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GEOMETRIC ENTROPY OF NON-RELATIVISTIC FERMIONS AND TWO DIMENSIONAL STRINGS

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Abstract

We consider the geometric entropy of free nonrelativistic fermions in two dimensions and show that it is ultraviolet finite for finite fermi energies, but divergent in the infrared. In terms of the corresponding collective field theory this is a *nonperturbative* effect and is related to the soft behaviour of the usual thermodynamic entropy at high temperatures. We then show that thermodynamic entropy of the singlet sector of the one dimensional matrix model at high temperatures is governed by nonperturbative effects of the underlying string theory. In the high temperature limit the "exact" expression for the entropy is regular but leads to a negative specific heat, thus implying an instability. We speculate that in a properly defined two dimensional string theory, the thermodynamic entropy could approach a constant at high temperatures and lead to a geometric entropy which is finite in the ultraviolet.

Recently the entropy of entanglement between different regions of space in quantum field theories have been intensively studied [1]-[6]. The motivation for this is its direct connection to the question of information loss due to black holes and black hole entropy [7]-[12]. A significant feature of this entanglement entropy, or "geometric entropy" is that it is ultraviolet divergent in typical field theories. This has been interpreted to imply that at least at the semiclassical level information loss due to the formation of a horizon is inevitable in quantum field theories. The divergence of the entropy is a reflection of short distance singularities in quantum field theories. Alternatively [13]-[16] the divergence is related to the behaviour of the usual thermodynamic entropy at high temperatures since, as we shall see below, the geometric entropy effectively involves an *integral* of the *thermodynamic entropy density* over all temperatures.

One may hope that in string theories this divergence disppears because of a soft ultraviolet behaviour [11]. However, to leading order in the string perturbation expansion, the thermodynamic free energy of a string is equal to the sum of the free energies of the physical modes of the string and one would obtain the same divergence in each term of the sum. Furthermore, unlike in a field theory of a finite number of fields, the thermodynamic free energy of free strings is itself divergent at the Hagedorn temperature. This is an *infrared* diveregence and might signal an *instability* of the theory and one might obtain a finite answer once one takes into account interactions and shift to a stable vacuum [17]. As argued in [13, 14] the geometric entropy in string theory is afflicted by this Hagedorn transition and what appears as an ultraviolet divergence in each term of the sum over all string modes may be interpreted as an infrared problem in the full answer.

The need to include string interactions calls for a formulation of the problem in some well defined and tractable string field theory. While this appears to be an almost impossible task at present, there is one string theory where a tractable nonperturbative formulation exists, at least for some bulk quantities. This is the two dimensional string defined via the one dimensional matrix model [18]. The singlet sector of the matrix model [1] may be written as a two dimensional collective field theory of the density variable.The fluctuations of the collective field represent a massless particle which is the only

[1] As we will see soon the nonsinglet contributions are irrelevant for a calculation of the geometric entropy in the ground state

propagating degree of freedom of the two dimensional string [2] and the coupling is proportional to the inverse of the fermi energy. For nonperturbative considerations it is better to write the model as a field theory of nonrelativistic fermions in the presence of an inverted harmonic oscillator potential and no self-interactions. The idea, then, is to consider a geometric entropy in this model and use exact nonperturbative answers to understand stringy effects. Hopefully this will teach us something about higher dimensional string theories as well.

In this paper we take the first step in this program. We first consider the problem of *free* nonrelativistic fermions in two dimensions. By constructing the explicit expression for the ground state wave functional and the corresponding geometric density matrix we argue that the geometric entropy has no ultraviolet divergence for finite fermi energies. In terms of the collective field theory this is a *nonperturbative* phenomenon. In fact the result follows from the softer behaviour of the ordinary thermodynamic entropy at high temperatures (compared to relativistic fermions). We then consider the the high temperature limit of the "exact" expression for the thermodynamic partition function of the singlet sector of the one dimensional matrix model [21]. We will show that the genus expansion breaks down at high enough temperatures and hence the geometric entropy of this string theory is essentially nonperturbative. A naive high temperature expansion leads to a regular behaviour of the entropy, but the specific heat turns out to be *negative*, signifying a nonperturbative instability.

Consider a theory of fields $\phi(x,t)$ in $1+1$ dimensions. The ground state wave functional may be written as $\Psi_0[\phi_L, \phi_R]$, where ϕ_L (ϕ_R) denotes the field ϕ for $x<0$ ($x>0$). The density matrix which gives expectation values of operators localized in the $x>0$ region is

$$\rho(\phi_R, \phi'_R) = \int \mathcal{D}\phi_L \Psi_0[\phi_L, \phi_R] \; \Psi_0[\phi_L, \phi'_R] \tag{1}$$

As shown in [3, 4] the quantity $\mathrm{Tr}\rho^n$ may be represented as an euclidean path integral over a cone with a deficit angle $2\pi(1-n)$. The geometric entropy

[2] The relationship between the collective field and the massless scalar of the effective field theory is rather subtle and not completely clear at this moment. See [23, 24] for a recent discussion.

may be then written using a "replica trick" as

$$S_g = -\hat{\rho}\log\,\hat{\rho} = [(1-n\frac{d}{dn})\log\mathrm{Tr}\;\rho^n]_{n=1} \tag{2}$$

For relativistic systems this establishes the equality of the geometric entropy with the usual *thermodynamic* entropy of the field in Rindler space at the Rindler temperature, and hence the quantum correction to the entropy of a large mass black hole. For nonrelativistic systems the Rindler hamiltonian would depend on the Rindler time. However the geometric entropy defined above makes sense and $\mathrm{Tr}\rho^n$ is still a path integral on a cone.

Consider a system of N_F free nonrelativistic fermions contained in a box : $-L < x < L$. The dispersion relation for the single particle states is given by $e(k) = \frac{1}{2}k^2$ where k is the (spatial) momentum which is quantized as $k = \frac{\pi n}{L}$ with integer or zero n. However, below we will often use continuum notation. The second quantized fermion field operator can be expanded in terms of quasiparticle operators $\hat{b}(k)$ and $\hat{b}^+(k)$

$$\hat{\psi}(x,t) = \int_{-\infty}^{\infty}\frac{dk}{2\pi}e^{-ikx}[\hat{b}(k)\theta(|k|-k_F) + \hat{b}^+(-k)\theta(k_F-|k|)] \tag{3}$$

and similarly for $\hat{\psi}^+$. The operators $\hat{b}, \hat{b}^+$ satisfy the standard anticommutation relations $\{\hat{b}(k,t), \hat{b}^+(k',t)\} = \delta(k-k')$. The ground state is then described by the filled Fermi sea $\hat{b}(k)|0>= 0$

In terms of the coherent states of the b-oscillators

$$\hat{b}(k)|b(k)>= b(k)|b(k)> \qquad\qquad <b(k)|\hat{b}^+(k) = \bar{b}(k) < b(k)| \tag{4}$$

where $b(k), \bar{b}(k)$ are grassman numbers, the ground state wave functional is given by

$$\Psi_0[b(k),\bar{b}(k)] = exp[-\frac{1}{2}\int_{-\infty}^{\infty}\frac{dk}{2\pi}\bar{b}(k)b(k)] \tag{5}$$

The grassmann fields appearing in the path integral are, however, not $b(k)$ and $\bar{b}(k)$. Rather, they are the combinations

$$\psi(k) = b(k)\theta(|k|-k_F) + \bar{b}(-k)\theta(k_F-|k|) \tag{6}$$

and similarly for $\bar{\psi}(k)$. The fourier transforms of these fields are the original fields $\psi(x)$. For reasons which will be clear in a moment we will use a new field $\chi(q) = \psi(k_F + q)$. Then the wavefunctional (5) becomes

$$\Psi_0 = \exp\,[-\frac{1}{2}(\int_{-\infty}^{\infty}\frac{dq}{2\pi}\bar{\chi}(q)\chi(q) - 2\int_{-2k_F}^{0}\frac{dq}{2\pi}\bar{\chi}(q)\chi(q))] \tag{7}$$

The inverse fourier transform of $\chi(q)$, denoted by $\chi(x)$ is related to the original field $\psi(x)$ by $\chi(x) = e^{ik_F x}\psi(x)$. Since this relationship is *local* we can compute the geometric entropy using these fields.

In (7) the first term may be written as a integral over position space of a *local* quantity $\bar{\chi}(x)\chi(x)$ and do not contribute to the geometric entropy. The only momentum modes which contribute to the geometric entropy are those in the filled Fermi sea as in the second term in (7). The fermi momentum thus roughly acts as an ultraviolet cutoff and this is the essential reason why the geometric entropy turns out to be finite.

We can proceed similarly for a relativistic Weyl fermion. For example for right moving fermions the fermi momentum is zero and the Dirac sea consists of all negative momenta. It is easy to see that the wavefunctional is the expression (7) where the limit of integration $-2k_F$ in the second term is replaced by $-\infty$. Clearly, in the limit of large k_F the nonrelativistic expression reduces to the relativistic expression. This simply reflects the fact that the excitations very close to the fermi level behave as massless relativistic particles with the velocity of light replaced by the fermi velocity.

To compute the geometric entropy we now introduce the following technique which may be trivially generalized to other situations (e.g. relativistic bosons, fermions etc.). We first expand the field eigenvalues in terms of modes which are localized to the left region ($x < 0$), $f_L(\omega)$ and those localized to the right region ($x > 0$), $f_R(\omega)$ as follows

$$\chi(x) = \frac{1}{\sqrt{|x|}}[\theta(x)\int_{-\infty}^{\infty} d\omega(\frac{x}{a})^{-i\omega} f_R(\omega) + \theta(-x)\int_{-\infty}^{\infty} d\omega(\frac{-x}{a})^{-i\omega} f_L(\omega)] \tag{8}$$

where a denotes the lattice spacing (or some other ultraviolet cutoff). The factor of $\frac{1}{\sqrt{|x|}}$ arises from the dimension of the field $\chi(x)$ under rescalings of x. To obtain the expression for $\chi^+(x)$ replace f_L, f_R by $\bar{f}_L, \bar{f}_R$ and ω by $-\omega$ in the integrand of (8).

The fields $\chi(q)$ and $\chi^+(q)$ may be now expressed in terms of f_L and f_R. We get the following expressions, all valid for $q > 0$.

$$\chi(q) \;\; = \;\; \int_{-\infty}^{\infty}\frac{d\omega}{2\pi}[ie^{\pi\omega} f_L(\omega) + f_R(\omega)]G(q,\omega)$$

$$
\begin{aligned}
\bar{\chi}(q) &= \int_{-\infty}^{\infty} \frac{d\omega}{2\pi}[\bar{f}_L(\omega) + ie^{-\pi\omega}\bar{f}_R(\omega)]G(q,-\omega) \\
\chi(-q) &= \int_{-\infty}^{\infty} \frac{d\omega}{2\pi}[f_L(\omega) + ie^{\pi\omega}f_R(\omega)]G(q,\omega) \\
\bar{\chi}(-q) &= \int_{-\infty}^{\infty} \frac{d\omega}{2\pi}[ie^{-\pi\omega}\bar{f}_L(\omega) + \bar{f}_R(\omega)]G(q,-\omega) \qquad (9)
\end{aligned}
$$

where we have defined

$$
G(q,\omega) = \int_0^{\infty} \frac{dx}{\sqrt{x}} e^{iqx} (\frac{x}{a})^{-i\omega} = a^{\frac{1}{2}} (\frac{i}{qa})^{\frac{1}{2}-i\omega}\ \Gamma(\frac{1}{2} - i\omega) \qquad q > 0 \qquad (10)
$$

Note that $G(-q,\omega) = i\ e^{\pi\omega}\ G(q,\omega)$, which has been used in writing (9).

The ground state wave functional may be now rewritten explicitly in terms of f_L, f_R :

$$
\Psi_0[f_L, f_R] = \exp\ [-\frac{1}{2}(I(\infty) + \bar{I}(\infty) - 2\bar{I}(2k_F))] \qquad (11)
$$

where we have defined

$$
I(\gamma) \equiv \int_0^{\gamma} \frac{dq}{2\pi}\bar{\chi}(q)\chi(q); \qquad \bar{I}(\gamma) \equiv \int_{-\gamma}^{0} \frac{dq}{2\pi}\bar{\chi}(q)\chi(q) = \int_0^{\gamma} \frac{dq}{2\pi}\bar{\chi}(-q)\chi(-q) \qquad (12)
$$

In evaluating $I(\gamma)$ we need to perform the integral

$$
F_\gamma(\omega,\omega') = \int_0^{\gamma} \frac{dq}{2\pi} G(q,\omega)\ G(q,-\omega') \qquad (13)
$$

In the presence of a finite box size $2L$ the lower limit of the q integral is really $\frac{\pi}{L}$. Using (9) one may write the wave functional explicitly in terms of f_L and f_R and obtain the density matrix for fields localized in the $x > 0$ region by simply functionally integrating over $f_L, \bar{f}_L$

$$
\rho(f_R, f_R') = \int \mathcal{D}\bar{f}_L \mathcal{D}f_L\ \Psi_0[f_R, f_L]\ \Psi_0[f_R', f_L] \qquad (14)
$$

It is easy to evaluate the integral $F_\gamma(\omega,\omega')$ for $\gamma = \infty$. The result is

$$
F_\infty(\omega,\omega') = 2\pi i\ \mathrm{sech}(\pi\omega)\delta(\omega - \omega') \qquad (15)
$$

From this we can make the first consistency check on our technique. Using (15) one gets the wave functional for relativistic fermions

$$\begin{aligned}\Psi_0^{rel} &= \exp\,[-2\pi \textstyle\int d\omega \quad [\tanh\,\pi\omega(\bar{f}_R(\omega)f_R(\omega) - \bar{f}_L(\omega)f_L(\omega)) \\ &\qquad + i\ \mathrm{sech}\ \pi\omega(\bar{f}_L(\omega)f_R(\omega) - \bar{f}_R(\omega)f_L(\omega))]\end{aligned} \tag{16}$$

After a trivial rescaling of the fields by a constant [3] this is exactly the answer obtained by euclidean path integral methods in [6]. As discussed above, (16) is also the answer for nonrelativistic fermions exactly at $k_F = \infty$.

The density matrix and its various powers in this relativistic limit may be obtained as in [6] and the geometric entropy may be calculated using the replica trick to obtain the result

$$S_g \sim \log(L/a) \tag{17}$$

This expression has an ultraviolet as well as an infrared divergence. This result may be understood in a very simple way. The geometric entropy is a measure of the entanglement of the modes f_L and f_R. It is clear from the wave functional that the modes $f_L(\omega)$ and $f_R(\omega)$ mix for *all* values of ω. This means that all the ω-modes contribute to the geometric entropy, which should be then proportional to the total number of ω's. From (8) it follows that for large L the number of allowed ω's is proportional to $\log(L/a)$ - hence the answer (17).

For small values of k_F the integral $\bar{I}(2k_F)$ may be approximated as

$$\bar{I}(2k_F) \sim (2k_F - \frac{\pi}{L})\ \bar{\chi}(0)\chi(0) \tag{18}$$

It may be easily seen that for large $N = \frac{L}{a}$ the function $G(0,\omega)$ is peaked at $\omega = 0$, falls to zero at $\omega \sim \frac{1}{\log N}$ and then oscillates rapidly around zero. This means that the mode $\chi(0)$ expressed as an integral over ω essentially receives contribution from a few values of ω around $\omega = 0$. To the lowest order we can then approximate the wave functional by

$$\begin{aligned}\Psi_0[f_L, f_R] &\sim \exp \quad [-\int_{-\infty}^{\infty} d\omega(\bar{f}_L(\omega)f_L(\omega) + \bar{f}_R(\omega)f_R(\omega)) \\ &\qquad + 2\log\,(\frac{k_F L}{\pi})\ [-\bar{f}_R(0)f_R(0) - \bar{f}_L(0)f_L(0) \\ &\qquad + i\bar{f}_R(0)f_L(0) - i\bar{f}_L(0)f_L(0)]\end{aligned} \tag{19}$$

[3]It may be easily verified from the definitions that a rescaling of fields by a constant does not change the geometric entropy, as one would expect.

In (19) the original integrals over ω have been replaced by sums and the resulting factors of $\log N$ have been suitably absorbed by a redefinition of the fields.

In (19) the left and right modes mix only for $\omega = 0$. For small but finite k_F only a few modes around $\omega = 0$ mix, and these alone contribute to the geometric entropy. Let us evaluate the entropy due to the $\omega = 0$ mode alone. The final result will be this contribution multiplied by a factor of order unity. The unnormalized density matrix for this mode is easily seen to be

$$\begin{aligned}\rho(f_R, f'_R) \;=\exp[- & \;(1-\frac{1}{2}\mathrm{sech}^2\eta)(\bar{f}_R(0)f_R(0)+\bar{f}'_R(0)f'_R(0)) \\ & +\frac{1}{2}\mathrm{sech}^2\eta(\bar{f}_R(0)f'_R(0)+\bar{f}'_R(0)f_R(0))]\end{aligned} \tag{20}$$

where we have defined

$$\mathrm{sech}\ \eta = \frac{\log\frac{k_F L}{\pi}}{\log\frac{k_F L}{\pi}+2} \tag{21}$$

By a redefinition of the fields $f_R \rightarrow (2\tanh\ \eta)^{\frac{1}{2}}\ f_R$ one may rewrite the density matrix in a form which facilitates the calculation of $\mathrm{Tr}\rho^n$ for any n. The final result for the geometric entropy is

$$S_g = 2\log(2\cosh\ \eta) - 2\eta\ (\tanh\ \eta) \tag{22}$$

For $k_F L \sim \pi$ one has $\eta \rightarrow \pm\infty$ and the leading order expression for the geometric entropy follows from (22)

$$S_g^{(0)} = (\log\frac{k_F L}{\pi})^2[1+\log\ 2 - 2\ \log\ (\log\frac{k_F L}{\pi})] \tag{23}$$

The total geometric entropy for small k_F is $S_g^{(0)}$ multiplied by a factor of order unity. As advertized above the result does not involve the ultraviolet cutoff essentially because only a few modes contribute to the entropy, rather than all of the $\log N$ modes. In a similar fashion we expect that for finite k_F the role of the ultraviolet cutoff is replaced by k_F.

There is an alternative way to view the quantity $\mathrm{Tr}\rho^n$. Consider dividing up the region $x > 0$ into small cells. In the thermodynamic limit the path integral which represents $\mathrm{Tr}\rho^n$ would be a product of path integrals for each individual cell. However the path integral for the cell centered at the point x

is the standard *thermodynamic* partition function at a temperature $T(x) = \frac{1}{2\pi n x}$. Thus the geometric entropy may be obtained by simply calculating the thermodynamic entropy *density* at a temperature $T(x)$ and integrating the result from $x = 0$ to $x = \infty$. This is the procedure used to compute the genus one contribution to the entropy of strings in [13, 14].

The ultraviolet divergence of the geometric entropy in relativistic field theories may be now understood in terms of the behaviour of the standard thermodynamic entropy at high and low temperatures. For example, a free massless boson in d space dimensions at temperature $T = \frac{1}{\beta}$ has an entropy density $s = \frac{A}{\beta^d}$. This diverges for low β, or high temperature for all d. In our problem we have to put $\beta = 2\pi x$ and integrate this entropy density over x, so that there is a diveregence of the geometric entropy from the lower limit of integration. This is the ultraviolet divergence in the entropy. In the corresponding Rindler problem, the divergence arises because the local temperature is very high near the horizon and the corresponding contribution to the entropy density is large. For $d = 1$ there is an additional infrared divergence coming from large x. Thus using an ultraviolet cutoff ϵ and an infrared cutoff L in the integral over x one has $S \sim \frac{1}{\epsilon^{d-1}}$ for $d > 1$ and $S \sim \log(\frac{L}{\epsilon})$ for $d = 1$.

Consider now our system of N_F free nonrelativistic fermions in one spatial dimension in the grand canonical ensemble with an inverse temperature $\beta = \frac{1}{T}$ and fugacity z. The fugacity is determined in terms of N_F by $N_F = z\partial_z \log \mathcal{Z}$ (where $\mathcal{Z}$ is the partition function) which leads to

$$N_F = L\sqrt{\frac{2}{\pi\beta}} f_{\frac{1}{2}}(z) \qquad f_{\frac{1}{2}}(z) = z\partial_z f_{\frac{3}{2}}(z) \tag{24}$$

where the function $f_{\frac{3}{2}}$ is defined as

$$f_{\frac{3}{2}}(z) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} dx \log\,(1 + z\, e^{-x^2}) \tag{25}$$

The expression for the entropy follows from usual thermodynamics and is given by

$$S = N_F\left(\frac{3 f_{\frac{3}{2}}(z)}{2 f_{\frac{1}{2}}(z)} - \log z\right) \tag{26}$$

The function $f_{\frac{1}{2}}(z)$ is a monotonically increasing function of z. Thus for large values of $\frac{\sqrt{\beta} N_F}{L}$ (i.e. low temperatures or high densities) one may use the standard expansions of the functions $f_r(z)$ for large z [19] to obtain the leading order term in the entropy density

$$s = \frac{N_F}{L}[\frac{\pi^2}{6\log\, z} + \frac{13\pi^4}{360(\log\, z)^3} + \cdots] = \frac{\pi}{6\beta\sqrt{2\epsilon_F}} + \frac{8\pi^3}{45\ \beta^3\ k_F^5} + \cdots \tag{27}$$

The leading term is the same expression for a free relativitsic boson or a free relativistic fermion in two dimensions, apart from a factor of $\sqrt{2\epsilon_F}$. The reason is simple. For a given temperature and size of the system, large values of $\frac{\sqrt{\beta}N_F}{L}$ mean large fermi momenta. In this case the excitations are restricted to particle-hole excitations near the fermi level. The dispersion for these excitations are given precisely by $E(k) = \sqrt{2\epsilon_F}|k|$ which is the same as that of a massless relativistic boson in two dimensions. The factor of $\sqrt{2\epsilon_F}$ in the dispersion relation explains the same factor in the entropy density. What is more significant for our considerations is the fact that the same result applies for a given μ_F and size, but low temperatures.

On the other hand, for small values of $\frac{\sqrt{\beta}N_F}{L}$ one has to use the power series expansions of $f_r(z)$ for small z [19] and one easily gets for the entropy density

$$s = \frac{N_F}{L} - \frac{N_F}{2L}\log(\frac{2\pi\beta N_F^2}{L^2}) + O(\beta^{\frac{1}{2}}) \tag{28}$$

To obtain the geometric entropy one has to integrate $s(2\pi x)$ over all positive x. If we use the low temperature expansion we would get an ultraviolet divergent answer from the behaviour of the integrand near $x = 0$. In fact it may be easily checked that the lowest order answer agrees completely with the direct calculation of the geometric entropy of relativistic fermions discussed above. However to treat the region $x = 0$ we have to use the high temperature behaviour in (28). This shows that the integrand has an *integrable* singularity at $x = 0$ whereas there is a logarithmic divergence coming from large values of x. Thus the geometric entropy is finite in the ultraviolet. This is in marked contrast with relativistic bosons or fermions. Essentially as one approaches the point $x = 0$ the temperature becomes large compared to the scale set by the fermi energy. Thus the fermi energy provides a cutoff to the relativistic behaviour for large values of x.

The true significance of this result can be appreciated if one rewrites the model in terms of the collective field theory of the density of fermions $\rho(x,t)$. Consider a more general system of N_F fermions in an external static potential

$V(x)$. The density $\rho(x,t)$ may be expanded around the "classical" average value

$$\rho(x,t) = \rho_0(x) + \partial_x \xi(x,t) \qquad \rho_0 = \frac{1}{\pi}\sqrt{2(\mu_F - V(x))} \tag{29}$$

where μ_F is the fermi level. Introduce the "time of flight" variable $\tau(x) = \frac{1}{\pi}\int^x \frac{dx}{\rho_0(x)}$. Then the hamiltonian which governs the fluctuations $\xi(\tau,t)$ is given by

$$H_{coll} =: \frac{1}{2}\int d\tau [\Pi_\xi^2 + (\partial_\tau \xi)^2 - \frac{1}{\pi^{\frac{3}{2}}\ \rho_0^2}(\Pi_\xi(\partial_\tau\xi)\Pi_\xi + \frac{1}{3}(\partial_\tau\xi)^3) - \frac{1}{\pi^{\frac{5}{2}}}(\frac{\rho_0''}{3\rho_0^3} - \frac{(\rho_0')^2}{2\rho_0^4})] : \tag{30}$$

where Π_ξ denotes the field momentum. Note that the hamiltonian already comes in the normal ordered form [20] and leads to finite answers. The density thus behaves as a scalar field with in general position dependent coupling given by $\frac{1}{\rho_0^2}$. For free fermions $V(x) = 0$. Then $\rho_0 = \sqrt{2\mu_F}$ and $\tau = \frac{\sqrt{2\mu_F}}{\pi}$. The coupling is a constant and equal to $\frac{1}{2\mu_F}$. For μ_F large compared to typical energies, one has a *free* relativistic scalar field, which is why in this limit we obtained the relativistic answer for the entropy. For small μ_F the theory is strongly coupled and the perturbation expansion does not make sense. Thus the finiteness of the entropy demonstrated above is a *non-perturbative* effect in this collective field theory.

Let us now consider the one dimensional matrix model which provides a nonperturbative definition for the two dimensional noncritical string. This is defined by the action

$$A = \lambda \int dt \mathrm{Tr}\ (\frac{1}{2}[\partial_t M(t))^2 + V(M(t))] \tag{31}$$

where $M(t)$ is a $N \times N$ matrix and the potenial function $V(M)$ has to be chosen such that it has a quadratic maximum.The detailed form of the potential is not important in the double scaling limit. As is well known the singlet sector of this model may be written exactly in terms of nonrelativistic fermions $\psi(x,t)$ where x denotes the space of eignevalues of the matrix M. These fermions have no self interactions, but move in an external potential $V(x)$. Let us denote the fermi energy by μ_F. Then as the coupling $g = \frac{N}{\lambda}$ of the theory approaches a critical value g_c the fermi level μ_F appraches μ_c which is the energy at the top of the potential hump. The double scaling continuum limit of this problem is given by $\mu = \mu_c - \mu_F \to 0$ and $\lambda \to \infty$

with $\kappa = \lambda\mu =$ fixed. This is known to describe the two dimensional string. In this limit only the quadratic hump of the potential is relevant and one has a problem of nonrelativistic fermions in two dimensions in an external potential $V(x) \sim -x^2$ - the inverted harmonic oscillator. The collective field theory hamiltonian is given by (30) with this potential. This is now a version of string field theory and the string coupling is $g_{st} \sim \frac{1}{\kappa}$.

The geometric entropy of this model in the ground state may be calculated in principle by the technique discussed above. The expansion for the fermion fields are no longer in terms of plane waves, but in terms of the eigenfunctions of the Schrodinger operator in the inverted harmonic oscillator potential, which are parabolic cylinder functions. The main modification would be to replace the plane waves e^{iqx} by parabolic cylinder functions in the definition of $G(q,\omega)$ in (10). The resulting expressions are rather difficult to analyze.

We have seen, however, that the divergence of the geometric entropy is related to the high and low temperature behaviors of the ordinary thermodynamic entropy. The complete thermodynamic partition function of the matrix model is a formidable task since this includes contributions from the non-singlet states. However if we are interested in computing the geometric entropy for the *ground state*, which is a singlet, the trace involved in $\mathrm{Tr}\rho^n$ is a trace over *singlet* states alone. The thermodynamics in the singlet sector has been completely solved and we can use the known results of [21]

Let us define $\Delta = g_c - g$ and let F denote the free energy. Then the chemical potential μ is determined by the equation

$$\frac{\partial \Delta}{\partial \mu} = \frac{1}{2\pi}\mathrm{Re}\,[\int_0^\infty \frac{dt}{t}\; e^{-it}\; \frac{(t/\kappa)}{\sinh(t/\kappa)}\; \frac{(\pi t/\kappa\beta)}{\sinh(\pi t/\kappa\beta)}] + \frac{1}{2\pi}[\log\frac{\lambda}{2} - \gamma] \qquad (32)$$

The infinite constant γ is necessary for (32) to reproduce the correct WKB expansion in $\frac{1}{\lambda}$, but would be unimportant in what follows. The free energy F is then obtained by integrating the equation

$$\frac{\partial F}{\partial \Delta} = \lambda^2(\mu - \mu_c) \qquad (33)$$

Recalling that the number of fermions is $N = \lambda g$ one may write down the expression for the entropy using standard relations in the grand canonical ensemble

$$S = \beta^2[(\frac{\partial F}{\partial \beta})_\mu - \lambda^2\mu(\frac{\partial \Delta}{\partial \beta})_\mu] \qquad (34)$$

The second term in (34) arises because we are considering derivatives with fixed μ rather than fixed N.

The above expressions have an important duality symmetry. From (32) it follows that

$$\frac{\partial \Delta}{\partial \mu}(\beta, \lambda) = \frac{\partial \Delta}{\partial \mu}(\frac{\pi^2}{\beta}, \frac{\lambda\beta}{\pi}) \tag{35}$$

It then follows from (33) that the canonical partition function βF is *invariant* under this duality transformation.

The standard genus expansion is obtained by considering κ to be large and expanding the hyperbolic sine functions in a power series expansion. The result for this asymptotic expansion is

$$\frac{\partial \Delta}{\partial \mu} = \frac{1}{2\pi}[-\log\,\mu + \sum_{m=1}^{\infty} \frac{f_m(\beta)}{(\beta\kappa^2)^m}] \tag{36}$$

where the functions $f_m(\beta)$ are symmetric under the duality transformation $\frac{\beta}{\pi} \rightarrow \frac{\pi}{\beta}$ and has the form

$$f_m(\beta) \sim \sum_{k=0}^{m} \; C(m,k) \; (\beta)^{m-2k} \tag{37}$$

$C(m,k)$ are numbers related to Bernoulli coefficients. The $m = 1$ in the sum in the above expression is the one loop contribution in string theory. The corresponding one loop free energy is

$$F = \frac{\log\,\mu}{12\pi\beta}(\frac{\beta}{\pi} + \frac{\pi}{\beta}) \tag{38}$$

This result is identical to the free energy of an ideal gas of massless bosons in one spatial dimension of size log μ apart from a constant which is in fact the *finite* one loop correction to the ground state energy of the system. This is expected since the only propagating mode of the two dimensional string is a massless scalar. The same result is obtained by performing the Polyakov path integral in the continuum $d = 2$ string theory [22] and is in fact the result of the following modular invariant integral

$$F = \int_{\mathcal{F}} \frac{d^2\tau}{\tau_2^2} \sum_{m,n} \; exp[-\frac{\beta^2|n - m\tau|^2}{4\pi\tau_2}] \tag{39}$$

where τ is the complex modular parameter on the torus and the integration is over the fundamental domain. The term in the sum with $m, n = 0$ is the zero temperature free energy and is separately modular invariant. The temperature dependent term evaluates to the second term in (38). As noted this is the contribution from a single massless scalar. This is nevertheless modular invariant as it should be since it follows from a string theory [4] . Note further that unlike in higher dimensions the one loop answer does not have a Hagedorn behavior at any finite temperature simply because the 2d string has only one propagating degree of freedom.

The contribution to the geometric entropy from the one loop term alone gives the standard logarithmically divergent answer for a free relativistic boson in two dimensions. One may regard the entire divergence to be *infrared* since the answer is modular invariant, as argued in [14]. However the answer is really the same as a single boson. The string interpretation then gives a relation between the infrared and ultraviolet cutoff in terms of the cutoff in the modular parameter Imτ.

Returning to our description of the usual thermodynamics of the matrix model let us record below the genus expansion for the thermodynamic entropy.

$$S = -\frac{1}{3\beta}[\log \mu + 1] + \frac{1}{\lambda^2}[\frac{1}{18\beta} + \frac{7}{90\beta^3}] + \cdots \tag{40}$$

The asymptotic expansion (36) makes sense for small temperatures, i.e. large β. Indeed, for large β, $\frac{\partial \Delta}{\partial \mu}$ approaches a constant, and the leading correction from *all the terms in the sum in* (36) is of order $\frac{1}{\beta^2}$, as is clear from (37). The leading term in the free energy is then seen to be of order $\frac{1}{\beta^2}$ as well and has contributions from all genus.

We have seen, however, that the ultraviolet divergence of the geometric entropy is related to the *high* temperature behaviour of the free energy. The asymptotic expansion of (32) leading to the genus expansion (36) clearly breaks down for high temperatures. More specifically for $\beta < \frac{\pi}{\kappa}$ one should not expand the hyperbolic functions in (32) in power series. This means that for high temperatures strong coupling effects (in the sense of string theory) become important. Indeed the genus expansion in (36) contain ever increasing powers of $\frac{1}{\beta}$ with higher and higher genus giving rise to higher and

[4] See [16] for related remarks

higher powers. It is clearly necessary to look at the high temperature limit *for arbitrary values of the string coupling* g_{st}.

Before considering the behaviour at high temperatures let us look at the behavior at low temperatures,$\beta >> \frac{\pi}{\kappa}$, but independent of the genus expansion. This means we expand the factor $(\pi t/\kappa\beta)/\sinh(\pi t/\kappa\beta)$ in (32) in a power series, but *not* the first factor. This gives a power series in $\frac{1}{\beta^2}$ for all values of κ. A straightforward evaluation of the integrals yield the first few terms

$$\begin{aligned} \frac{\partial\Delta}{\partial\mu} &= -\frac{1}{2\pi}\mathrm{Re}\ \Psi[\frac{1}{2}(1+i\kappa)] + \frac{\pi}{48\beta^2}\mathrm{Re}\ \Psi''[\frac{1}{2}(1+i\kappa)] \\ & -\frac{7\pi^3}{11520\beta^4}\Psi''''[\frac{1}{2}(1+i\kappa)] + \cdots \end{aligned} \tag{41}$$

where $\Psi(x)$ denotes the digamma function. It may be checked that if one now considers the large κ asymptotic expansion of the digamma functions in (41), the result agrees with the large-β limit of the genus expansion in (36).

Since we have duality invariance $\frac{\pi}{\beta} \to \frac{\beta}{\pi}$ we can *deduce* the high temperature behaviour from the low temperature expansion (41). Using (35) and (41) one gets

$$\begin{aligned} \frac{\partial\Delta}{\partial\mu} &= -\frac{1}{2\pi}\mathrm{Re}\ \Psi[\frac{1}{2}(1+i\frac{\kappa\beta}{\pi})] + \frac{\beta^2}{48\pi^3}\mathrm{Re}\ \Psi''[\frac{1}{2}(1+i\frac{\kappa\beta}{\pi})] \\ & -\frac{7\beta^4}{11520\pi^5}\Psi''''[\frac{1}{2}(1+i\frac{\kappa\beta}{\pi})] + \cdots \end{aligned} \tag{42}$$

The expansion in (42) is now valid for small β. For very high temperatures $\beta << \frac{\pi}{\kappa}$ it is senseless to perform asymptotic expansions of the digamma functions in (42) for large values of the argument. Rather one could perform a Taylor expansion leading to the result

$$\frac{\partial\Delta}{\partial\mu} = -\frac{1}{2\pi}\Psi(\frac{1}{2}) + \frac{\beta^2}{48\pi^3}\Psi''(\frac{1}{2})\ [1+3\kappa^2] - \frac{\beta^4}{768\pi^5}\Psi''''(\frac{1}{2})[\kappa^4+2\kappa^2+\frac{7}{15}] + \cdots \tag{43}$$

Alternatively one may work with the full expression (32). For high temperatures one may expand the factor factor $(\pi t/\kappa\beta)/\sinh(\pi t/\kappa\beta)$ in powers of $e^{-\frac{2\pi t}{\kappa\beta}}$. One gets

$$\frac{\partial\Delta}{\partial\mu} = \frac{1}{\beta}\sum_{n=0}^{\infty}\mathrm{Re}\ \Psi'[\frac{1}{2}(1+\frac{\pi(2n+1)}{\beta}+i\kappa] \tag{44}$$

For small β we can use the asymptotic expansions for $\Psi'(z)$ for large z and thus obtain an expansion for $\frac{\partial\Delta}{\partial\mu}$ in powers of β^2. The result agrees entirely with the expansion (43).

The expression for the entropy is obtained by integrating the expression (43) and using (34)

$$S = -\frac{\beta^3\Psi''(\frac{1}{2})}{48}[2\kappa^2 + \kappa^4] + \frac{\beta^5\Psi''''(\frac{1}{2})}{1440}[7\kappa^2 + 5\kappa^4 + \kappa^6] + \cdots \tag{45}$$

Note that $\Psi''(\frac{1}{2})$ and $\Psi''''(\frac{1}{2})$ are negative so that the leading contribution to the entropy is positive.

One important feature of the high temperature limit is that there is no term proportional to the "volume", which is $\log\mu$ in this model.

Contrary to the results of the genus expansion the entropy has a regular behaviour at high temperatures. However the specific heat $C_v = -\beta\frac{\partial S}{\partial\beta}$ is negative. This means that the system is unstable. We should not really trust the above thermodynamic expressions for all temperatures.

We believe that this instability is related to an inherent nonperturbative instability of the model defined naively as an inverted harmonic oscillator. In fact it is quite unclear how should one define the matrix model so that it satisfies all the basic physical requirements [23]. In a properly defined model there should be no instability of the kind discussed above. One possible scenario could be : the specific heat should remain positive, but the entropy saturates at high temperatures. It is also possible that there is a phase transition in the singlet sector itself at $\beta \sim \frac{1}{\kappa}$. Such a phase transition is distinct from the usual KT transition in this model which is driven by the non-singlet states and which is a *perturbative* phenomenon.

What is clear from the above discussion, however, is that the genus expansion is a bad guide to the behaviour of the thermodynamic entropy at high temperatures and that nonperturbative effects are of crucial importance in a discussion of the geometric entropy.

Even if we obtain a physically reasonable expression for the thermodynamic entropy it is still not clear how one could use this result to obtain the geometric entropy. This is because in this theory interactions are not translationally invariant and it is far from obvious how one could extract an entropy density which we expect to be position dependent as well. From the point of view of the direct calculation of the geometric entropy discussed in the earlier part of this paper this issue is related to the fact that the entropy would depend on the region of space which is integrated out to obtain the density matrix.

To obtain the geometric entropy of the underlying string theory, one further needs to address the question of the exact correspondence of the collective field and the massless scalar of the string theory. The point is, the massless scalar of the string theory seems to be related to the collective field in a *non-local* (in space) way [23]. Hence the geometric entropy obtained by integrating out the string theory scalar field in some region of the liouville space is very different from that obtained by integrating out the fermions or the collective fields in some region of the x or τ space. Nevertheless our results strongly indicate that in all these quantities nonperturbative stronng coupling effects play a crucial role.

Finally to really understand black hole entropy in this string theory one needs a clear understanding of the black hole solution in the matrix model. There have been several suggestions about this [25], but the situation is rather unclear. In particular there is no contact with known black hole solutions of the low energy effective field theory with nontrivial tachyon backgrounds [26]. All these issues are intimately tied with a deeper understanding of the nature of space of time in the matrix model.

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References

[1] L. Bombelli, R. Koul, J. Lee and R. Sorkin, *Phys. Rev.* **D 34** (1986) 373

[2] M. Srednicki, *Phys. Rev. Lett.* **71** (1993) 666.

[3] C.G. Callan and F. Wilczek, *Phys. Lett.* **B333** (1994) 55.

[4] D. Kabat and M.J. Strassler, *Phys. Lett.* **B329** (1994) 46.

[5] J.S. Dowker, University of Manchester Preprint, hep-th/9401125.

[6] F. Larsen and F. Wilczek, hep-th/9408089

[7] J.D. Beckenstein, *Phys. Rev.* **D7** (1973) 2333; *Phys. Rev.* **D9** (1974) 3292

[8] S.W. Hawking, *Phys. Rev.* **D14** (1976) 2460; *Comm. Math. Phys.* **43** (1975) 199; *Comm. Math. Phys.* **87** (1982) 395; *Phys. Rev.* **D18** (1978) 1747.

[9] G. Gibbons and S. Hawking, *Phys. Rev.* **D15** (1977) 2752.

[10] G. 't Hooft, *Nucl. Phys.* **B256** (1985) 727.

[11] L. Susskind, Rutgers Preprint, hep-th/9309145; L. Susskind and J. Uglum, *Phys. Rev.* **D50** (1994) 2700; D. Lowe, L. Susskind and J. Uglum, *Phys. Lett.* **B327** (1994) 226.

[12] E. Keski-Vakkuri and S. Mathur, *Phys. Rev.* **D50** (1994) 917; T. Fiola, J. Preskill, A. Strominger and S. Trivedi, *Phys. Rev.* **D50** (1994) 3987.

[13] J. Barbon, *Phys. Rev.* **D50** (1994) 2712.

[14] A. Dabholkar, hep-th/9409158

[15] A. Dabholkar, hep-th/9408098; D. Lowe and A. Strominger, hep-th/9410215.

[16] E. Elizalde and S.D. Odintsov, hep-th/9411024.

[17] J. Atick and E. Wiiten, *Nucl. Phys.* **B310** (1988) 291.

[18] For reviews and references to the original literature, see I. Klebanov, in *String Theory and Quantum Gravity*, (Trieste Spring School proceedings, 1991), hep-th/9108019; S.R. Das, in *String Theory and Quantum Gravity* (Lectures in Trieste Spring School, 1992),hep-th/9211085; P. Ginsparg and G. Moore, in *Recent directions in Particle theory* (Proceedings of TASI Summer School, Boulder, 1992), hep-th/9304011;J. Polchinski, hep-th/9411028

[19] See e.g. K. Huang, *Statistical Mechanics*, J. Wiley (1963).

[20] see e.g. S.R.Das in [18]

[21] D. Gross and I. Klebanov, *Nucl. Phys.* **B344** (1990) 475; *Nucl. Phys.* **B354** (1991) 459.

[22] M. Bershadsky and I. Klebanov, *Phys. Rev. Lett.* **65** (1990) 3088.

[23] J. Polchinski, *Nucl. Phys.* , **B424** (1994) 137; hep-th/9409168; hep-th/9411028.

[24] M. O'Loughlin, hep-th/9501039

[25] E. Martinec and S. Shatashvilli, *Nucl. Phys.* **B 368** (1992) 338; S.R. Das, *Mod. Phys. Lett.* , **A8** (1993) 69 and *Mod. Phys. Lett.* , **A8** (1993) 1331; A. Dhar, G. Mandal and S.R. Wadia, *Mod. Phys. Lett.* , **A7** (1992) 3703; *Mod. Phys. Lett.* , **A8** (1993) 1701; J.G. Russo, *Phys. Lett.* **B300** (1993) 336; A. Jevicki and T. Yoneya, *Nucl. Phys.* , **B411** (1994) 64.

[26] S. Kalyana Rama, *Phys. Rev. Lett.* , **70** (1993) 3186; *Nucl. Phys.* **B407** (1993) 191; *Phys. Lett.* , **B321** (1994) 358; A. Peet, L. Susskind and L. Thorlacius, *Phys. Rev.* , **D48** (1993) 2415.