf{x}_p$ is an integral vector and there do not exist constants $b_1, \ldots, b_q$, not all zero, such that $b_1\mathbf{f}_1 + \ldots + b_q\mathbf{f}_q$ is an integral vector. Then $C_T(\mathbf{v}) = X$; that is, any element of $X$ can be approached starting from $\mathbf{v}$.

Clearly when $q = 0$ Theorem 1.3 reduces to Theorem 1.2. The study of dynamics of horospherical flows involved in Dani and Raghavan[2] was extended in Dani[3] proving a certain conjecture regarding the orbits of such flows in a very general set up. Theorem 1.3 is to be deduced from the main dynamical theorem in Dani[3]. The details of the deduction are included in §3, after introducing the necessary technical background in §2.

Of course, it would also be of interest to describe $C_T(\mathbf{v})$ for $\mathbf{v}$ in $E^n$ which are not contained $X$. However the available techniques do not yield comparable results for points outside $X$. It is indeed possible to prove certain weaker results, which we shall not take up here. (cf. Dani[4], for instance).

We note, on the other hand, that here we have concentrated on describing results which can be understood without much mathematical background and hence dealt only with the above simple classes of transformations. Similar results can in fact be deduced for a wider class of sets, (or rather, groups), of transformations using the dynamical approach outlined in the forthcoming sections. (cf. Theorem 4.1, for instance).

### §2. Dynamics on homogeneous spaces

Let $G = SL(m, \mathbb{R})$, the group of all $m \times m$ matrices with real entries, and determinant equal to 1. We consider $G$ equipped with the usual topology. Let $D$ be the subgroup $SL(m, \mathbb{Z})$ of $G$, consisting of the matrices in $G$ whose entries are integers. $D$ is a discrete subgroup of $G$. We form the quotient space $G/D$ and equip it with the quotient topology. Such a quotient space is also sometimes called a homogeneous space. On $G/D$ there is a natural action of $G$ on the left; an element $g$ in $G$ acts by taking the coset $xD$ to $gxD$. It turns out that on $G/D$ there is unique Borel measure $\mu$ such that $\mu(G/D) = 1$ which is invariant under the action of $G$ as above (cf. Raghunathan[5], Chapter X).

Let $H$ be a closed subgroup of $G$. Then the action of $G$ on $G/D$ yields an action of $H$ on $G/D$, by restriction. It turns out that whenever $H$ is non-compact this $H$-action on $G/D$ is ergodic with respect to the measure $\mu$; this means that any Borel subset of $G/D$ which is invariant under the $H$-action has measure either 0 or 1 (cf. Zimmer[6], Theorem 2.2.6). As a consequence it follows that for *almost all* $x$ in $G/D$ (with respect to the measure $\mu$) the orbit $Hx$ is dense in $G/D$., (cf. Zimmer[6], Proposition 2.1.7). It must be emphasized that the assertion is only for almost all $x$. Only for certain subgroups, e.g. the group of all upper triangular matrices in $G$, the assertion holds for all $x$ in $G/D$ (cf. Dani and Raghavan[2], Proposition 1.5). In general, not only it may not hold for all $x$ but there is also no information available about the set of $x$ for which the assertion holds. However, for a class of subgroups called horospherical subgroups a characterization of the set of points with dense orbits was obtained in Dani[3]. We shall describe the result below and use it to prove Theorem 1.3. But before that we shall make some general observations so that the result may be viewed in a proper perspective.

In the above discussion it is possible to allow $G$ to be any simple Lie group with finite centre and $D$ to be any discrete subgroup such that the homogeneous (quotient) space $G/D$ admits a finite measure invariant under the action of $G$ on the left. The case $G = SL(m,\mathbf{R})$ and $D = SL(m,\mathbf{Z})$ is but an example of such a set up. We note that while in this example $G/D$ happens to be noncompact, there are other discrete subgroups $D$ of $G = SL(m,\mathbf{R})$, and also of other simple Lie-groups, such that $G/D$ is compact.

A subgroup $U$ of a Lie group $G$ is said to be *horospherical* if there exists $g$ in $G$ such that

$$U = \{u \in G \mid g^j u g^{-j} \to e \text{ as } j \to \infty\}, \quad (6)$$

where $e$ is the identity element. If $G = SL(m,\mathbf{R})$, a subgroup of $G$ is horospherical if and only if it is conjugate to a subgroup of the following form: For any subset $A = \{a_1, \ldots, a_l\}$ of $\{1, \ldots, m-1\}$, where $a_1 < a_2 < \ldots < a_l$, let

$$U_A = \{(u_{ij}) \mid u_{ij} = \delta_{ij} \text{ if } i > a_k \text{ and } j \le a_{k+1} \text{ for some } k = 0, \ldots, l\}, \quad (7)$$

where $a_0 = 0$ and $a_{l+1} = m$ by convention and $\delta_{ij}$ is 1 or 0 according to whether $i = j$ or $i \neq j$. We urge the reader to interpret the condition in terms of blocks of matrices, which is easier to understand but clumsy for printing. In particular if $A = \{1, \ldots, m-1\}$ then $U_A$ is the group of all upper triangular matrices with 1's on the diagonal. If $A = \{p\}$, the set consisting of only one element $p$, where $1 \le p \le m-1$, we shall write $U_p$ for $U_A$; clearly

$$U_p = \left\{ \begin{pmatrix} I_p & R \\ 0 & I_{n-p} \end{pmatrix} \middle| \, R \text{ an arbitrary } p \times (n-p) \text{ matrix} \right\}, \quad (8)$$

where $I_p, I_{n-p}$ are identity matrices of sizes $p \times p$ and $(n-p) \times (n-p)$ respectively and 0 is the $(n-p) \times p$ zero matrix.

The action of a horospherical subgroup of $G$ on a homogeneous space $G/D$ as above is called a horospherical flow. When $G = SL(2, \mathbf{R})$, the horospherical subgroup $U_1$ as above is one-dimensional and in this case the corresponding horospherical flow is called the horocycle flow. The dynamics of the horocycle flow is under study for over half a century now. In the last two decades various authors have also studied the general horospherical flows. The reader is referred to the survey of Dani[7] for various details in this regard. In particular it is known that if $G/D$ (as above) is compact then every orbit of any nontrivial horospherical flow is dense in $G/D$. But recall that in the example of our interest here, namely $G = SL(m,\mathbf{R})$ and $D = SL(m,\mathbf{Z})$, the homogeneous space $G/D$ is noncompact. In this case the corresponding assertion is in fact false. A characterization was obtained[3] for any $G$ and $D$ as above (actually in even somewhat greater generality, which we shall not go into here) of the set of points whose orbits under a given horospherical flow are dense in $G/D$. It was also deduced that when the orbit is not dense the closure of the orbit is still a homogeneous space with finite invariant measure, but now with respect to a proper closed subgroup.

We now go back to the case of $G = SL(m,\mathbf{R})$ and $D = SL(m,\mathbf{Z})$ as before. Recall that $E^m$ is viewed as the space of row vectors with $m$ real entries or, equivalently, $1 \times m$ real matrices. Consider the action of $G$ on $E^m$ defined by

$$g(\mathbf{v}) = \mathbf{v}\,{}^t g \text{ for all } v \text{ in } E^m, \qquad (9)$$

where ${}^t g$ denotes the transpose of the matrix $g$. Observe that $S_m$ as in §1 consists precisely of transformations as above corresponding to all the elements $g$ in $D$. For the case at hand the result in Dani[3] implies the following.

**2.1 Theorem.** Let $G = SL(m,\mathbf{R})$, $D = SL(m,\mathbf{Z})$ and $U = U_A$ for some subset $A$ of $\{1, \ldots, m-1\}$. Let $x$ be an element of $G$. Then there exists a closed subgroup $H$ containing $U$ such that

i) $HxD/D$ is the closure of $UxD/D$,
ii) $HxD/D$ admits a finite $H$-invariant measure,
iii) either $H = G$ or there exists a proper nonzero subspace $W$ or $E^m$

such that the following conditions are satisfied: a) $W$ is a rational subspace; that is, $W$ is defined within $E^m$ by a system of linear equations with rational coefficients and, b) $W$ is invariant under the action of all $g$ in $x^{-1}Hx$ (and in particular all $g$ in $x^{-1}Ux$); that is, $g(\mathbf{w})$ is in $W$ for all $\mathbf{w} \in W$ and $g$ in $x^{-1}Hx$.

The conditions i) and ii) are the same as in the statement of the main theorem in Dani[3]. A scrutiny of the proof would show that for the particular case at hand condition iii) also holds. It is also possible to conclude condition iii) from conditions i) and ii) using certain results in the theory of algebraic groups; it would however be pointless to go into the details of this.

## §3. Proof of Theorem 1.3.

In this section we shall give a proof of Theorem 1.3. In preparation, we first note the following.

**3.1 Lemma.** Let $U$ be a subgroup of $G = SL(m,\mathbf{R})$ of the form $xU_px^{-1}$ for some $x$ in $G$ and $1 \leq p \leq m-1$. Then the following conditions are satisfied:

i) the vector subspace consisting of the fixed points of the action of $U$ viz $V = \{\mathbf{w} \mid \mathbf{w} \text{ in } E^m,\ g(\mathbf{w}) = \mathbf{w} \text{ for all } g \text{ in } U\}$, is of dimension $p$.
ii) if $W$ is a vector subspace of dimension at most $p$ and invariant under the action of all $g$ in $U$ then $g(\mathbf{w}) = \mathbf{w}$ for all $\mathbf{w}$ in $W$ and $g$ in $U$.
iii) the subgroup consisting of transposes of all elements of $U$ is conjugate to $U_{m-p}$; that is, it equals $yU_{m-p}y^{-1}$ for some $y$ in $G$.

**Proof.** We first observe that each of these conditions would hold for any $xU_px^{-1}$, $x$ in $G$, if they hold for $U_p$ itself. For i) and ii) this may be seen by noting that for $\mathbf{w}$ in $E^m$, $g(\mathbf{w}) = \mathbf{w}$ for all $g$ in $xU_px^{-1}$ if and only if $u[x^{-1}(\mathbf{w})] = x^{-1}(\mathbf{w})$ for all $u$ in $U_p$ and a similar assertion for invariant subspaces. For iii) it is even more obvious.

Now for $U = U_p$ condition i) is obvious from the definition of $U_p$ (see (8)). In fact, in this case $V = \{(v_1, \ldots, v_p, 0, \ldots, 0) \mid v_1, \ldots, v_p \text{ real}\}$. Condition ii) can be deduced by verifying by direct computation that if a subspace $W$ is invariant under the action of all $g$ in $U_p$ and contains an element $\mathbf{w}$ not in $V$ then it contains the subspace spanned by $\mathbf{w}$ and $V$. Condition iii) can be seen to hold for $U = U_p$ for the choice $y = (y_{ij})$ where $y_{ij} = 0$ if $i+j \neq m$ and $y_{ij} = \pm 1$ if $i+j = m$, where the signs are chosen so that the determinant of $y$ is 1.

We shall now deduce Theorem 1.3 from Theorem 2.1. We first note that by including more vectors in $E^m$ as components we may without loss of generality assume that $p+q = m$. Now consider the action of $G = SL(m, \mathbf{R})$ on $E^n$, where $n = m(p+q)$, defined for all $g$ in $G$ by

$$g[(\mathbf{x}_1, \ldots, \mathbf{x}_p, \mathbf{f}_1, \ldots, \mathbf{f}_q)] = [g(\mathbf{x}_1), \ldots, g(\mathbf{x}_p), {}^t g^{-1}(\mathbf{f}_1), \ldots, {}^t g^{-1}(\mathbf{f}_q)], \tag{10}$$

where $\mathbf{x}_1, \ldots, \mathbf{x}_p, \mathbf{f}_1, \ldots, \mathbf{f}_q$ are the component vectors in $E^m$, and the action on each component is as defined by (9). Let $X$ be the subset of $E^n$ as before (see (5)), namely the set of $\mathbf{v} = (\mathbf{x}_1, \ldots, \mathbf{x}_p, \mathbf{f}_1, \ldots, \mathbf{f}_q)$, such that each $\mathbf{x}_k$ is perpendicular to each $\mathbf{f}_l$ for $k = 1, \ldots, p$ and $l = 1, \ldots, q$. Let $\mathbf{e}$ be the element $(\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_m)$ in $E^n$, where $\mathbf{e}_1, \ldots, \mathbf{e}_n$ is the standard orthonormal basis of $E^m$; namely $e_i$ has 1 in $i$th place and 0 elsewhere. Then $e$ is in $X$. Further, it can be verified that the orbit of $\mathbf{e}$ under the action of $G$, namely $Y = \{g(\mathbf{e}) | g \text{ in } G\}$, consists precisely of $\mathbf{v} = (\mathbf{x}_1, \ldots, \mathbf{x}_p, \mathbf{f}_1, \ldots, \mathbf{f}_q)$ in $X$ such that $\mathbf{x}_1, \ldots, \mathbf{x}_p$ and $\mathbf{f}_1, \ldots, \mathbf{f}_q$ are linearly independent. We note that, in particular, $Y$ is locally compact and dense in $X$. Let $\mathbf{v} = (\mathbf{x}_1, \ldots, \mathbf{x}_p, \mathbf{f}_1, \ldots, \mathbf{f}_q)$ be as in the hypothesis of the theorem. Then $\mathbf{v}$ belongs to $Y$ since any linear combination with at least one nonzero coefficient is not even an integral vector let alone be zero. The assertion of the theorem is equivalent to saying that the set $\{g(\mathbf{v}) | g \text{ in } D\}$, the orbit of $\mathbf{v}$ under the action of $D$, is dense in $X$. Since $Y$ is dense in $X$ it is enough to show that the orbit under $D$ (which is indeed contained in $Y$) is dense in $Y$. Let us suppose that this is not true. We shall complete the proof by showing that this leads to a contradiction.

Let $U$ be the subgroup of $G$ consisting of those elements which fix $\mathbf{e}$. It is easy to see that in fact $U = U_p$ in the notation above; in particular it is a horospherical subgroup. As each element of $U$ fixes $\mathbf{e}$ we get a well-defined bijective map $\phi$: $G/U \to Y$ such that $\phi(gU) = g(\mathbf{e})$ for all $g$ in $G$. As $Y$ is locally compact it follows from a standard result that the map $\phi$ is in fact a homeomorphism. Let $g_0$ in $G$ be such that $\mathbf{v} = g_0(\mathbf{e})$. Since $\phi$ is a homeomorphism and, by assumption, the $D$-orbit of $\mathbf{v}$ is not dense in $Y$ it follows that $Dg_0U/U$ is not dense in $G/U$. This is equivalent to saying that the set $Dg_0U$ is not dense in $G$. Hence $Ug_0^{-1}D$, which consists precisely of inverses of elements of $Dg_0U$, is not dense in $G$. Hence $Ug_0^{-1}D/D$ is not dense in $G/D$. Since $U$ is a horospherical subgroup, by Theorem 2.1 this implies that there exists a proper nonzero rational subspace $W$ of $E^m$ invariant under the action of all $g$ in $g_0Ug_0^{-1}$. Let $d$ = dimension of $W$. Then $1 \le d \le m-1$.

Suppose first that $d \le p$. Since $U = U_p$, by Lemma 3.1, i), $V$ is $p$-dimensional. On the other hand, by choice, $V$ contains $\mathbf{x}_1, \ldots, \mathbf{x}_p$. Hence $V$ is spanned by $\mathbf{x}_1, \ldots, \mathbf{x}_p$. But $W$ is under the action of all $g$ in $g_0Ug_0^{-1}$. Then by Lemma 3.1, i), $V$ is $p$-dimensional. On the other hand, by choice, $V$ contains $\mathbf{x}_1, \ldots, \mathbf{x}_p$. Hence $V$ is spanned by $\mathbf{x}_1, \ldots, \mathbf{x}_p$. But $W$ is contained in $V$. Hence every element of $W$ is a linear combination of $\mathbf{x}_1, \ldots, \mathbf{x}_p$. Also since $W$ is a non-zero rational subspace, it contains nonzero integral vectors. Thus a certain linear combination $a_1\mathbf{x}_1 + \ldots + a_p\mathbf{x}_p$ must be a nonzero integral vector; this contradicts the hypothesis.

Next suppose, if possible, that $d > p$. Let $W^\perp$ be the orthocomplement of $W$ in $E^n$. Since $W$ is a rational subspace so is $W^\perp$. Also the dimension of $W^\perp$ is $m - d \le m - p = q$. Since $W$ is invariant under the action of all $g$ in $g_0Ug_0^{-1}$, $W^\perp$ is invariant under the action of all $g$ such that ${}^t g^{-1}$ is in $g_0Ug_0^{-1}$. Recall that by Lemma 3.1, iii), the subgroup consisting of all $g$ satisfying that condition is conjugate to $U_q$. Hence, by Lemma 3.1, ii), we get that $g(\mathbf{w}) = \mathbf{w}$ for all $w$ in $W$ and $g$ such that ${}^t g^{-1}$ is in $g_0Ug_0^{-1}$. Equivalently ${}^t g^{-1}(\mathbf{w}) = \mathbf{w}$ for all $\mathbf{w}$ in $W^\perp$ and $g$ in $g_0Ug_0^{-1}$. An argument similar to the one above now shows that every element of $W^\perp$ is a linear combination of $\mathbf{f}_1, \ldots, \mathbf{f}_q$. Since $W^\perp$ contains nonzero integral vectors, this also contradicts the hypothesis. Since the assumption to the contrary leads to a contradiction, we conclude that the orbit of $\mathbf{v}$ under $D$ is dense in $Y$ and hence in $X$. This proves Theorem 1.3.

## §4. Comments and questions

1. Theorem 2.1 and the corresponding general result can also be used to prove similar results

for actions of other discrete linear groups, following the method involved in the proof of Theorem 1.3. In particular, the result that all orbits of horospherical flows are dense if $G/D$ is compact can be seen to lead to the following:

**4.1. Theorem.** Let $G = SL(m,\mathbf{R})$ and $D$ be discrete subgroup of $G$ such that $G/D$ is compact. Let $n = m(p+q)$ where $p,q \geq 1$ and $p+q \leq m$ and consider the $G$-action on $E^n$ defined by (10). Let $X$ be the subset of $E^n$ as in (5) and let $\mathbf{v} = (\mathbf{x}_1, \ldots, \mathbf{x}_p, \mathbf{f}_1, \ldots, \mathbf{f}_q)$ be in $X$. Then the $D$-orbit of $\mathbf{v}$ is dense in $X$ if and only if $\mathbf{x}_1, \ldots, \mathbf{x}_p$ and $\mathbf{f}_1, \ldots, \mathbf{f}_q$ are linearly independent.

2. If one is interested in approaching zero, rather than an arbitrary vector in $E^n$, one can also use certain results in dynamics which are known in a more general setting, but have weaker conclusions. An illustration of this may be found in Dani[4].

3. One may like to relate the degree of approximation achieved to the 'size' of the matrix, (namely, how large it is, in a suitable sense) and look for optimal possibilities. The reader is referred to Dani[8] for certain results on this question in respect of the approach to zero in Theorem 1.2 starting from $\mathbf{v}$ satisfying the condition of that theorem.

4. Though Theorem 1.3 describes the vectors that can be approached starting from a typical element in $X$, given a vector that can be approached, there is no algorithm to achieve the purpose. In the simplest case of Theorem 1.1 the approach to zero can be achieved algorithmically. However not much can be said in general, in this respect.

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