TIFR-TH/00-52

Supergravity couplings to Noncommutative Branes, Open Wilson lines and Generalized Star Products

Sumit R. Das [a,b] [1] and Sandip P. Trivedi [a] [2]

[a] *Tata Institute of Fundamental Research,*

Homi Bhabha Road, Bombay 400 005, INDIA.

[b] *Harish-chandra Research Institute* [3],

Jhusi, Allahabad 211 019, INDIA.

Noncommutative gauge theories can be constructed from ordinary $U(\infty)$ gauge theories in lower dimensions. Using this construction we identify the operators on noncommutative D-branes which couple to linearized supergravity backgrounds, from a knowledge of such couplings to lower dimensional D-branes with no B field. These operators belong to a class of gauge invariant observables involving open Wilson lines. Assuming a DBI form of the coupling we show, to second order in the gauge potential but to all orders of the noncommutativity parameter, that our proposal agrees with the operator obtained in terms of ordinary gauge fields by considering brane actions in backgrounds and then using the Seiberg-Witten map to rewrite this in terms of noncommutative gauge fields. Our result clarify why a certain *commutative* but *non-associative* "generalized star product" appears both in the expansion of the open Wilson line, as well as in string amplitude computations of open string - closed string couplings. We outline how our procedure can be used to obtain operators in the noncommutative theory which are holographically dual to supergravity modes.

November 2000

[1] das@theory.tifr.res.in

[2] sandip@theory.tifr.res.in

[3] formerly Mehta Research Institute of Mathematics and Mathematical Physics

1. Introduction and summary

In noncommutative gauge theory, space and color space are intertwined. As a result there are no local position space gauge invariant observables. However since the theories typically have translation invariance there are such operators with definite momentum. These are open Wilson lines, constructed by Ishibashi, Iso, Kawai and Kitazawa (IIKK) [1]. Consider a noncommutative Yang-Mills (NCYM) theory in $d+1 = p+2n+1$ space-time dimensions with the noncommutativity matrix θ^{AB} given by

$$\begin{aligned} &\theta^{AB} = \theta^{ij} \qquad (A,B) = (i,j) \quad i,j = 1 \cdots 2n \\ &\theta^{AB} = 0 \qquad\quad \text{otherwise} \end{aligned} \tag{1.1}$$

In the following we will use $i, j = 1, \cdots 2n$ to label noncommutative directions, which are taken to be space-like, $\mu, \nu = 1, \cdots p$ label the spatial commutative directions and $A, B = 1, \cdots p + 2n + 1$ to label all directions collectively. An open Wilson line $W(k, C)$ along some open contour C given by $y^A(\lambda)$ with momenta k_A is defined in the star product language as

$$W(k,C) = \int d^{d+1}x \ \mathrm{tr}\ [P_\star \exp[i \int_C d\lambda \frac{dy^A(\lambda)}{d\lambda} A_A(x + y(\lambda))] \ \star \ e^{ik_B x^B}] \tag{1.2}$$

The trace in (1.2) is over the nonabelian gauge group. λ is a parameter that increases along the path. In our conventions the path ordering is defined so that points at later values of λ occur successively to the left. Note also that all products in (1.2), including those in the path ordered exponential, are star products. The open Wilson line (1.2) is gauge invariant if the end points of the contour are separated by an amount Δx^A where

$$\Delta x^A = k_B \theta^{BA}. \tag{1.3}$$

Clearly the separation is nonzero only along the noncommutative directions. When $\theta = 0$ this is just the fourier transform of an ordinary Wilson loop with a marked point. For $\theta \neq 0$ one can perform a fourier transform along the commutative directions to obtain an operator which has a definite marked point in the commutative directions and a definite momentum along the noncommutative directions. Various aspects of open Wilson lines have been discussed in [2]. In [3] it was argued that these operators (with a modification to include scalar fields) form a complete set (in fact an overcomplete set) of operators of the theory made from gauge fields and scalars. They can be interpreted as macroscopic fundamental strings [4].

In [5] gauge invariant operators were written down which reduce to local gauge invariant operators in the commutative limit. These are defined as

$$\tilde{\mathcal{O}}(k) = \int d^{d+1}x \ \ \mathrm{tr} \ \ \mathcal{O}(x + k \cdot \theta) \star P_\star \ \exp[i \int_0^1 d\lambda \ k_A \theta^{AB} A_B(x + k \cdot \theta \ \lambda)] \star e^{ik\cdot x}. \quad (1.4)$$

Note, the contour is now a *straight* path transverse to the momentum along the direction

$$\eta^A = k_B \theta^{BA}. \quad (1.5)$$

$\mathcal{O}(x+k\cdot\theta)$ is a local operator constructed from the fields which is inserted at the endpoint of the path, and

$$(k \cdot \theta)^A \equiv k_B \theta^{BA}. \quad (1.6)$$

The path ordered Wilson loop factor above will be referred to as the "tail" of the operator. It clearly extends in the noncommutative direction. This represents a deformation of the fourier transforms of local gauge invariant operators to the noncommutative theory. Correlation functions of these operators have been calculated in [5] yielding interesting results - in particular a universal behavior at high momenta. It is also possible to construct other operators, e.g. open wilson lines with self intersections and open wilson lines ending with a closed wilson loop [6], and the operators (1.4) are in fact special cases of the latter.

Recently, a smeared version of the operator (1.4) has been introduced in [7]. This relates to the situation where the operator $\mathcal{O}(x)$ is itself a product of operators

$$\mathcal{O}(x) = \prod_{\alpha=1}^{n} \mathcal{O}_\alpha(x) \quad (1.7)$$

Then the operators $\mathcal{O}_\alpha$ can be smeared over the Wilson tail

$$\hat{\mathcal{O}}(k) = \int d^{d+1}x \int \prod_{\alpha=1}^{n} d\tau_\alpha \ \ P_* \ \mathrm{tr} \ [\prod_{\alpha=1}^{n} O_\alpha(x^i + \theta^{ji}k_j\tau_\alpha) W_t(k, A, \phi, x)] \star e^{ik_i x^i} \quad (1.8)$$

where W_t denotes the Wilson tail

$$W_t(k, A, \phi, x) = \exp[i \int_0^1 d\lambda \ k_A \theta^{AB} A_B(x + k \cdot \theta \ \lambda)] \quad (1.9)$$

For normal p-branes without B fields on them, coupling to a linearized supergravity background yields a natural set of gauge invariant operators of the worldvolume $(p+1)$-dimensional gauge theory. The same should be true with B fields and should therefore naturally lead to a set of gauge invariant operators of the noncommutative gauge theory, as emphasized in [8]. Such couplings are useful in various contexts, e.g. absorption or Hawking radiation [4] [9] or discovery of expanded brane configurations in the presence of backgrounds [10]. A reasonably exhaustive set of such couplings have been obtained by matrix theory techniques in [11] and from T-duality consistencies in [10].

While it is obvious that these gauge invariant operators couple to general closed string modes, so far it has not been possible to determine in a precise fashion which operator couples to which supergravity mode. In this paper we propose a way to do this. We use the construction of noncommutative gauge theories from ordinary $U(\infty)$ gauge theories in lower dimensions or matrix models [12] [1] [2] [13] [14] [15], which was used to write down these operators in the first place [1]. We propose that once we know the linearized couplings of a set of ordinary Dp branes to supergravity backgrounds, we can use the above construction to find the couplings of these backgrounds to noncommutative $D(p+2n)$ branes with noncommutativity in $2n$ of the directions. These operators turn out to be exactly of the type (1.8) constructed in [7].

There is another way one could obtain the couplings to noncommutative branes. One can, by direct calculation, obtain these operators by considering the coupling of a single closed string with several open strings in the presence of a nonzero B field and express them in terms of an "ordinary" gauge field $f_{\mu\nu}$. For example for a single noncommutative brane one may take the coupling given by the DBI-WZ action in an arbitrary background written in terms of ordinary gauge fields. On the other hand, the gauge field $f_{\mu\nu}$ is related to the noncommutative gauge field $F_{\mu\nu}$ by the Seiberg-Witten map [16]. Using this map one can in principle obtain the operators in terms of the noncommutative gauge fields $F_{\mu\nu}$. One would, of course, get an infinite series and any finite truncation would not be gauge invariant under noncommutative gauge transformations. Furthermore the Seiberg-Witten map is not known to all orders. Nevertheless one may carry out this procedure in an expansion in powers of the noncommutative gauge field A_μ. This has been carried out for the DBI action in [17] where it has been argued that the answer correctly reproduces the simplest amplitudes involving open and closed strings obtained in [18] and [17].

In our proposal the operators are obtained directly in terms of the noncommutative gauge fields and are gauge invariant by construction. However, the answer, when expanded

[4] For absorption/radiation by black holes the coupling is sometimes to effective theories rather than fundamental brane theories

in powers of A_μ, should agree with the answer obtained via ordinary gauge fields and the Seiberg-Witten map. As a concrete check of our proposal we carry out this comparison explicitly for the dilaton coupling to noncommutative branes. A nonabelian version of the Dirac-Born-Infeld action coupled to backgrounds was discussed in [10] [11] [19]. We assume this form for the lower dimensional brane used to construct the higher dimensional noncommutative brane. We show that the resulting operator is identical, to second order in the noncommutative gauge potential, to the one obtained from the DBI action written in terms of ordinary gauge fields and transformed by the Seiberg-Witten map. For simplicity, we do the calculation where we have a single *euclidean* noncommutative $(2n-1)$ brane which is obtained from the action of an infinite number of D-instantons (in DBI form). However the calculation may be easily generalized to lorentzian branes (with magnetic type B fields on them). Extension of our results to arbitrary number of noncommutative branes requires a solution to the Seiberg-Witten map for nonabelian gauge fields.

The solution to the Seiberg-Witten map yields an interesting structure : the result appears in terms of a "generalized star product" which are *commutative* but *non-associative* [17]and a triple product [20]. These generalized products therefore appear in the open-closed string couplings as well. The same generalized products appear in one loop effective actions of NCYM theories [21] [22] and in the study of anomalies [23]. Recently it has been shown [20],[7] that these generalized products *also* appear in the expansion of the open Wilson lines considered in [1] - [6]. Our result therefore provides an explanation as to why the same structure appears in open-closed interactions as well in the gauge invariant open Wilson lines [5].

Gauge invariant operators also appear in the context of holography. The states created by such operators would have a dual description as normal modes in the dual supergravity background. It turns out that the asymptotic geometry for the $p+2n+1$ dimensional non-commutative theory is *identical* to that for the $p+1$ dimensional ordinary theory at a particular point in the Coulomb branch where the p-branes are spread out uniformly along the $2n$ directions. This is in fact the dual manifestation of the relationship between commutative and noncommutative Yang-Mills theories discussed above [24].This connection may be possibly used to tackle the problem of mode mixing in such supergravity backgrounds. We do not have definitive results about this at present.

[5] In [17] it was proposed that to obtain the coupling of a mode to a noncommutative brane (in the DBI approximation) one has to first write down the usual coupling, replace ordinary products by generalized star products and then use the Seiberg-Witten map. We have not been able to see why this prescription is correct.

2. Noncommutative Yang-Mills from lower dimensional ordinary Yang-Mills

In this section we review how noncommutative Yang-Mills theories are obtained from lower dimensional ordinary $U(\infty)$ Yang Mills theories. This is how how space-time emerges in Eguchi-Kawai models [25], particularly in its "twisted" version, [26] and how branes arise in matrix models [27]. In modern Matrix theory, both of the BFSS [28] as well as the IKKT type [29] branes arise in a similar way [30]. This has led to the discovery of noncommutativity in string theory [31] and has been useful in providing valuable insights [32][16]. Several useful aspects of this connection are contained in [1] [2] [13] [33] [34] [35]. We will use the framework of [12] and [15].

Consider a $U(\infty)$ ordinary gauge theory in $(p+1)$ dimensions with the usual gauge fields $\mathbf{A}_\mu(\xi) \quad , \mu = 1, \cdots p+1$ and $(9-p)$ scalar fields $\mathbf{X}^I(\xi),\ I = 1, \cdots (9-p)$ in the adjoint representation, together with their fermionic partners. In this paper we will restrict ourselves to only bosonic components of operators, consequently, fermions will not enter the subsequent discussion. The bosonic part of the action is

$$S = \mathrm{Tr} \int d^{p+1}\xi [\mathbf{F}_{\mu\nu}\mathbf{F}^{\mu\nu} + D_\mu \mathbf{X}^I D^\mu \mathbf{X}^J g_{IJ} + [\mathbf{X}^I, \mathbf{X}^J][\mathbf{X}^K, \mathbf{X}^L] g_{IK} g_{JL}] \tag{2.1}$$

where g_{IJ} are constants and the other notations are standard. Boldface has been used to denote $\infty \times \infty$ matrices.

The action has a nontrivial classical solution

$$\begin{aligned} &\mathbf{X}^i(\xi) = \mathbf{x}^i \qquad && i = 1, \cdots 2n \\ &\mathbf{X}^a = 0 && a = 2n+1 \cdots 9-p \\ &\mathbf{A}_\mu = 0 \end{aligned} \tag{2.2}$$

where the constant (in ξ) matrices $\mathbf{x}^i$ satisfy

$$[\mathbf{x}^i, \mathbf{x}^j] = i\theta^{ij}\mathbf{I} \tag{2.3}$$

The antisymmetric matrix θ^{ij} has rank p and $\mathbf{I}$ stands for the unit $\infty \times \infty$ matrix. The inverse of the matrix θ^{ij} will be denoted by B_{ij}

The idea is then to expand the various fields as follows.

$$\begin{aligned} &\mathbf{C}_i = B_{ij}\mathbf{X}^j = \mathbf{p}_i + \mathbf{A}_i \\ &\mathbf{X}^a = \phi^a \\ &\mathbf{A}_\mu = \mathbf{A}_\mu \end{aligned} \tag{2.4}$$

where

$$\mathbf{p}_i = B_{ij}\mathbf{x}^j \tag{2.5}$$

We will expand any matrix $\mathbf{O}(\xi)$ as follows

$$\mathbf{O}(\xi) = \int d^{2n}k \ \exp[i\theta^{ij}k_i\mathbf{p}_j] \ O(k,\xi) \tag{2.6}$$

where $O(k,\xi)$ are ordinary functions. Regarding these $O(k,\xi)$ as fourier components of a function $O(x,\xi)$, where x^i are the coordinates of a $2n$ dimensional space we then get the following map between matrices and functions.

$$\begin{aligned}
&\mathbf{O}(\xi) \rightarrow O(x,\xi)\\
&[\mathbf{p}_i, \mathbf{O}(\xi)] = i\partial_i O(x,\xi)\\
&\mathrm{Tr}\mathbf{O}(\xi) = \frac{1}{(2\pi)^n}[\mathrm{Pf\ B}]\int d^{2n}x \ \ O(x,\xi)
\end{aligned} \tag{2.7}$$

The product of two matrices $\mathbf{O}_1(\xi)$ and $\mathbf{O}_2(\xi)$ is then mapped to a star product

$$\mathbf{O}_1(\xi)\mathbf{O}_2(\xi) \rightarrow O_1(x,\xi) * O_1(x,\xi) \tag{2.8}$$

where

$$O_1(x,\xi) * O_2(x,\xi) = \exp\ [\frac{\theta^{ij}}{2i}\frac{\partial^2}{\partial s^i \partial t^j}] \ O_1(x+s,\xi)O_2(x+t,\xi) \ |_{s=t=0} \tag{2.9}$$

A quick way to see this is to consider the operators

$$\mathbf{O}(k) = \exp\ [i\theta^{ij}k_i\mathbf{p}_j] \tag{2.10}$$

which form a complete basis. Then the commutation relations of $\mathbf{x}^i$ and hence $\mathbf{p}_i$ show

$$\mathbf{O}(k)\mathbf{O}(k') = e^{-\frac{i}{2}\theta^{ij}k_ik'_j} \ \mathbf{O}(k+k') \tag{2.11}$$

With these rules, one can easily verify

$$\begin{aligned}
&\mathbf{F}_{\mu\nu} \rightarrow \partial_\mu A_\nu - \partial_\nu A_\mu - iA_\mu * A_\nu + iA_\nu * A_\mu \equiv F_{\mu\nu}\\
&D_\mu \mathbf{X}^i \rightarrow \theta^{ij}(\partial_\mu A_j - \partial_j A_\mu - iA_\mu * A_j + iA_j * A_\mu) \equiv \theta^{ij}F_{\mu j}\\
&D_\mu \mathbf{X}^a \rightarrow \partial_\mu\phi^a - iA_\mu * \phi^a + i\phi^a * A_\mu \equiv D_\mu\phi^a\\
&[\mathbf{X}^i, \mathbf{X}^j] \rightarrow i\theta^{ik}\theta^{jl}(F_{kl} - B_{kl})\\
&[\mathbf{X}^i, \mathbf{X}^a] \rightarrow i\theta^{ij}(\partial_j\phi^a - iA_j \star \phi^a + i\phi^a \star A_j) \equiv i\theta^{ij}D_j\phi^a
\end{aligned} \tag{2.12}$$

where we have defined

$$F_{ij} = \partial_i A_j - \partial_j A_i - iA_i \star A_j + iA_j \star A_i \tag{2.13}$$

In the above equations the quantities appearing in the right hand side are ordinary functions of (x, ξ).

The action (2.1) becomes the action of $U(1)$ noncommutative gauge theory in the $p + 2n + 1$ dimensions spanned by x, ξ. The noncommutativity is entirely in the $2n$ directions. In addition to the gauge fields we also have $(9 - p - 2n)$ "adjoint" scalars ϕ^a. The gauge field appears in the combination

$$F_{AB} - B_{AB} \tag{2.14}$$

where B_{AB} is an antisymmetric matrix whose (ij) components are B_{ij} and the rest zero. This corresponds to a specific choice of "description" in the NCYM theory [15]. Furthermore the upper and lower indices of various quantities some contracted with the "open string metric" whose components in the nocommutative directions are

$$G^{ij} = -\theta^{ik} g_{kl} \theta^{lj} \tag{2.15}$$

The componments of the open string metric in the commutative directions are the same as the original metric g_{ab}. Finally the coupling constant which appears in front is the open string coupling G_s which is related to the closed string coupling g_s by

$$G_s = g_s \left(\frac{\det(G - B)}{\det(g + B)}\right)^{\frac{1}{2}} = g_s \left(\frac{\det\ B}{\det\ g}\right)^{\frac{1}{2}} \tag{2.16}$$

It may be also easily verified that

$$\frac{1}{G - B} = -\theta + \frac{1}{g + B} \tag{2.17}$$

(Recall that $\theta^{-1} = B$ as matrices.)

It is straightforward to extend the above construction to obtain a nonabelian noncommutative theory. The classical solution which one starts with is now

$$\mathbf{X}^i(\xi) = \mathbf{x}^i \otimes I_M \tag{2.18}$$

where I_M denotes the unit $M \times M$ matrix. Now the various $\infty \times \infty$ matrices map on to $M \times M$ matrices which are functions of x, in addition to ξ. With this understanding the formulae above can be almost trivially extended. The star product would now include matrix multiplication and the map for the trace becomes

$$\mathrm{Tr}\mathbf{O}(\xi) = \frac{1}{(2\pi)^n}[\mathrm{Pf\ B}]\int d^{2n}x \ \ \mathrm{tr}\ O(x,\xi) \tag{2.19}$$

where tr now denotes trace over $M \times M$ matrices. Instead of obtaining a $U(1)$ noncommutative theory one now obtains a $U(m)$ noncommutative theory.

Finally the expression for the open Wilson line (1.2) is easily seen to be [1]

$$\begin{aligned} W(C,k) &= \int d^{p+1}\xi \ \mathrm{Lim}_{M\to\infty} \ \ \mathrm{Tr}\ [\prod_{n=1}^{M} U_j]\ e^{ik_\mu \xi^\mu} \\ U_j &= \exp\ [i\vec{\mathbf{C}} \cdot (\vec{\Delta} d)_n] \end{aligned} \tag{2.20}$$

where Δd_n denotes the n-th infinitesimal line element along the contour C. The momentum components k_μ along the commutative directions appear explicitly in (2.20), while the components along the noncommutative directions k^i are given by

$$k_i = B_{ji} d^j \tag{2.21}$$

where d^j are the components of the vector

$$\vec{d} = \sum_{n=1}^{M} \vec{\Delta d} \tag{2.22}$$

Operators with straight Wilson line tails given by (1.4) are similarly represented by

$$\mathcal{O}(k) = \int d^{p+1}\xi \ e^{ik_\mu \xi^\mu} \ \mathrm{Tr}\ [e^{ik_i \mathbf{X}^i} \ \mathcal{O}(\mathbf{X}, \mathbf{A}, \xi)] \tag{2.23}$$

3. Branes in supergravity backgrounds

Consider a large number of coincident p branes with no B field in the presence of a weak supergravity background. Let us denote a supergravity mode in momentum space by $\Phi(k_I, k_\mu)$ where k_μ denotes the momentum along the brane and k_I denotes the momentum transverse to the brane. Let X^I denote the transverse coordinate and $\mathbf{A}_\mu$ the gauge field on the brane. Then in the brane theory, the transverse coordinates are represented by scalar fields $\mathbf{X}^I(\xi)$. Suppose a linearized coupling of the mode to the set of branes is given by

$$\Phi(k_\mu, k_I) \int d^{p+1}\xi \ e^{ik_\mu \xi^\mu} \ \mathrm{Tr}\ [e^{ik_I \mathbf{X}^I} \ \mathcal{O}_\phi(\mathbf{X}, \mathbf{A}, \xi)] \tag{3.1}$$

Such operators can be derived by various methods, for example by using T-duality on Matrix Theory results [11]. Note, this is a coupling to an operator quite similar to (2.23). The only difference is that among the $\mathbf{X}^I$'s some of them, $\mathbf{X}^a$ are expanded around the trivial solution, while the $\mathbf{X}^i$ are expanded around the nontrivial solution $\mathbf{x}^i$. A straightforward extension of the manipulations performed in [1] allows us to rewrite this in the language of functions and star products. This leads to the generalization of the Wilson line given in [3]. The final expression for the coupling of the *same* supergravity mode to a set of M noncommutative $(p+2n+1)$ branes

$$S_{int} = \Phi(k) \int \frac{d^{p+1}\xi \ d^{2n}x}{(2\pi)^n} \ (\mathrm{Pf}B) \ e^{ik_\mu \xi^\mu} \ \mathrm{tr} \ [O_\phi(x + k\cdot\theta, \xi) \star P_\star(W_t(k, A, \phi)) \star e^{ik_i x^i}]$$
$$W_t(k, A, \phi) = \exp[i \int_0^1 d\lambda \ k_i \theta^{ij} A_j(x + \eta(\lambda)) + i \int_0^1 d\lambda \ k_a \phi^a(x + \eta(\lambda))] \tag{3.2}$$

where $\eta^i(\lambda) = \theta^{ji} k_j \lambda$. The operator $O_\phi(x,\xi)$ is obtained from $\mathcal{O}_\phi$ by the mapping discussed in the previous section. In the rest of the paper we will set $k_a = 0$ for simplicity, so that the supergravity mode has no momentum in directions transverse to the resulting noncommutative brane. A nonzero k_a can be restored easily using the above formula.

The coupling to the branes is, however, not quite given by (3.1) [6]. It was found in [11] that the trace appearing in (3.1) is in fact a *symmetrized* trace defined as follows [7]. The operator $\mathcal{O}_\phi(\mathbf{X}, \mathbf{A}, \xi)$ is in general a composite operator made out of field strengths, $\mathbf{F}_{\mu\nu}$, the covariant derivatives of the scalar fields $D_\mu \mathbf{X}^I$ and $[X^I, X^J]$. That is

$$\mathcal{O}_\phi(\mathbf{X}, \mathbf{A}, \xi) = \prod_{\alpha=1}^{n} \mathcal{O}_\alpha(\mathbf{X}, \mathbf{A}, \xi) \tag{3.3}$$

where each of the $\mathcal{O}_\alpha$ denotes a $\mathbf{F}_{\mu\nu}$, $D_\mu \mathbf{X}^I$ or a $[X^I, X^J]$. Then imagine expanding the exponential in $e^{ik\cdot\mathbf{X}}$ in (3.1). For some given term in the exponential we thus have a product of a number of $\mathbf{X}$'s and $\mathcal{O}_\alpha$'s. Finally we symmetrize these various factors of $\mathbf{X}$'s and $\mathcal{O}_\alpha$'s and average. The resulting symmetrized trace will be denoted by the symbol "STr" below. The coupling is then of the form

$$\Phi(k_\mu, k_I) \int d^{p+1} \ \xi \ e^{ik_\mu \xi^\mu} \ \mathrm{STr} \ [e^{ik_I \mathbf{X}^I} \ \prod_{\alpha=1}^{n} \mathcal{O}_\alpha(\mathbf{X}, \mathbf{A}, \xi)] \tag{3.4}$$

[6] Some of the arguments in this section arose out a conversation with M. van Raamsdonk.

[7] A similar symmetrized trace appears in Tsyetlin's prescription for the nonabelian DBI action [36]

Following the same steps as above, it is straightforward to write down the corresponding operator in the star product language. Then the effect of symmetrized trace is to place these various operators $\mathcal{O}_\alpha$ along the path C defining the Wilson line and performing a path ordering. The final result is an operator of the form (1.8),

$$S_{int} = \Phi(k) \int \frac{d^{p+1}\xi \; d^{2n}x}{(2\pi)^n} \; (\mathrm{Pf}B) \; \int \prod_{\alpha=1}^{n} d\tau_\alpha \; e^{ik_\mu \xi^\mu} \; \mathrm{tr} \; P_\star[\prod_{\alpha=1}^{n} O_\alpha(x^i + \theta^{ji} k_j \tau_\alpha) W_t] \star e^{ik_i x^i} \tag{3.5}$$

where W_t has been defined above. This is precisely an operator of the form (1.8).

These couplings are at the linearized level and the backgrounds produced by the branes are ignored, which is the situation at weak string coupling. This means one can couple the p brane to any on-shell supergravity fluctuation about flat space. We see that once this coupling is known the coupling to the $p+2n$ brane is determined uniquely.

4. Generalized star products

The generalized star product is defined by

$$f(x) \star' g(x) = \frac{\sin(\frac{\partial_1 \wedge \partial_2}{2})}{\frac{\partial_1 \wedge \partial_2}{2}} \; f(x_1) g(x_2)|_{x_1 = x_2 = x} \tag{4.1}$$

and the triple product is defined by

$$[f(x)g(x)h(x)]_{*3} = [\frac{\sin(\frac{\partial_2 \wedge \partial_3}{2}) \sin(\frac{\partial_1 \wedge (\partial_2 + \partial_3)}{2})}{\frac{(\partial_1 + \partial_2) \wedge \partial_3}{2} \; \frac{\partial_1 \wedge (\partial_2 + \partial_3)}{2}} + (1 \leftrightarrow 2)] \; f(x_1) g(x_2) h(x_3)|_{x_1 = x_2 = x_3 = x} \tag{4.2}$$

where

$$\partial_1 \wedge \partial_2 = \theta^{ij} \frac{\partial}{\partial x_1^i} \frac{\partial}{\partial x_2^j} \tag{4.3}$$

This $\star'$ product is symmetric in f and g and $\star_3$ is invariant under all permutations of f, g and h [20]. However

$$(f(x) \star' g(x)) \star' h(x) \neq f(x) \star' (g(x) \star' h(x)) \tag{4.4}$$

so that it is *commutative but nonassociative*. Nevertheless it may be verified that

$$\int dx (f(x) \star' g(x)) \star' h(x) = \int dx f(x) \star' (g(x) \star' h(x)) \tag{4.5}$$

so that if the generalized star product appears in an action it does make sense. If one of the three functions in the product $[fgh]_{*3}$ is a constant this reduces to a $\star'$ product

$$[A\ g(x)h(x)]_{*3} = Ag(x) \star' h(x) \qquad A = \text{constant} \tag{4.6}$$

Another property of the $\star'$ product will be useful in the following [20]

$$\theta^{ij}\partial_i f \star' \partial_j g = -i(f * g - g * f) \tag{4.7}$$

As shown in [20], these generalized star products and triple products appear in the expansion of the gauge invariant Wilson line in powers of A. This may be seen by directly expanding the expression (1.4) or equivalently (2.23). The following identity is responsible for the appearance of the $\star'$ product :

$$\int_0^1 d\sigma \mathbf{O}(k)\mathbf{O}(k')e^{i(k\wedge k')\sigma} = \frac{\sin(\frac{k\wedge k'}{2})}{\frac{k\wedge k'}{2}}\ \mathbf{O}(k+k') \tag{4.8}$$

where the operators $\mathbf{O}(k)$ have been defined in (2.10). This leads to the identity

$$\int d\sigma e^{i(k\wedge k')\sigma}\ e^{ikx} \star e^{ik'x} = e^{ikx} \star' e^{ik'x} \tag{4.9}$$

The expansion of $\tilde{\mathcal{O}}(k)$ (defined in (1.4)) to second order in A is (for $k_a = 0$)[20],

$$\mathcal{O}(k) = \int d^{p+1}\xi \frac{d^{2n}x}{(2\pi)^n} e^{ik_\mu\xi^\mu}\ (\mathrm{PfB})\mathrm{tr}\,[\mathcal{O}(x,\xi) + \theta^{ij}\partial_j(\mathcal{O}\star' A_i) + \frac{1}{2}\theta^{ij}\theta^{kl}\partial_j\partial_l[\mathcal{O}A_iA_k]_{*3}] \star e^{ik_ix^i} \tag{4.10}$$

One would expect that at higher orders different structures will emerge.

The operators which are obtained from symmetrized traces as in (3.4) and (3.5) can be similarly expanded in terms of generalized products [7].

In a sense the generalized star product is not a fundamentally different structure : it appears because of the integration over the parameter σ in (4.8). If we retain this integral the answer is always written in terms of the conventional star product. However it is more natural to perform the σ integral to make the operator look local in position space : in that case the $\star'$ product appears. It should be emphasized that whatever the notation, the operator is actually nonlocal in position space - since translations are equivalent to specific noncommutative gauge transformations [3],[5] and the operator has to be gauge invariant.

These generalized star products also appear in the explicit solution of the Seiberg Witten map which relates an ordinary gauge field f_{ij} to the noncommutative gauge field

F_{ij} and the noncommutative gauge potential A_i. In a $U(1)$ theory the map is given, upto two powers of A_i, by [17]

$$f_{ab} = F_{ab} + \theta^{kl}(A_k \star' \partial_l F_{ab} - F_{ak} \star' F_{bl}) + O(A^3) \tag{4.11}$$

Again to higher orders the triple product appears [20]. As explained in the introduction this leads to the apperance of these products in the closed string - open string interactions when expressed in a power series in A_i.

Our proposal for operators which couple to supergravity modes then provides a natural explanation why these same products also appear in the direct string amplitude calculations of [18] and [17]. This is simply because these operators are precisely appropriate momentum space operators with straight Wilson tails.

Our proposal also explains why the generalized star products appeared in one loop effective action calculations [21], [22] in the first place. For usual gauge theories, this one loop effective action for the massless fields obtained in the symmetry breaking $U(N_1 + N_2) \rightarrow U(N_1) \times U(N_2)$ can be alternatively viewed as the potential between a set of N_1 branes and another set of N_2 branes separated by a distance [37] due to exchange of supergravity modes. If the same is true for noncommutative gauge theories, generalized products in the supergravity couplings naturally lead to their presence in the effective action.

5. Dilaton couplings to noncommutative branes

In this section we perform a test of our proposal. We consider the coupling of the dilaton to noncommutative branes in a DBI approximation and show that our proposal is consistent with the opertors which would be obtained by starting out with ordinary gauge fields and using the Seiberg Witten map.

For simplicity we consider a *single* noncommutative *euclidean* $D(2n-1)$ brane (with $2n$ dimensional worldvolume) and we will construct this from a large number N of $D(-1)$ branes. Following [10],[11]and [19] we will assume that the action in the presence of a dilaton field $D(x)$ (with all backgrounds trivial) is given by

$$S = \frac{1}{g_s} \mathrm{STr}\ e^{-D(\mathbf{X}^I)} \sqrt{\det(\delta^I_J - i[\mathbf{X}^I, \mathbf{X}^K] g_{KJ})} \tag{5.1}$$

Here, as before, X^I denote all the d transverse coordinates. g_{IJ} is a constant closed string metric which is taken to be diagonal as well [8] and g_s is the closed string coupling. The meaning of the symmetrized trace has been explained in the previous section. As in the previous sections we will write the background in terms of its fourier transform so that for a given space-time momentum k the linearized coupling is

$$S_{int} = \frac{D(k)}{g_s} \mathrm{STr}\ e^{ik\cdot\mathbf{X}} \sqrt{\det(\delta^I_J - i[\mathbf{X}^I, \mathbf{X}^K]g_{KJ})} \tag{5.2}$$

where $D(k)$ is the fourier transform of $D(x)$ The classical solution which leads to a non-commutative $(2n-1)$ brane

$$\begin{aligned} \mathbf{X}^i &= x^i \qquad i = 1\cdots 2n \\ \mathbf{X}^a &= 0 \qquad a = (2n+1)+\cdots d \end{aligned} \tag{5.3}$$

We then expand around this classical solutions as in (2.3) and (2.4). To simplify things further we will assume that $k_a = 0$ so that we have dependence only on X^i and also set the scalar fields to zero, $\phi^a = 0$. It is trivial to repeat the following for nonzero k_a and ϕ^a.

In the following we will be interested in terms upto $O(A^2)$ in the noncommutative gauge fields. In the language of matrices we will be interested in terms which contain at most two matrices. For such terms there is no distinction between the symmetrized trace and ordinary trace. We will therefore replace STr in (5.2) with Tr. Using the results of section 2, this interaction is then written in terms of noncommutative gauge fields F_{ij}

$$S_{int} = \frac{D(k)}{g_s} |\sqrt{\det B}| \int d^{2n}x\ e^{ikx}\ P_*[\exp\ (i\int d\eta^i A_i(x+\eta(\lambda)))]\ \sqrt{\det(I - \theta(F-B)\theta g)} \tag{5.4}$$

where in (5.4) the quantities θ, F, B, g are written as $(2n)\times(2n)$ matrices and I stands for the identity matrix, in a natural notation. In the following whenever these quantities appear without indices they denote these matrices. We now use (2.15) and (2.16) to write this in terms of the open string metric G_{ij} and the open string coupling G_s as

$$S_{int} = \frac{D(k)}{G_s} \int d^{2n}x\ e^{ikx}\ P_*[\exp\ (i\int d\eta^i A_i(x+\eta(\lambda)))]\ \sqrt{\det(G+F-B)} \tag{5.5}$$

Here the path used is given by (1.5) and all products are star products.

[8] the diagonal nature of the closed string metric has been used to arrive at (5.1) starting from the form of action given e.g. in [10].

In terms of the ordinary gauge fields f_{ij}, the closed string metric and the closed string coupling, the interaction may be read off from the standard Dirac-Born-Infeld action

$$\tilde{S}_{int} = \frac{D(k)}{g_s} \int d^{2n}x \; e^{ikx} \; \sqrt{\det(g + f + B)} \tag{5.6}$$

The strategy is now to express (5.6) in terms of the noncommutative gauge field F_{ij} using the Seiberg-Witten map in a series involving powers of the potential A_i and compare the result with (5.5) which is also expanded in a similar fashion.

For zero momentum operators this is the comparison done in [16], where it is shown that

$$\frac{1}{g_s}\sqrt{\det(g + f + B)} = \frac{1}{G_s}\sqrt{\det(G - B + F)} + O(\partial F) + \text{total derivatives} \tag{5.7}$$

which shows the equivalence of the two actions in the presence of constant backgrounds. The crucial aspect of our comparison is the presence of these total derivative terms in (5.7), which cannot be ignored if $k \neq 0$. We will find that these total derivative terms are in precise agreement with similar terms coming from the expansion of the Wilson tail in (5.5), upto $O(A^2)$.

Since we are using the DBI action, the field strengths should be really treated as constant. In carrying out the comparison though some caution must be exercised. Since the Seiberg-Witten map contains gauge potentials as well as field strengths a term containing a derivative of a field strength multiplied by a gauge potential without a derivative on it, cannot be set automatically to zero, as emphasised in [16].

5.1. $O(A)$ comparison

First let us do the comparaison to $O(A_i)$. To this order the Seiberg-Witten map in (4.11) simply reduces to $f_{ij} = \partial_i A_j - \partial_j A_i + O(A^2)$. Thus it is sufficient to expand the determinant in (5.6) to linear order in f. One obtains

$$\tilde{S}^{(1)}_{int} = \frac{D(k)}{g_s}\sqrt{\det(g + B)} \int d^{2n}x \; e^{ikx} \; [1 + \frac{1}{2}(\frac{1}{(g+B)})^{ij}(\partial_j A_i - \partial_i A_j) + O(A^2)] \tag{5.8}$$

Using (2.16) and (2.17) this may be written as

$$\tilde{S}^{(1)}_{int} = \frac{D(k)}{G_s}\sqrt{\det(G - B)} \int d^{2n}x \; e^{ikx} \; [1 + \frac{1}{2}(\frac{1}{(G-B)} + \theta)^{ij}(\partial_j A_i - \partial_i A_j) + O(A^2)] \tag{5.9}$$

We have to compare this with the expansion of the expression (5.5) to $O(A)$. In this expression all products are star products. To do this we can use (4.10) with the function $\mathcal{O}$ being replaced by the quantity $\sqrt{\det(G+F-B)}$. To linear order in A we have

$$\sqrt{\det(G+F-B)} = \sqrt{\det(G-B)}[1+\frac{1}{2}(\frac{1}{G-B})^{ij}(\partial_i A_j - \partial_j A_i) + O(A^2)] \qquad (5.10)$$

The various products appearing on the left hand side of the above equation are star products. However to this order these collapse to ordinary products since G, B etc. are constants. Also to this order one has

$$\theta^{ij}\partial_j(\sqrt{\det(G+F-B)} \star' A_i) = \theta^{ij}\sqrt{\det(G-B)}\partial_j A_i = \frac{1}{2}\sqrt{\det(G-B)}\theta^{ij}(\partial_j A_i - \partial_i A_j) \qquad (5.11)$$

Thus substituting (5.10) and (5.11) in S_{int} we have after using (4.8)

$$S_{int}^{(1)} = \frac{D(k)}{G_s}\sqrt{\det(G-B)} \int d^{2n}x e^{ikx} \, [1+\frac{1}{2}(\frac{1}{(G-B)}+\theta)^{ij}(\partial_j A_i - \partial_i A_j) + O(A^2)] \qquad (5.12)$$

which is exactly the same as (5.9).

Note that the term proportional to θ in (5.9) came because of the relation (2.17), while the corresponding term in (5.12) came from the "Wilson tail" involved in the gauge invariant operator. To this order one is sensitive only to the linear term of the Seiberg Witten map. However the agreeement of the two derivations of the interaction term is still nontrivial and the importance of the open Wilson line is evident.

5.2. $O(A^2)$ comparison

To next order, several points have to be remembered. First the Seiberg-Witten map is nontrivial. Secondly, we have to be careful about where we can ignore star products and where we can not. The strategy is the same as in the previous subsection. We expand the expression in terms of ordinary gauge fields to the required order and reexpress the terms using Seiberg Witten map, after using (2.17). Finally we compare the resulting expression with the expansion of (5.5). It will be necessary to write the noncommutative gauge field strength (2.13) in terms of $\star'$ products by using (4.7),

$$F_{ij} = \partial_i A_j - \partial_j A_i + \theta^{kl}(\partial_k A_i \star' \partial_l A_j) \qquad (5.13)$$

The details of the calculation are given in the Appendix. Here we quote the final result. The expansion of (5.6) becomes

$$\frac{\tilde{S}_{int}}{D(k)} = \frac{\sqrt{\det(G-B)}}{G_s}\int d^{2n}x\ e^{ikx}[1+\frac{1}{2}M^{ij}F_{ji}-\frac{1}{4}M^{ij}F_{jk}M^{kl}F_{li}+\frac{1}{8}M^{ij}F_{ji}M^{kl}F_{lk}$$
$$+\theta^{ij}\partial_j A_i+\frac{1}{4}\theta^{ij}F_{ji}M^{kl}F_{lk}+\frac{1}{2}\theta^{ij}M^{kl}(\partial_j F_{lk}\star' A_i)$$
$$+\frac{1}{2}\theta^{ij}\theta^{kl}(\partial_l F_{ji}\star' A_k)+\frac{1}{8}\theta^{ij}F_{ji}\theta^{kl}F_{kl}$$
$$+\frac{1}{2}\theta^{ij}\theta^{kl}(\partial_l A_i\star'\partial_j A_k)+O(A^3)] \tag{5.14}$$

where we have defined

$$M^{ij} = (\frac{1}{G-B})^{ij} \tag{5.15}$$

The result is *exact to all orders in* θ, but to $O(A^2)$.

On the noncommutative side, (5.5) may be written as

$$S_{int} = \frac{D(k)}{G_s}\int d^{2n}x\, e^{ikx}[P(x)+\theta^{ij}\partial_j(P(x)\star' A_i)+\frac{1}{2}\theta^{ij}\theta^{kl}\partial_j\partial_l[P(x)A_i(x)A_k(x)]_{*3}]+O(A^3) \tag{5.16}$$

where

$$P(x) = \sqrt{\det(G+F(x)-B)} \tag{5.17}$$

Here, in the expansion of the determinant in powers of F we can replace the star product by ordinary products. We perform this expansion to the requisite power of A and find that the result is in exact agreement with (5.14).

It should be possible to extend the discussion in this section can be extended to any other mode, to include fluctuations of the ϕ^a and to Lorentzian signature. It will be particularly interesting to study the couplings to the RR fields : these Chern-Simons couplings for *constant* backgrounds have been obtained in [38] and one has to extend this to nonconstant backgrounds using the proposal of this paper.

5.3. Higher orders

To higher order terms in A, the difference between symmetrized trace and ordinary trace becomes important. Because of this, it is important to extend the above comparison to higher orders. We have not performed this calculation yet.

5.4. Other Descriptions

Let us make one comment before proceeding. Constructing the noncommutative Yang Mills theory from the $D(-1)$ branes leads (in the notation of [16]) to the $\Phi = -B$ description. One conjecture for the coupling of the Dilaton in a general description is:

$$S_{int} = \frac{D(k)}{G_s} \int d^{2n}x \; e^{ikx} \; P_*[\exp\,(i \int d\eta^i A_i(x+\eta(\lambda)))] \; \sqrt{\det(G+F+\Phi)} \qquad (5.18)$$

The $\Phi = B$ description corresponds to the case with ordinary gauge field f_{ij}, closed string metric and closed string coupling and to the dilaton coupling (5.6).

Above we showed that upto $O(A^2)$ the coupling in the $\Phi = -B$ description agreed with that in the $\Phi = B$ case. It is interesting to note that the calculations can be repeated for other descriptions, i.e. other values of Φ in a very straightforward fashion. One finds that the ansatz (5.18) in other descriptions agrees with the coupling (5.6) as well, upto $O(A^2)$. This follows in a straightforward fashion by repeating the calculation above and noting that in the general case, (2.16) and (2.17) are replaced by

$$\frac{1}{G+\Phi} = -\theta + \frac{1}{g+B}, \qquad (5.19)$$

and

$$\frac{\sqrt{det(g+B)}}{g_s} = \frac{\sqrt{det(G+\Phi)}}{G_s}. \qquad (5.20)$$

6. Holographic duals

Gauge invariant operators also appear in the context of holography. Extending the well known AdS/CFT correspondence [39], it has been proposed in [40][41]that noncommutative gauge theories are holographic descriptions of string theories living in appropriate backgrounds . Then the supergravity (or more generally string theory) modes should be dual to momentum space operators, as emphasized in [41],[42],[8]. Naturally these momentum space operators should be related to the set of gauge invariant operators discussed above [3],[5],[6]. In fact for $d = 3$ it has been argued in [3] that the relationship between the momentum and the extent in the noncommutative directions encoded in the defintion of the open Wilson loop operators is visible in dual supergravity. Similarly [5] argue that the universal large momentum behavior of the operators (1.4) is in agreement with similar behavior in dual supergravity found in [41]. In fact supergravity predicts an interesting crossover in the behavior of *closed* Wilson loops [43]

In principle these operators are logically distinct from operators coupled to linearized supergravity about flat space, though in many cases, they are related to the operators obtained by lineraization around the *background geometry*[44] Moreover as argued in [11], it is possible to obtain the correlation function of the holographically dual operators from those of the operators obtained by coupling to linearized supergravity (around *flat* spacetime) by solving the scattering problem in the full geometry.

Consider a non-commutative Yang Mills theory in $p+2n+1$ dimensions. We remind the reader that in our notation, (see comments following equation (1.1)) the non-commutativity parameter has rank $2n$; p denotes the remaining spatial directions with no noncommutativity. In the following we will define

$$d = p + 2n \tag{6.1}$$

to save clutter in the formulae.

6.1. Dual backgrounds

The supergravity duals were discussed in [40],[41], [45], [24]. The metric in these backgrounds are given by:

$$ds^2 = (\frac{r}{R})^{(\frac{7-d}{2})}\big(-dt^2 + \sum_{i=1}^{p} dx_i^2 + \sum_{i \text{ odd}}^{2n-1} h_i(dy_i^2 + dy_{i+1}^2)\big) + (\frac{R}{r})^{(\frac{7-d}{2})}(dr^2 + r^2 d\Omega^2_{8-p-2n}). \tag{6.2}$$

where

$$r^2 = \sum_{i=2n+1}^{9-p} y_i y^i \tag{6.3}$$

is the radial coordinate in the $9-p-2n$ directions transverse to the brane and

$$\begin{aligned} h_i =& \frac{1}{1 + a_i^{7-d} r^{7-d}} \\ a_i^{7-d} =& \frac{b_i^2}{R^{7-d} l_s^4} \\ R^{7-d} =& (4\pi)^{(\frac{7-d-2}{2})}\Gamma(\frac{7-d}{2})\ g_s N l_s^{(7-d)} \prod_{i \text{ odd}} b_i. \end{aligned} \tag{6.4}$$

N above refers to the number of $p+2n$ branes. Similarly the two-form NS field B and the dilaton are

$$\begin{aligned} B_{i,i+1} =& \frac{l_s^2}{b_i} \frac{a^{7-d} r^{7-d}}{1 + a^{7-d} r^{7-d}}, \quad i = \{1, 3, \cdots, 2n-1\} \\ e^{2\phi} =& g_s^2 (\frac{R^{7-d}}{r^{7-d}})^{(\frac{7-d-4}{2})} \prod_{i \text{ odd}} \frac{b_i}{l_s^2} h_i \end{aligned} \tag{6.5}$$

Note that in (6.2) $x_i, i = 1, \cdots p$ and $y_i = 1, \cdots 2n$ denote the $p+2n$ directions parallel to the brane. The corresponding gauge theory also lives in $p+2n$ space directions with non-commutativity parameters turned on along the $2n$ directions, $y = 1, \cdots 2n$. Let us now consider what happens to the metric in the asymptotic region, $a_i r \gg 1, i = \{1, 3 \cdots 2n-1\}$. In this region

$$h_i \to \frac{1}{a^{7-d} r^{7-d}} = \frac{R^{7-d}}{r^{7-d}} \frac{l_s^4}{b_i^2}. \tag{6.6}$$

Rescaling $y_{i,i+1} \to \frac{b_i}{l_s^2} y_{i,i+1}$ then gives (6.2)

$$ds^2 = (\frac{r}{R})^{\frac{7-d}{2}} \big(-dt^2 + \sum_{i=1}^{p} dx_i^2\big) + (\frac{R}{r})^{\frac{7-d}{2}} \big(\sum_{i=1}^{2n} dy_i^2 + dr^2 + r^2 d\Omega_{8-p-2n}^2\big). \tag{6.7}$$

The NS field, B, goes to constant asymptotically and the dilaton is given by

$$e^{2\phi} = g_s^2 (\frac{R^{7-p}}{r^{7-p}})^{\frac{3-p}{2}}. \tag{6.8}$$

In comparison the metric and dilaton background dual to a p dimensional ordinary gauge theory are

$$\begin{aligned} ds^2 =& H^{-1/2}(dt^2 + \sum_{i=1}^{p} dx_i^2) + H^{1/2}(dy_i^2) \\ e^{2\phi} =& g_s^2 H^{\frac{3-p}{2}}. \end{aligned} \tag{6.9}$$

Here, H denotes the appropriate harmonic function which in general can depend on the $9-p$ transverse coordinates, y_i. When the p branes are uniformly distributed in $2n$ of these $9-p$ transverse directions the harmonic function is given by

$$\begin{aligned} H =& \frac{R_p^{7-p}\rho}{r^{7-p}} \\ \text{where} \quad R_p^{7-p} =& (4\pi)^{\frac{5-p}{2}} \Gamma(\frac{7-p}{2}) g_s l_s^{7-p}. \end{aligned} \tag{6.10}$$

In (6.10) ρ is the number density of p branes along the $2n$ directions and r is the transverse distance in the remaining $9-p-2n$ transverse directions.

Comparing (6.7) (6.8) with (6.9) (6.10) shows that they are identical, with,

$$\rho = \frac{1}{(4\pi)^n} \frac{\Gamma(\frac{m}{2})}{\Gamma(\frac{7-p}{2})} \frac{N}{l_s^{4n}} \prod_{i \quad \text{odd}} b_i. \tag{6.11}$$

In short, asymptotically, the background (6.2) (6.5), becames identical to multicene-terd version of the dual for an ordinary $p+1$ dimensional Yang Mills theory, with the branes distributed uniformly along $2n$ transverse directions. This behavior is in accord with our description in section 2 of the non-commutative $p+2n+1$ diemsnional gauge theory as a particular *state* in the p dimensional ordinary Yang Mills theory. In the $N \to \infty$ limit, matrices which satisfy (2.2) have eigenvalues which are uniformly distributed between $-\infty$ and $+\infty$. This is in agreement with what we have found above where the branes are uniformly distributed in the $2n$ directions.

The discussion above implies that supergravity modes which are perturbations about (6.2) (6.5) must asymptotically map in a one to one manner to modes about the background (6.7) (6.8). The latter background is considerably simpler, and being the dual of the multicentered ordinary Yang Mills theory, in some cases better understood. This simplies the task of classifying the sugra modes in the noncommutative background.

6.2. Normal modes and dual operators

The background (6.2)(6.5) has nonzero values for several of the supergravity fields. As a result the analysis of small fluctuations around such a background is rather complicated and it is difficult to find normal modes which satisfy decoupled equations. On the other hand these normal modes should be dual to independent gauge invariant operators of the holographic theory on the boundary.

One such mode is known for supergravity duals of $3+1$ dimensional NCYM with noncommutativity matrix of rank 2. In the notation of (6.2) we now have $p=1, n=1$ and the noncommutativity is in the (y_1, y_2) direction. In that case denote the component of the ten dimensional graviton polarization along the (t, x_1) directions by $h_{tx_1}(k_\mu; k_i; k_a)$. We have used the notation of section 3 : k_μ denotes momenta along the commutative directions t, x_1, $k_i, i=1,2$ denote momenta along the noncommutative directions y_1, y_2 and $k_a, a=1,\cdots 6$ denote the momenta transeverse to the three-brane which may be also written in terms of a radial momentum along r and the angular momenta on the S^5. When $k_a=0$ the angular momentum along the S^5 is zero and this is a decoupled mode [41]. To extract the dual operator in the gauge theory, we use the fact that the asymptotic geometry is that of an infinite set of $D1$ branes along x_1 which are smeared in the two transverse directions y_1, y_2. In the $D1$ brane the operator which couples to the s-wave graviton with polarization along (t, x_1) would be given by

$$\int d^2\xi \ \mathrm{Tr}\ [e^{ik_i\mathbf{X}^i}\ \mathcal{T}_{\mathbf{tx_1}}] \tag{6.12}$$

where $\mathcal{T}_{tx_1}$ is the operator whose trace gives the energy momentum tensor component T_{tx_1}. The exponential factor gives the the operator a R-charge or equivalently momentum along the directions y_1, y_2.

Such decoupled modes are, however, rare. In general it is rather difficult to find these from the supergravity equations. The observation of the previous subsection, however, relates this problem to a possibly easier problem of decoupling the equations around the background of a set of lower dimensional branes with no B fields. It would be intersting to see whether this approach is indeed fruitful.

7. Conclusions

We have proposed a definitive way to identify operators which couple supergravity modes to noncommutative branes. These are operators smeared along *straight* Wilson tails. It is gratifying that these operators involve the simplest form of nonlocality required by gauge invariance. We have tested our proposal in a rather simple setting, viz. for an abelian theory and in the DBI approximation. Contrary to naive expectations even this test is rather nontrivial. We expect the couplings of supergravity modes to non-commutative branes found in this paper to be true in general, beyond these approximations as well, since it only relies on the construction of non-commutative branes theory from lower dimensional non-abelian branes. It is important to check this by comparing with direct string amplitude computations.

By the nature of our construction, we obtain the operators in the $\Phi = -B$ description of the noncommutative gauge theory. The operators in some other description may be in principle obtained by using the Seiberg Witten map between these two descriptions. In fact, in the DBI approximation we have argued that the operators in any other description may be written down by a straightforward replacement of the parameters. It is not clear what happens beyond the DBI approximation. One possibility is to consider the Seiberg-Witten low energy limit. In this case, one may hope to obtain the operators in some other description - in particular the $\Phi = 0$ description by using the Seiberg Witten map in a low energy approximation. It would be interesting to see whether the resulting operators have again a simple form.

One might hope that our proposal can be used to identify the operators which are involved in the holographic map as well. Here the fact that the supergravity duals asymptote to geometries which are those of lower dimensional branes smeared over some of the directions may be helpful.

8. Appendix

In this appendix we give the details of the calculations which lead to the result (5.14) both from the expansion of (5.6) and (5.16).

We will use the following properties

(1) The noncommutative field strength may be written in terms of the $\star'$ product as in (5.13).

(2) The $\star'$ product is commutative

Furthermore since we are dealing with the DBI approximation and working to only $O(A^2)$

(3) In terms which are $O(A)$ we have to keep the full expression for F_{ij} as in (5.13). However in terms which are $O(A^2)$ we can replace F_{ij} by $\partial_i A_j - \partial_j A_i$.

(4) In terms which involve only the F_{ij} with no explicit A_i we can replace the $\star$ and $\star'$ product by ordinary products.

In the following we will refer to these as rules (1)-(4) respectively.

Consider first the expansion of the integrand of (5.6), which we denote by I_{com}

$$I_{com} = \frac{\sqrt{\det(g+B)}}{g_s}[1 + \frac{1}{2}N^{ij}f_{ji} - \frac{1}{4}N^{ij}f_{jk}N^{kl}f_{li} + \frac{1}{8}N^{ij}f_{ji}N^{kl}f_{lk}] + O(f^3) \tag{8.1}$$

where we have defined

$$N^{ij} = (\frac{1}{g+B})^{ij} \tag{8.2}$$

We have to now use (2.17) to write

$$N^{ij} = M^{ij} + \theta^{ij} \tag{8.3}$$

where M is defined in (5.15). Now use the Seiberg-Witten map to express this in terms of F_{ij}. In the second term of (8.1) we have to use equation (4.11) to get

$$\begin{aligned}\frac{1}{2}N^{ij}f_{ji} =& \frac{1}{2}M^{ij}F_{ji} + \frac{1}{2}\theta^{ij}[(\partial_j A_i - \partial_i A_j) + \theta^{kl}(\partial_k A_j \star' \partial_l A_i)] \\ &+ \frac{1}{2}M^{ji}\theta^{kl}(A_k \star' \partial_l F_{ij} - F_{ik} \star' F_{jl}) \\ &+ \frac{1}{2}\theta^{ji}\theta^{kl}(A_k \star' \partial_l F_{ij} - F_{ik} \star' F_{jl}) + O(A^3)\end{aligned} \tag{8.4}$$

In the first line we have kept the F_{ji} as it is when it multiplies the matrix M, but have used the expansion in terms of A_i in (5.13) when it multiplies θ. The reason will become clear soon. Using the observations (1)-(4) we can now write this (after some simplification)

$$\begin{aligned}\frac{1}{2}N^{ij}f_{ji} =& \frac{1}{2}(M^{ij}F_{ji}) + \theta^{ij}(\partial_j A_i)\\ &+ \frac{1}{2}M^{ji}\theta^{kl}(A_k \star' \partial_l F_{ij}) + \frac{1}{2}(M^{ji}F_{ik}\theta^{kl}F_{lj})\\ &+ \frac{1}{2}\theta^{ji}\theta^{kl}(A_k \star' \partial_l F_{ij}) + \frac{1}{2}\theta^{ij}\theta^{kl}(\partial_k A_j \star' \partial_l A_i) - \frac{1}{2}\theta^{ji}\theta^{kl}F_{ik}F_{jl} + O(A^3)\end{aligned} \tag{8.5}$$

In the third and fourth terms of (8.1) already contain two powers of F. Thus to $O(A^2)$ we can set $f_{ij} = F_{ij}$ and we get

$$\begin{aligned}-\frac{1}{4}N^{ij}f_{jk}N^{kl}f_{li} &= -\frac{1}{4}(M^{ij}F_{jk}M^{kl}F_{li}) - \frac{1}{4}(\theta^{ij}F_{jk}\theta^{kl}F_{li}) - \frac{1}{2}(M^{ij}F_{jk}\theta^{kl}F_{li})\\ \frac{1}{8}(N^{ij}f_{ji})(N^{kl}f_{lk}) &= \frac{1}{8}(M^{ij}F_{ji})(M^{kl}F_{lk}) + \frac{1}{8}(\theta^{ij}F_{ji})(\theta^{kl}F_{lk}) + \frac{1}{4}(M^{ij}F_{ji})(\theta^{kl}F_{lk})\end{aligned} \tag{8.6}$$

Adding these various contributions and using the realation between g_s and G_s in (2.15) we get (5.14).

Now consider the expansion of the integrand of (5.16)

$$I_{nc} = \frac{1}{G_s}[P(x) + \theta^{ij}\partial_j(P(x) \star' A_i) + \frac{1}{2}\theta^{ij}\theta^{kl}\partial_j\partial_l[P(x)A_i(x)A_k(x)]_{*3}] \tag{8.7}$$

where

$$P(x) = \sqrt{\det(G - B + F(x))} \tag{8.8}$$

In the first term of (8.7) we will need the expansion of the determinant to $O(F^2)$, in the second term we need the expansion to $O(F)$ and in the last term to $O(F^0)$. The first term becomes

$$P(x) = \sqrt{\det(G-B)}[1 + \frac{1}{2}(M^{ij}F_{ji}) - \frac{1}{4}(M^{ij}F_{jk}M^{kl}F_{li}) + \frac{1}{8}(M^{ij}F_{ji})(M^{kl}F_{lk})] \tag{8.9}$$

We have replaced star products by ordinary products in accordance to our rule (4) above. The second term in (8.7) becomes

$$\begin{aligned}\theta^{ij}\partial_j(P \star' A_i) = \sqrt{\det(G-B)}[&\theta^{ij}\partial_j A_i + \frac{1}{2}M^{kl}\theta^{ij}(\partial_j F_{lk} \star' A_i)\\ &+ \frac{1}{2}M^{kl}\theta^{ij}(F_{lk} \star' \partial_j A_i)]\end{aligned} \tag{8.10}$$

Using rules (2)-(4) above this may be written as

$$\begin{aligned}\theta^{ij}\partial_j(P \star' A_i) = \sqrt{\det(G-B)}[&\theta^{ij}\partial_j A_i + \frac{1}{2}M^{kl}\theta^{ij}(\partial_j F_{lk} \star' A_i)\\ &+ \frac{1}{4}(M^{kl}F_{lk})(\theta^{ij}F_{ji})]\end{aligned} \tag{8.11}$$

Finally we consider the third term in (8.7). Here we can replace P by $\sqrt{\det(G-B)}$ since the other terms can contribute only $O(A^3)$ terms. Then the triple product collapses to a $\star'$ product by virtue of (4.6). Using this one finds

$$\begin{aligned}\frac{1}{2}\theta^{ij}\theta^{kl}\partial_j\partial_l[PA_iA_k]_{*3} = \frac{1}{2}\sqrt{\det(G-B)}\theta^{ij}\theta^{kl}[&\partial_l\partial_j A_i \star' A_k + A_i \star' \partial_l\partial_j A_k \\ &+ \partial_j A_i \star' \partial_l A_k + \partial_l A_i \star' \partial_j A_k]\end{aligned} \tag{8.12}$$

Since this is already $O(F^2)$ we can use rule (3) above and then use the rule (4) to write

$$\begin{aligned}\frac{1}{2}\theta^{ij}\theta^{kl}\partial_j\partial_l[PA_iA_k]_{*3} = \sqrt{\det(G-B)}[&\frac{1}{2}\theta^{ij}\theta^{kl}(\partial_l F_{ji} \star' A_k) \\ &+ \frac{1}{8}(\theta^{ij}F_{ji})(\theta^{kl}F_{lk}) + \frac{1}{2}\theta^{ij}\theta^{kl}(\partial_l A_i \star' \partial_j A_k) + O(A^3)]\end{aligned} \tag{8.13}$$

Adding the contributions (8.9),(8.10)and (8.13) and using the commutative nature of the $\star'$ product one again gets (5.14).

9. Acknowledgements

We would like to thank S. Mukhi for discussions. We thank M. van Raamsdonk for comments about the first version of this paper, especially for pointing out the importance of symmetrized traces in supergravity coupling to branes.

References

[1] N. Ishibashi, S. Iso, H. Kawai and Y. Kitazawa, Nucl. Phys. B573 (2000) 573, hep-th/9910004.

[2] J. Ambjorn, Y. Makeenko, J. Nishimura and R. Szabo, JHEP 11 (1999) 029, hep-th/9911041; J. Ambjorn, Y. Makeenko, J. Nishimura and R. Szabo, Phys. Lett B 480 (2000) 399, hep-th/0002158; J. Ambjorn, Y. Makeenko, J. Nishimura and R. Szabo, JHEP 05 (2000) 023, hep-th/0004147.

[3] S.R. Das and S.J. Rey, hep-th/0008042.

[4] S.J. Rey and R. von Unge, hep-th/0007089.

[5] D. Gross, A. Hashimoto and N. Itzhaki, hep-th/0008075.

[6] A. Dhar and S.R. Wadia, hep-th/0008144.

[7] H. Liu, hep-th/0011125

[8] S.R. Das and B. Ghosh, JHEP 06 (2000) 043, hep-th/0005007.

[9] C. Callan and J. Maldacena, Nucl. Phys. B475 (1996) 645, hep-th/9602043; A. Dhar, G. Mandal and S. wadia, Phys. Lett. 388B (1996) 51, hep-th/9605234; S.R. Das and S. Mathur, Nucl. Phys. B478 (1996) 561, hep-th/9606185; I. Klebanov, Nucl. Phys. B496 (1997) 231, hep-th/9702076; S. Gubser, I. Klebanov and A. Tseytlin, Nucl. Phys. B499 (1997) 217; I. Klebanov, W. Taylor and M. van Raamsdonk, Nucl.Phys. B560 (1999) 207, hep-th/9905174.

[10] R. Myers, JHEP 9912 (1999) 022, hep-th/9910053.

[11] W. Taylor and M. van Raamsdonk, Nucl. Phys. B558 (1999) 63, hep-th/9904095; W. Taylor and M. van Raamsdonk, Nucl. Phys. B573 (2000) 703, hep-th/9910052

[12] H. Aoki, N. Ishibashi, S. Iso, H. Kawai, Y. Kitazawa and T. Tada, Nucl.Phys.B565 (2000) 176, hep-th/9908141

[13] N. Ishibashi, hep-th/9909176; N. Ishibashi, Nucl.Phys. B539 (1999)107, hep-th/9804163

[14] L. Cornalba and R. Schiappa, hep-th/9907211; L. Cornalba, hep-th/9909081.

[15] N. Seiberg, hep-th/0008013.

[16] N. Seiberg and E. Witten, JHEP 9909 (1999) 032, hep-th/9908142.

[17] M. Garousi, Nucl. Phys. B579 (2000) 209, hep-th/9909214

[18] S. Hyun, Y. Kiem, S. Lee and C.Y. Lee, hep-th/9909059

[19] M. Garousi and R. Myers, hep-th/0010122.

[20] T. Mehen and M. Wise, hep-th/0010204.

[21] H. Liu and J. Micheslon, hep-th/0008205

[22] D. Zanon, hep-th/0009196; A. Santambrogio and D. Zanon, hep-th/0010275

[23] F. Ardalan and N. Sadoghi, hep-th/0009233.

[24] J.X. Lu and S. Roy, Nucl.Phys. B579 (2000) 229, hep-th/9912165; R. Cai and N. Ohta, JHEP 0003 (2000) 009, hep-th/0001213.

[25] T. Eguchi and H. Kawai, Phys. Rev. Lett. 48 (1982) 1063; G. Parisi, Phys. Lett. 112B (1982) 463; G. Bhanot, U. Heller and H. Neuberger, Phys. Lett 113B (1982) 47; D. Gross and Y. Kitazawa, Nucl. Phys. B 206 (1982) 440; S.R. Das and S.R. Wadia, Phys. Lett. 117B (1982) 228; S.R. Das, Rev. Mod. Phys. 59 (1987) 235.

[26] A. Gonzalez-Arroyo and M. Okawa, Phys. Rev. D27 (1983) 2387; T. Eguchi and R. Nakayama, Phys. Lett. 122B (1983) 59; A. Gonzalez-Arroyo and C. Korthals-Altes, Phys. Lett. 131B (1983) 396.

[27] B. deWit, J. Hoppe and H. Nicolai, Nucl. Phys B305 (1989) 135.

[28] T. Banks, W. Fischler, S. Shenker and L. Susskind, Phys. Rev. D55 (1997) 5112, hep-th/9610043.

[29] N. Ishibashi, H. Kawai, Y. Kitazawa and A. Tsuchiya, Nucl. Phys. B498 (1997) 467, hep-th/9612115.

[30] O.J. Ganor, S. Ramgoolam and W. Taylor, Nucl. Phys. B492 (1997) 191, hep-th/9611202; T. Banks, N. Seiberg and S. Shenker, Nucl. Phys. B490 (1997) 91, hep-th/9612157.

[31] A. Connes, M. Douglas and A. Schwarz, JHEP 9802 (1998) 003, hep-th/9711162; M. Douglas and C. Hull, JHEP 9802 (1998) 008, hep-th/9711165.

[32] V. Schomerus, JHEP 9906 (1999) 030

[33] M. Li, Nucl. Phys. B499 (1997) 149, hep-th/9612222.

[34] I. Bars and D. Minic, hep-th/9910091

[35] L. Alvarez-Gaume and S.R. Wadia, hep-th/0006219; A.H. Fatollahi, hep-th/0007023.

[36] A. Tseytlin, Nucl. Phys. B501 (1997) 41, hep-th/9701125.

[37] M. Douglas, D. Kabat, P. Pouliot and S. Shenker, Nucl. Phys. B485 (1997) 85, hep-th/9608024; M. Douglas and W. Taylor, hep-th/9807225; S.R. Das, JHEP 9902 (1999) 012, hep-th/9901004; S.R. Das, JHEP 9906 (1999) 029, hep-th/9905037.

[38] S. Mukhi and N. Suryanarayana, hep-th/0009101.

[39] J. Maldacena, Adv. Math. Theo. Phys. 2 (1998) 231, hep-th/9711200; S.S. Gubser, I.R. Klebanov and A.M. Polyakov, Phys. Lett. 428B (1998) 105, hep-th/9802109; E. Witten, Adv. Theo. Math. Phys. 2 (1998) 253, hep-th/9802150.

[40] A. Hashimoto and N. Itzhaki, Phys. Lett. B465 (1999) 142, hep-th/9907166

[41] J. Maldacena and J. Russo, JHEP 09 (1999) 025, hep-th/9908134.

[42] S.R. Das, S. Kalyana Rama and S.P. Trivedi, JHEP 03 (2000) 004, hep-th/9911137.

[43] A. Dhar and Y. Kitazawa, hep-th/0010256.

[44] See e.g. S.R. Das and S.P. Trivedi, Phys.Lett. B445 (1998) 142, hep-th/9804149; S. Ferrara, M. A. Lled, A. Zaffaroni, Phys.Rev. D58 (1998) 105029, hep-th/9805082.; I.Y. Park, A. Sadrzadeh and T.A. Tuan, hep-th/0010116.

[45] M. Alishalia, Y. Oz and M. M. Sheikh-Jabbari, hep-th/9909215