QUARK CONFINEMENT AND DUAL REPRESENTATION IN 2+1 DIMENSIONAL PURE YANG-MILLS THEORY

Sumit R. Das and **Spenta R. Wadia**

Tata Institute of Fundamental Research

Homi Bhabha Road, Bombay 400005, INDIA

e-mail: das@theory.tifr.res.in, wadia@theory.tifr.res.in

Abstract

We study the quark confinement problem in 2+1 dimensional pure Yang-Mills theory using euclidean instanton methods. The instantons are regularized and dressed Wu-Yang monopoles. The dressing of a monopole is due to the mean field of the rest of the monopoles. We argue that such configurations are stable to small perturbations unlike the case of singular, undressed monopoles. Using exact non-perturbative results for the 3-dim. Coulomb gas, where Debye screening holds for arbitrarily low temperatures, we show in a self-consistent way that a mass gap is dynamically generated in the gauge theory. The mass gap also determines the size of the monopoles. In a sense the pure Yang-Mills theory generates a dynamical Higgs effect. We also identify the disorder operator of the model in terms of the Sine-Gordon field of the Coulomb gas.

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I. INTRODUCTION

The problem of quark confinement is one of the "old" unsolved problems in theoretical physics. Despite intense activity over the past two decades, and several approaches to the problem, it is surprising how little we know about this phenomenon. Lattice gauge theories, together with the theory of renormalization group (which provides the basic conceptual framework for all other approaches as well) is the only known *quantitative* and reliable method of attacking this problem and Monte Carlo simulations have indeed almost "demonstrated" confinement in pure gauge theories in four dimensions. However there is the nagging feeling that we can "demonstrate" confinement, even calculate relevant quantities to some degree of accuracy - but we still dont "understand" confinement. Despite impressive progress lattice methods remain a black box. More specifically we do not understand in full generality whether (if any) some specific type of gauge field configurations are responsible for confinement and whether we can arrive at a consistent picture of the vacuum of strongly coupled gauge theories.

Physical mechanisms for confinement are, of course, as old as the idea itself. One of the most significant ideas, proposed by Mandelstam [1], 't Hooft [2] and Nambu [3] is that of dual superconductivity - that the gauge theory vacuum is a condensate of magnetic monopoles. By the dual version of Meissner effect, quarks would then be naturally confined. The formulation of this idea gave rise to deep results concerning duality between electric and magnetic fluxes and its implications to the phase diagram of general gauge theories [4]. The idea is very convincing, and lattice results indeed seem to support it : but is rather difficult to establish in usual gauge theories. Recently, however, there has been significant progress in supersymmetric gauge theories where holomorphy and global properties of the moduli space of vacua have been used to argue for non-perturbative phenomena like quark confinement and chiral symmetry breaking [5]. This approach is essentially Hamiltonian : one tries to obtain a picture of the vacuum wave functional.

A complementary viewpoint stems from the Euclidean approach to the problem. This is the idea that Euclidean instantons essentially disorder the vacuum and lead to color confinement. The only known successful implementation of this idea is the classic work of Polyakov [6] who showed that confinement indeed occurs by this mechanism in three dimensional $SU(2)$ Yang-Mills theory coupled to adjoint representation Higgs field. The Higgs breaks the gauge group to $U(1)$. The instantons of this model are the 't Hooft-Polyakov monopoles. (In the three dimensional Euclidean theory these are of course not solitons, but tunnelling configurations). When the mass scale of this symmetry breaking - the mass of the W boson, m_W is large compared to the mass scale set by the dimensional gauge coupling, a dilute gas of monopoles provides a self-consistent picture. The resulting monopole plasma leads to Debye screening. Wilson loops in the fundamental representation obey an area law and a careful treatment shows that the adjoint representation Wilson loops obey a perimeter law [7]. This is exactly what one expects. The argument can be extended to the $SU(N)$ theory as well.

In this paper we extend this approach to *pure* Yang-Mills theory. This is a much more nontrivial system for several reasons. Firstly, perturbation theory, though ultraviolet finite, is hopelessly infrared divergent. Secondly, because this theory has only one length scale set by the gauge coupling g, one does not have the luxury of having another length scale m_W to enable a controlled semiclassical approximation. The classical monopole configurations are *singular* configurations in the continuum limit and hence the renormalization of the monopole gas is very nontrivial.

It is possible to regulate the singularity by modifying the fields inside a "core" of some size λ. The classical action now depends on λ. We do not vary the size λ of the monopoles. Rather we treat it as a parameter of the theory to be self consistently determined by the mass gap.

If the fluctuations around a regularised monopole solution are decomposed in terms of representations of the direct product of spin and color groups, the even parity S-wave fluctuations are unstable [8]. This is because the background magnetic field has a long range Coulomb tail. However the fluctuation problem in the Yang-Mills theory should be performed not around a single monopole solution, but around the neutral plasma of monopoles which populate the vacuum. These monopoles have long range Coulomb interactions : other monopoles affect the field around a typical monopole in a nontrivial way. More significantly the monopole positions and charges are fluctuating, which make the *charge density* field of the monopoles a dynamical variable. Such a fluctuation problem is difficult to solve exactly.

In this work we address this question in an approximation guided by the physics of the problem. We incorporate the fluctuations of the charge density $\rho(x)$ by invoking known rigorous results of the three dimensional Coulomb gas due to Brydges [9]. In [9] it has been shown that for a given arbitrarily low temperature there is a chemical potential (fugacity) such that the correlation of the density operator cluster. In other words there is Debye screening at all temperatures. The field around a given charge thus decays exponentially over a *debye length* l_D as opposed to a power law decay.

Our strategy is as follows. The aim is to show that Debye screening is *self consistently* realized in the plasma of magnetic monopoles. The main complication is that the fugacity of the Coulomb gas is itself a functional of the density of monopoles $\rho(x)$. However using the results of [9] we can argue that for a given value g^2 of the gauge coupling there exists a fugacity of the monopoles for which the plasma has a finite Debye length. The mean 'magnetic field' is also screened with a fall off given by the Debye length. In this sense a monople configuration in a plasma does not have a Coulomb tail and such a configuration which incorporates this collective property we call a 'dressed' monopole. Small fluctuations of the gauge field around a dressed monopole are expected to be stable for reasons similar to the Yang-Mills-Higgs theory. There the potential appearing in the stability operator decays exponentially with a scale $\frac{1}{m_W}$ due to contribution from the Higgs field. Hence in the pure Yang-Mills theory one seems to have a dynamical Higgs effect that is produced by the monopole plasma. We also indicate how the mass gap determines the size of the monopoles.

Finally we discuss the relation of our approach with the work of 't Hooft [4]. We give an explicit representation of the disorder operators in $2+1$ dimensional Yang-Mills theory and indicate that the dual theory is a Z_2 non-linear sigma model.

Our discussion of the confinement problem gives a picture of the dominant configurations in the Euclidean framework. Feynman [10] has given qualitative arguments for the ground state wave function of this gauge theory in analogy with his work on the roton spectrum in liquid helium. It would be interesting to relate these two approaches.

The plan of this paper is as follows. In Section II we discuss the topology of the vacuum in both the Georgi-Glashow model and the pure gauge theory. In Section III we briefly review the generation of mass gap and confinement of N-ality in the Yang-Mills Higgs system. In Section IV we discuss self-consistent Debye screening of the monopole plasma in the pure Yang-Mills theory. Section V is devoted to the dual representation in terms of disorder operators. Section VI is devoted to conclusions. In Appendix I we state the main result of [9].

II. TOPOLOGY OF THE GAUGE CONDITION

In this section we shall review the topological properties of the vacuum of $SU(2)$ Yang-Mills theory in $2+1$ dimensions following [13]. The physical degrees of freedom are most transparent in a unitary type of gauge. For pure gauge theories this is defined as follows. Consider some local operator $X(x)$ which transforms according to the adjoint representation of $SU(2)$. The unitary gauge is now defined by

$$[X(x), \tau_3] = 0 \tag{1}$$

(τ^i , $i = 1, 2, 3$ stands for the Pauli matrices). This gauge condition retains the $U(1)$ generated by τ^3 as an unbroken symmetry. If the model contains adjoint Higgs fields in addition to the gluons, X may be the Higgs field itself and (1) is the conventional unitary gauge in such a Georgi-Glashow model. In pure Yang Mills theory X has to be constructed out of the gauge fields alone.

We may write the matrix operator X in the form

$$X = \lambda\ I + \sum_{a=1}^{3} \epsilon_a(x) \tau^a \tag{2}$$

where I is the identity matrix. Then the points x_0 where

$$\epsilon^a(x_0) = 0 \tag{3}$$

are singularities of the gauge condition (1). As shown in [13] these singularities are nothing but magnetic monopoles with respect to the unbroken $U(1)$. In $3+1$ dimensions these monopoles are "particles" which respresent dynamical degrees of freedom (in this gauge) other than the conventional fields. In $2+1$ dimensions these monopoles are instanton configurations which will play a crucial role in determining the properties of the vacuum.

In the Georgi-Glashow model the monopoles are given by the well known 't Hooft-Polyakov solution, while for pure gauge theory they are described by the Wu-Yang solution.

III. CONFINEMENT IN 2+1 DIMENSIONAL GEORGI GLASHOW MODEL

We now briefly discuss the salient features of the mechanism of quark confinement in the Georgi-Glashow model (for details see Ref. [6] and [7]).

The model is described by the lagrangian density

$$\mathcal{L} = -\frac{1}{2g^2}\, tr\; F_{\mu\nu}F^{\mu\nu} - tr(\nabla_\mu \Phi)^2 - \frac{\lambda}{4}\, \left(2\mathrm{tr}\; \Phi^2 - v^2\right)^2 \tag{4}$$

where A_μ denotes the gauge field and Φ a Higgs field in the adjoint representation of $SU(2)$. The instantons of this model are 't Hooft-Polyakov monopoles. For reasons to be discussed later we are interested in monopoles with minimal charge :$q = \pm 1$ These are the 't Hooft-Polyakov monopoles. In the unitary gauge $[\Phi, \tau_3] = 0$ the fields are given by:

$$\tilde{A}_\mu = \frac{1}{2} \begin{bmatrix} q\tilde{A}^3_\mu & \tilde{A}^1_\mu - iq\tilde{A}^2_\mu \\ \tilde{A}^1_\mu + iq\tilde{A}^2_\mu & -q\tilde{A}^3_\mu \end{bmatrix}$$

$$\tilde{\Phi} = q\,\frac{H(vr)}{r}\,\frac{\tau^3}{2} \tag{5}$$

where $q = \pm 1$ and

$$\tilde{A}^1_\mu = = -\frac{K(rm_W)}{r}\, \left[\hat{\phi}\cos\phi + \hat{\theta}\sin\phi\right]$$

$$\tilde{A}^2_\mu = \frac{K(rm_W)}{r} \ \left[-\hat{\phi}\sin\phi + \hat{\theta}\cos\phi\right] \tag{6}$$

$$\tilde{A}^3_\mu = -\frac{1}{r} \ \tan\frac{\theta}{2} \ [\hat{\phi}]_\mu = D_\mu \tag{7}$$

where (r, θ, ϕ) denote polar coordinates in space-times. The functions $K(\xi)$, $H(\xi)$ obey well known differential equations for (5) - (6) to be a classical Euclidean solution. For $r \gg m_W^{-1}$ one has $K(rm_W) \simeq 0$ and $H(vr) \simeq vr$. m_W^{-1} denotes the "size" of the monopole.

Note that given a configuration of monopoles and anti-monopoles, the Weyl group changes *each* monopole into an anti-monopole and vice-versa. In principle one may fix the gauge further to remove this discrete degeneracy. We, however, prefer not to do so and average over the Weyl group. Then one may freely perform a sum over all the $q = \pm 1$.

As shown in [6], in the dilute gas approximation the path integral may written as a grand canonical partition function of a gas of monopoles:

$$Z = \sum_{N=0}^{\infty} \frac{1}{N!} \ J^N Q_N \tag{8}$$

where

$$Q_N = \sum_{\{q_a\}} \int \prod dx_a \exp\left(-\frac{2\pi}{g^2} \sum \frac{q_a q_b}{|x_a - x_b|}\right) \tag{9}$$

and the fugacity J is:

$$J = \frac{16}{\sqrt{\pi}} \ g^6 s^{3/2} e^{-s} \left(\frac{\det D^2}{\det -\partial^2}\right)^{-1/2} \left(\frac{\det \Delta_{FP}}{\det -\partial^2}\right) \tag{10}$$

D^2 denotes the small fluctuation operator around single monopole field, Δ_{FP} is the Fadeev-Popov operator, and s is the action of a single monopole.

In (8) - (10) we have treated the various zero modes of D^2 by the standard procedure of collective coordinates [14]. The Weyl degeneracy of the unitary gauge condition has been accounted for by summing over $q_a = 0, \pm 1$ for each space time point, as noted earlier.

The Coulomb gas of eqns. (8-10) may be expressed as a massive scalar field theory using the sine-Gordon transform:

$$Z = \int \mathcal{D}\chi(x) \exp\left[-\frac{g^2}{32\pi^2} \int d^3x \left\{(\nabla\chi)^2 - 2M^2(1 - \cos\chi)\right\}\right] \tag{11}$$

where

$$M^2 = \frac{16\pi^2 J}{g^2}. \tag{12}$$

Let us now examine self-consistency of the above scheme. When J is small, so that the scalar field theory is weakly coupled, M is precisely the mass gap. Equation (11) expresses the long distance behavior of the theory in a dual representation. The classical action of a single monopole has the form

$$s \sim \frac{m_W}{g^2} \tag{13}$$

When $m_W/g^2 \gg 1$, it follows from (10) that $J/g^6 \ll 1$. Further the "classical" piece in J dominates over the contribution from one-loop fluctuations. Thus the scalar field theory (11) is indeed weakly coupled; M is the mass gap and from (12) M/g^2 is small.

Since M is the mass gap, the correlation length $\xi = 1/M$. Also, J denotes the probability of occurence of a single monopole and hence the number of monopoles in a Debye volume of size ξ, N_ξ is given by

$$N_\xi = \frac{J}{M^3} \sim \frac{g^2}{M} \gg 1 \tag{14}$$

where we have used (12).

We immediately see that there is a large number of monopoles in one Debye volume and hence the potential χ may be treated classically. The situation here is identical to the Debye-Huckel theory of electrolytes in the limit of high temperatures where the Debye length is large and the smoothly varying potential field χ satisfies the classical sine-Gordon equation.

The Wilson loop average is given by

$$\langle W(c) \rangle = \frac{1}{2} \langle Tr\ P \exp\left(i \oint A_\mu dx_\mu \right) \rangle \tag{15}$$

where $A_\mu = A_\mu^a \frac{\tau^a}{2}$ when the external quarks are in the fundamental representation of $SU(2)$ and $A_\mu = A_\mu^a T^a$ when the quarks are in the adjoint representation. Here T^a are the generators of the adjoint representation.

The "classical" contribution from the Wilson loop factors out:

$$\langle W(c)\rangle = \langle W(c)\rangle_{c\ell}[\omega(c)]_{qu} \tag{16}$$

where $\langle W(c)\rangle_{c\ell} = \frac{1}{2}\langle Tr\ P \exp\left(i \oint \tilde{A}^a_\mu \frac{T^a}{2} dx_\mu\right)\rangle$, the average being evaluated in the ensemble given by (11). $[\omega(c)]_{qu}$ obeys a perimeter law:

$$[\omega(c)]_{qu} \sim \exp(-\alpha P) \tag{17}$$

where α is a constant and $P =$ perimeter of loop.

Let us first discuss the Wilson loop in the fundamental representation. In the ensemble given by (11) one has

$$\langle W(c)\rangle^{\text{fund}}_{c\ell} = \int \mathcal{D}\chi(y) \exp[-\frac{g^2}{32\pi^2 M} \int d^3y(\nabla(\chi - \frac{\eta}{2})^2 - 2(1-\cos\chi))] \tag{18}$$

Where $\eta(y)$ is the magnetic scalar potential due to a dipole layer of unit strength on sheet S, which is the solid angle subtended by the loop at the point y In (18) we have scaled the distances: $y \equiv Mx$.

Since M is small, (18) may be evaluated by stationary phase approximation. The result is

$$\langle W(c)\rangle^{\text{fund}}_{c\ell} = \exp(-\sigma A) \tag{19}$$

where

$$\sigma = \frac{g^2 M}{32\pi^2} \int_{-\infty}^{+\infty} dy_3 \left\{ \left(\frac{d\bar{\chi}}{dy_3}\right)^2 + 2\left(1-\cos\bar{\chi}\right) \right\} = \frac{3g^2 M}{8\pi^2} \ldots \tag{20}$$

and $A =$ area of the loop in physical units.. Then, using (16) it is easily seen that the Wilson loop obeys an area law, indicating that quarks in the fundamental representation are confined.

As shown in [7] the Wilson loop in the adjoint representation obeys a perimeter law instead. This happens because there is no distinction between the quantum of magnetic flux created by the Wilson loop and that created by the monopoles in the vacuum. The picture of confinement discussed in this section is valid for any SU(N) gauge group. The mechanism is essentially Debye screening in a gas of "non-Abelian" monopoles belonging to the adjoint representation of a dual group *SU(N) [7].

IV. CONFINEMENT IN PURE YANG-MILLS THEORY

In pure Yang Mills theory there is no Higgs field. This has two consequences. First, in the absence of the second scale (i.e. m_W), the perturbation expansion is infrared divergent and the theory cannot be defined perturbatively in the infinite volume limit. Secondly the classical monopole solutions are Wu-Yang monopoles [15] which have zero size and infinite action. We will regulate these monopoles by assigning a size λ, which is explained in the next subsection. We will then construct an expansion around a plasma of such monopoles and, as explained in the introduction, argue that these monopoles are stable against fluctuations.

A. The monopole solution

We will consider monopoles which have some size λ. The field due to single monopole at $\vec{x} = 0$ is given in the unitary gauge by:

$$\begin{aligned}
\tilde{A}_\mu(x) &= \frac{1}{2}\begin{bmatrix} q\tilde{A}^3_\mu & \tilde{A}^1_\mu - iq\tilde{A}^2_\mu \\ \tilde{A}^1_\mu + iq\tilde{A}^2_\mu & -q\tilde{A}^3_\mu \end{bmatrix} \\
\tilde{A}^1_\mu(x) &= -\frac{K(r/\lambda)}{r}\left[\hat{\phi}\cos\phi + \hat{\theta}\sin\phi\right]_\mu \\
\tilde{A}^2_\mu(x) &= \frac{K(r/\lambda)}{r}\left[-\hat{\phi}\sin\phi + \hat{\theta}\cos\phi\right]_\mu \\
\tilde{A}^3_\mu(x) &= -\frac{1}{r}\,\tan\,\frac{\theta}{2}\,[\hat{\phi}]_\mu = D_\mu
\end{aligned} \tag{21}$$

in spherical coordinates. Here $K(r/\lambda)$ is a structure function regulating the fields at $r = 0$. The function $K(r/\lambda)$ goes to 1 as $r \to 0$ as follows

$$K(r/\lambda) \sim 1 - \frac{r^2}{\lambda^2} \;\text{ for } r \to 0 \tag{22}$$

while at $r = \lambda$, $K(r/\lambda) = K'(r/\lambda) = 0$ and remains zero for $r > \lambda$. Furthermore $K'(r/\lambda)$ is continuous at $r = \lambda$. As shown by Banks, Myerson and Kogut [16], one can choose such a

$K(r/\lambda)$ so that the configuration (21) is a classical solution. The action of single monopole is

$$s = \frac{4\pi}{g^2}\int_0^\infty dr \left[\left(\frac{dK}{dr}\right)^2 + \left(\frac{K^2-1}{2r^2}\right)\right] \tag{23}$$

Note that the monopole field is abelian outside the monopole core. Consider a gas of such monopoles such that if the positions of the monopoles are x_a, x_b etc., one always has $|x_a - x_b| > 2\lambda$. The field configuration in the regions outside the core of the monopoles is approximated by

$$\mathcal{A}^{c\ell}_\mu(x) = \sum_{a=1}^{N} \tilde{A}_\mu(x - x_a) \tag{24}$$

where x_a denotes a monopole position and N is the number of monopoles. In our self-consistent approach, we assume that (24) represents the dominant field configuration in the euclidean path integral. The total action of this gas is

$$S_{cl} = Ns + \frac{2\pi}{g^2}\sum_{a\neq b}\frac{q_a q_b}{|x_a - x_b|} \tag{25}$$

Finally we record the single monopole field configuration in the radial gauge

$$A^a_\mu = -\frac{\epsilon_{\mu a j}x^j}{r^2}(1 - K(r/\lambda)) \tag{26}$$

B. The path integral

The monopole configuration is used to evaluate the path integral by the saddle point method. One expands the field around $\mathcal{A}^{c\ell}_\mu$ which is a classical solution outside the core. The form of the solution inside the core is unimportant for our purposes:

$$\mathcal{A}_\mu = \mathcal{A}^{c\ell}_\mu + g a_\mu \tag{27}$$

The path integral may be formally written as:

$$Z = \int \prod_x d\mathcal{A}_\mu(x) \exp(-S_{c\ell}) \exp\left(-\int a D^2 a d^3x\right) \tag{28}$$

where $S_{c\ell}$ is given by (25). Here D^2 is the stability operator for small fluctuations:

$$\int aD^2 a d^3x \equiv \int \mathcal{L}''(\mathcal{A}_{c\ell})\; a^2 d^3x \tag{29}$$

where $\mathcal{L}''(\mathcal{A}_{c\ell})$ is the second functional derivative of the Lagrangian density evaluated at $\mathcal{A}_{c\ell}$.

Equation (28) as it stands is meaningless since D^2 has a zero eigenvalue for each symmetry of the original Lagrangian. The local gauge symmetry is fixed by requiring the fluctuations to satisfy the background gauge condition

$$\nabla_\mu(\mathcal{A}^{c\ell})a_\mu(x) = 0 \tag{30}$$

where $\nabla_\mu(\mathcal{A}^{c\ell})$ is the covariant derivative evaluated at the configuration $\mathcal{A}^{c\ell}$:

$$\nabla_\mu(\mathcal{A}_{c\ell}) \equiv \partial_\mu + i\left[\mathcal{A}^{c\ell}_\mu, \quad\right] \tag{31}$$

This gives rise to the usual Fadeev-Popov determinant (det Δ_{FP}). The zero modes arising from the breaking of the global translation invariance and the $U(1)$ transformations obeying the background gauge condition but nonvanishing at infinity are replaced by integration over corresponding collective coordinates. Finally we have to sum over all N with the standard division by $N!$. One finally has the formal expressions

$$\begin{aligned} Z &= \sum_{N=0}^{\infty} \frac{1}{N!}\, Q_N \\ Q_N &= \sum_{\{q_a\}} \left(\frac{8}{\sqrt{\pi}}\, s^{3/2}\right)^N \int \prod_{a=1}^{N} dx_a e^{-S_{c\ell}}\; \mathcal{J} \end{aligned} \tag{32}$$

$$\mathcal{J} = \left(\frac{\det D^2}{\det -\partial^2}\right)^{-1/2} \left(\frac{\det \Delta_{FP}}{\det -\partial^2}\right) \tag{33}$$

C. Instability of the Single Undressed Monopole

In the usual semiclassical method the dilute instanton gas [17] is noninteracting and one writes

$$\det D^2 = (\det d^2)^N \tag{34}$$

where d^2 is the stability operator for the single monopole configuration.

We will see, however, that the d^2 has negative eigenvalues signifying the instability of a single monopole. In the background gauge (30) the operator d^2 has the form:

$$d^2 = \delta_{\mu\nu} \nabla_\alpha \left(\tilde{A}_{c\ell}\right) \nabla^\alpha \left(\tilde{A}_{c\ell}\right) + i \left[\tilde{F}^{c\ell}_{\mu\nu}, \ \right] \tag{35}$$

where $\tilde{A}_{c\ell}$ is defined in (21) and $\tilde{F}^{c\ell}_{\mu\nu}$ is the corresponding field. The Wu-Yang case corresponds to $K = 0$ in (21). Following Yoneya [8] we shall use a spherical basis in the product space [(space-time) $\otimes$ (isospin space)]. In this product space $\tilde{A}^{c\ell}_\mu$ and the fluctuations a_μ become tensors. Then the unstable modes of d^2 are given by

(a) Odd parity S waves:

$$\phi_1 = a_{rr}$$
$$\phi_4 = \frac{1}{\sqrt{2}} (a_{\phi\phi} + a_{\theta\theta})$$

(b) Even parity S wave:

$$\bar{\phi}_4 = \frac{1}{\sqrt{2}} \ (\phi_{\phi\theta} - \phi_{\theta\phi})$$

The tensor indices on ϕ refer to the abovementioned spherical basis. The corresponding eigenvalue equations are: (See Yoneya, Ref. [8])

$$\left(-\frac{d^2}{dr^2} + \frac{3K^2 - 1}{r^2}\right) \left(r\bar{\phi}_4\right) = \alpha_e^2 (r\bar{\phi}_4) \tag{36}$$

$$\frac{2K^2}{r^2} \phi_1 - \sqrt{2} \left(\frac{K}{r} \frac{d}{dr} + \frac{K - rK'}{r^2}\right) \phi_4 = \alpha_0^2 \phi_1 \tag{37}$$

$$\sqrt{2} \left(\frac{1}{r^2} \frac{d}{dr} (rK) - \frac{K - rK'}{r}\right) \phi_1 + \left(-\frac{d^2}{dr^2} - \frac{2}{r} \frac{d}{dr} + \frac{K^2 - 1}{r^2}\right) \phi_4 = \alpha_0^2 \phi_4 \tag{38}$$

$$\left(\frac{d}{dr}+\frac{2}{r}\right)\ \phi_1=\sqrt{2}\,\frac{K}{r}\,\phi_4 \tag{39}$$

For the Wu-Yang monopole $K=0$ and the instability is obvious from (36) and (38). In this case both $r\phi_4$ and $r\bar{\phi}_4$ obey the equation

$$\left(-\frac{d^2}{dr^2}-\frac{1}{r^2}\right)\ \psi=\alpha^2\psi,\ \ \psi=r\phi_4,r\bar{\phi}_4. \tag{40}$$

The above equation is the Schrodinger equation for a particle in a spherical potential $U_0(r)=-\frac{\gamma}{r^2}$ with $\gamma=1$. For such a potential it is known that when $\gamma>1/4$ there are bound states [18].

When the structure function $K(r/\lambda)\neq 0$ the odd parity S-waves become stable. Formally one can invert (39) to express ϕ_4 in terms of ϕ_1 and insert this into (37) to obtain:

$$\left[-\left(\frac{d^2}{dr^2}+\frac{2}{r}\frac{d}{dr}\right)+\frac{K^2+1}{r^2}\right]\left(\frac{r\phi_1}{K}\right)=\alpha^2\left(\frac{r\phi_1}{K}\right) \tag{41}$$

In the corresponding Schrodinger problem the "potential" is now positive everywhere and hence $\alpha^2>0$. Since one can now rewrite (39) as

$$\phi_1=\frac{\sqrt{2}}{r^2}\int_0^r yK(y)\phi_4(y)dy \tag{42}$$

stability of ϕ_4 also follows.

However, regularising the field near $r=0$ does not remove the even parity S-wave instability. In fact (36) has a potential that becomes asymptotically $-\frac{1}{r^2}$ as $r\to\infty$. For such a potential there are still an infinite number of negative eigenvalues. This result is independent of the detailed form of the potential near $r=0$. Hence the single regularised monopole is unstable and cannot be treated as a dominant configuration in the path integral.

It should be noted in the presence of Higgs fields the potential is replaced by

$$\frac{3K^2+H^2-1}{r^2} \tag{43}$$

where $H(vr)$ is the function in (5). As $r\to\infty$, $H(vr)=vr+e^{-m_Hr}$ (where $m_H=\frac{\sqrt{2\lambda}v}{g}$ is the Higgs mass) , and hence the $\frac{1}{r^2}$ Coulomb tail is cancelled and we have a screened potential $\frac{e^{-m_Hr}}{r^2}$, which removes the potential instability.

D. Debye Screening in the Monopole Gas

Our main point is that when the instantons are interacting, the fluctuation problem cannot in general be factored into N copies of the fluctuation problem for an isolated monopole. Rather one should consider the stability of the neutral plasma of monopoles as a whole. This statement also applies to the Yang-Mills-Higgs system in the previous section. The main reason behind this is that we have an integration over the positions of the monopole and the charges, or equivalently a functional integration over the charge density field. The effect of this averaging over the charge density is very nontrivial. The results of [9] show that the charge density field clusters so that the theory of the density field generates a mass gap dynamically. We summarize these results in Appendix I.

In fact the results of [9] mean that in the neutral plasma the fluctuations of the magnetic field are bounded and the $\frac{1}{r^2}$ magnetic field of a single monopole is Debye screened to $\frac{e^{-Mr}}{r^2}$, where $M(g^2, \lambda, z)$ is the non-perturbative mass gap, which depends on the coupling g^2, the monopole size λ which is effectively the cut-off of the Coulomb gas and the fugacity z. Recall that the source of instability for a single isolated monopole is the long range tail of the Coulomb potential. One might, therefore expect that in the screened neutral plasma a "dressed" monopole whose Coulomb tail has been screened can in fact be stable.

In the following we shall assume that the monopole gas is dilute. Using translation invariance we focus on one monopole at $x = x_\alpha$ and its neighbourhood. We are thus considering the problem in *the presence of a single source* at $x = x_\alpha$. Recall that the fields outside the monopole cores of size λ are abelian. Thus, in the unitary gauge it follows from (24) and (21) that outside the core of this monopole the field is

$$\tilde{A}^{\rm out}_\mu = \sum_{a=1}^{N} \frac{1}{2}\, q_a \begin{pmatrix} D_\mu(x - x_a) & 0 \\ 0 & -D_\mu(x - x_a) \end{pmatrix} \tag{44}$$

where we have set $x_N = x_\alpha$. Inside the core the effect of the core field of the other monopoles can be ignored and we have

$$\tilde{A}^{\rm in}_\mu = \sum_{a=1}^{N-1} \frac{1}{2}\, q_a \begin{pmatrix} D_\mu(x - x_a) & 0 \\ 0 & -D_\mu(x - x_a) \end{pmatrix}$$

$$+\frac{1}{2}\begin{pmatrix} qD_\mu(x-x_\alpha) & \tilde{W}^-_\mu(x-x_\alpha) \\ \tilde{W}^+_\mu(x-x_\alpha) & -qD_\mu(x-x_\alpha) \end{pmatrix} \tag{45}$$

where $\tilde{W}^\pm_\mu \equiv \tilde{A}^1_\mu \pm iq\tilde{A}^2_\mu$ and $\tilde{A}^1_\mu, \tilde{A}^2_\mu$ are as in (21). Introducing the charge density

$$\rho(x) \equiv \sum_{a=1}^{N} q_a \delta(x-x_a) \tag{46}$$

and assuming that in the mean, for large N, $\rho_N \simeq \rho_{N-1}$ we can rewrite (45) as

$$\begin{aligned} A_\mu(x;x_\alpha,[\rho]) &= \int d^3y\, \frac{\tau^3}{2}\, \rho(y) D_\mu(x-y) \\ &+ \quad \theta(\lambda - |x-x_\alpha|) \begin{pmatrix} D_\mu(x-x_\alpha) & W^-_\mu(x-x_\alpha) \\ W^+_\mu(x-x_\alpha) & -D_\mu(x-x_\alpha) \end{pmatrix} \end{aligned} \tag{47}$$

In the above expression the sharp θ function may be replaced by a smoother version.

Debye screening means that *in the presence of a source* the density $\rho(y)$ has a mean value $\bar{\rho}(y, x_\alpha)$, fluctuations around which are small. $A_\mu(x; x_\alpha, [\bar{\rho}])$ then represents a "dressed" monopole configuration. In our case this "source" is provided by the particular monopole at $x = x_\alpha$ in the plasma and the statement pertains to the field in the neighbourhood of this particular monopole.

The crucial point is that since Debye screening holds, we can assume *self-consistently* that the gas of "dressed" monopoles is weakly interacting, unlike the "bare" monopoles. The field around a dressed monopole decay exponentially over a distance scale set by the Debye screening length $l_D = \frac{1}{M}$. If the average distance between the monopoles is much larger than l_D then the interaction between such dressed monopoles vanishes and the operator D^2 has a N-fold degeneracy. In other words the potential appearing in the stability equation resembles N far separated potential wells. In this situation we have, using translation invariance,

$$\det D^2 \simeq (\det D^2[\bar{\rho}])^N \tag{48}$$

where $D^2[\bar{\rho}]$ now denotes the stability operator for a *single* dressed monopole which is the same for any monopole in the plasma. For finite distances between monopoles the exact degeneracy is lifted and eigenvalues of D^2 organize themselves in bands. This would lead to corrections to the result (48) which may be expanded in powers of $\frac{l_D}{l_m}$ where l_m denotes the average distance between the monopoles.

In this regard there is a difference between the pure Yang Mills system and the Yang-Mills-Higgs system in the Higgs phase. As our stability analysis in the previous subsection indicates (see e.g. equation (43)) the presence of the Higgs field means that the potential appearing in the stability operator around a single monopole decays over length scales of the order of $\frac{1}{m_W}$. Since the debye length is much larger than $\frac{1}{m_W}$ corrections to the extensivity of the small fluctuation determinant appear as powers of $\frac{1}{l_m m_W}$.

E. The Sine-Gordon Transform and Dynamical generation of Mass Gap

We can now rewrite the theory in terms of a sine-Gordon model. The path integral is written as

$$Z = \sum_{N=0}^{\infty} \frac{1}{N!} \left(\frac{8g^6}{\sqrt{\pi}} \, \bar{s}^{3/2} e^{-\bar{s}} \right)^N \sum_{\{q_a\}} \prod_{i=1}^{N} \int d\vec{x}_i \exp \left(-\frac{2\pi}{g^2} \sum_{a \neq b} \frac{q_a q_b}{|x_a - x_b|} \right) \, (\Theta[\bar{\rho}])^N \tag{49}$$

where we have defined $\Theta[\bar{\rho}]$ (functional of the charge densities) as

$$\Theta[\bar{\rho}] \equiv \left(\frac{\det D^2[\bar{\rho}]}{\det -\partial^2} \right)^{-1/2} \left(\frac{\det \Delta_{FP}(\bar{\rho})}{\det -\partial^2} \right) \tag{50}$$

and $\bar{s}$ is the action for a single monopole.

It may be noted that we are dealing with a superrenormalizable theory. Thus the expression (50) is ultraviolet finite.

We then have

$$Z = \sum_{N=0}^{\infty} \frac{1}{N!} \, \bar{J}^N \sum_{\{q_a\}} \prod_{i=1}^{N} \int d\vec{x}_i \, \exp \left(-\frac{2\pi}{g^2} \sum \frac{q_a q_b}{|x_a - x_b|} \right) \tag{51}$$

where the mean fugacity $\bar{J}$ is given by the formula

$$\bar{J} = \frac{8 \, g^6}{\sqrt{\pi}} \, \bar{s}^{\frac{3}{2}} \, e^{-\bar{s}} \, \Theta[\bar{\rho}] \tag{52}$$

We can now perform the sine-Gordon transform as in section 3, and we write (51) as

$$Z = \int \mathcal{D}\chi(x) \exp \left[-\frac{g^2}{32\pi^2} \int d^3x \left\{ (\nabla\chi)^2 - 2\bar{M}^2(1 - \cos\chi) \right\} \right] \tag{53}$$

where

$$\bar{M}^2 = \frac{16\pi^2 \bar{J}}{g^2}. \tag{54}$$

It is important to emphasize in accordance with the discussion of [9], that the quadratic term $\int d^3x(\nabla\chi)^2$ in (53) must be understood in a regularised sense, so that the Coulomb potential between the monopoles is valid only upto distances greater than the core size λ. In this sense λ is the cut off (lattice spacing) of the sine-Gordon theory.

F. Stability of the dressed monopole

We now discuss, in some more detail than previously, whether the function $\Theta[\bar{\rho}]$ which is used in the definition of the fugacity $\bar{J}$ in (52) is well defined. This issue is important because we have already indicated in Section 4 that as we average over $\rho(x)$ the operator $D^2[\rho]$ has negative eigenvalues when ρ corresponds to a single isolated monopole.

We will now argue that $D^2[\bar{\rho}]$ is a positive operator. Recall the form of $D^2[\bar{\rho}]$ in the unitary gauge, outside the core of the dressed monopole which we can choose to be at $x = 0$

$$D^2[\bar{\rho}] = -\delta_{\mu\nu} \nabla_\alpha(A(x, [\bar{\rho}]))\nabla^\alpha(A(x, [\bar{\rho}])) + i \left[\bar{F}^{(3)}_{\mu\nu}(x, [\bar{\rho}]), \quad \right] \tag{55}$$

where $\bar{F}^{(3)}_{\mu\nu}(x, [\bar{\rho}])$ is related to the magnetic field by $\bar{B}^{(3)}_\lambda = \epsilon_{\mu\nu\lambda}\bar{F}^{(3)}_{\mu\nu}$ and $\bar{B}^{(3)}_\lambda$ is given by

$$\bar{B}^{(3)}_\lambda = \frac{\partial}{\partial x_\lambda} \int d^3y \, \frac{\bar{\rho}(y)}{|x - y|} \tag{56}$$

Debye screening means that

$$\bar{B}^{(3)}_\lambda = \frac{x^\lambda}{r^3} \, e^{-Mr} \, f(Mr) \tag{57}$$

where the function $f(x)$ has the property that for $x >> 1$, $f(x) \sim 1$, and M is the mass gap related to the Debye length l_D by $M = \frac{1}{l_D}$

For a nonzero mass gap, the field outside the monopole core cannot be transformed to the radial form of (26). However close to the core and distances much smaller than the Debye length l_D the field is close to the single monopole field outside the core and may be cast in the radial gauge. Furthermore at distances much larger than the Debye length, the field is close to zero and once again one may cast the gauge potential in the radial gauge trivially (i.e. with $K(r) = 1$). As mentioned in Section 4 the source of the even parity S-wave instability is the long range Coulomb field of the monopole. Since screening cuts off this Coulomb field and replaces it approximately by an exponential, one expects stability.

The situation is in fact similar to that of the Yang-Mills-Higgs system in some respects. Recall that the S-wave stability of the 't Hooft-Polyakov monopole is ensured by the fact that the Higgs field rises *exponentially* to 1 beyond the core and cancels the negative $\frac{1}{r^2}$ tail of the gauge field, preventing the potential in the Schrodinger problem from being negative at large distances. In our problem the gauge field itself falls off exponentially to zero and thus the potential in (36) is positive at large distances. In this sense we have a dynamical Higgs effect.

It is indeed possible to argue that the above argument for stability is self-consistent. Recall that Debye screening means that the form of the magnetic field away from the monopole core is of the form 57. We are unable to determine the precise form of the function $f(x)$, but we can parametrize it in the following way. Inside the monopole core the magnetic field in the unitary gauge follows from the form of the single monopole vector potential

$$B_i^{(3)} = -\frac{x^i}{r^3}(1 - K(r\lambda)^2) \tag{58}$$

The function $K(x)$ is given by the solution in [16]. Let us assume that this form of the magnetic field extends upto $r = \alpha\lambda$ with some $\alpha > \lambda$. The magnetic potential far outside the core is an exponential, as required by Debye screening which gives

$$B_i^{(3)} = -\frac{x^i}{r^3}(1 + Mr)\, e^{-Mr} \tag{59}$$

where M is the mass gap generated in the plasma. We assume that this form of the magnetic field extends from infinity upto the point $r = \alpha\lambda$. This gives the function K in the region $r > \alpha\lambda$

$$K^2(r) = 1 - (1 + Mr)e^{-Mr} \qquad\qquad r > \alpha\lambda \tag{60}$$

We further simplify the problem by replacing the function $K(r)$ inside the core by an exponential so that we get

$$K^2(r) = e^{-2r/\lambda} \qquad\qquad r < \alpha\lambda \tag{61}$$

The potential which appears in the s-wave stability operator is $V(r) = \frac{3K^2-1}{r^2}$. This potential must be continuous at $r = \alpha 1$ which determines α implicitly through the equation

$$e^{-2\alpha} = 1 - (1 + M\alpha\lambda)e^{-M\alpha\lambda} \tag{62}$$

It may be now seen that the potential $V(r)$ is negative in the region $\frac{1}{2}\log\, 3\ \lambda < r < r_2$ where r_2 may be determined by the above considerations to be the solution of $V(r) = 0$ for $r > \alpha\lambda$. This is approximately $\frac{1.2}{M}$

We now test whether such a potential can have bound states. This may be done by applying the Bargmann criterion for absence of bound states

$$\int dr\ rV^-(r) < 1 \tag{63}$$

where $V^-(r)$ stands for the negative part of the potential. A straightforward numerical integration then gives the result that the Bargmann bound is satisfied for $\alpha < 1.38$. The fact that the critical value of α came out to be greater than one is an evidence for the self consistency of the picture.

The above considerations are rough : we have made several simplifying assumptions. However these assumptions are dictated by the physics of the problem. The numbers quoted above are to be considered as indicative since they will change with different approximations to the function $f(r)$ and $K(r)$ inside the core. However the above calculation gives a self consistent argument in favor of the stability of monopoles in a neutral plasma.

G. Self consistent dynamical generation of monopole size

As mentioned in Section the fugacity of the monopole gas depends on $\bar{\rho}$ and hence it has a dependence on the mass gap M and the cutoff λ of the sine-Gordon theory. This means that the cutoff λ is not independent, but determined in terms of M and g^2, i.e.

$$M = M(\lambda, g^2, z(\lambda, M)) \tag{64}$$

implies that

$$\lambda = \frac{1}{g^2} F(\frac{M}{g^2}) \tag{65}$$

where F is a function obtained by inverting (64). This equation now fixes the monopole core size λ as a function of the mass gap M in a self-consistent manner. The above considerations must be considered as qualitative because the calculation of the function F in (65) is beyond our present technology.

V. DUAL REPRESENTATION AND THE DISORDER OPERATOR

We now relate the euclidean formalism in terms of disorder operators introduced by 't Hooft [4] . In the hamiltonian formalism the Schrodinger picture disorder operator $\Phi_D(\vec{x}_0)$ is defined as an operator which creates a Z_2 magnetic vortex at the point $\vec{x}_0$ in two dimensional space. More specifically it implements a singular gauge transformation $\Omega^{[x_0]}$ which has the property that if we consider a closed spatial loop C parametrized by an angle θ one has

$$\Omega^{[x_0]}(\theta + 2\pi) = -\Omega^{[x_0]}(\theta) \tag{66}$$

when the loop C encloses the point $\vec{x}_0$. If C does not enclose $\vec{x}_0$ the gauge transformation is single valued. Consider now the two point function of the Heisenberg picture disorder operators

$$< \Phi_D^\dagger(x) \Phi_D(y) > \tag{67}$$

Here x and y stand for three dimensional coordinates (including the euclidean time). This two point function is then a sum over all configfurations of the gauge fields which have a Dirac string singularity along a line joining x and y with a monopole of charge $\frac{1}{2}$ at the point y and an antimonopole of charge $-\frac{1}{2}$ at the point x. It is crucial that the magnetic charges of these monopoles is half that of the monopoles which populate the vacuum. They have magnetic charges so that the Dirac string is *visible* by the lowest electrically charged quarks which couple to the gauge field.

Repeating the steps which led to the sine-gordon representation with the difference that we have two external magnetic sources with charges $\pm\frac{1}{2}$ we easily get

$$< \Phi_D^\dagger(x)\Phi_D(y) >=< e^{\frac{i}{2}(\chi(y)-\chi(x))} > \tag{68}$$

the average on the right hand side in (68) being performed in the sine-gordon theory. A similar identification holds for all higher point correlation functions of the disorder operators. Hence we can identify the disorder operator with

$$\Phi_D(x) = e^{\frac{i}{2}\chi(x)} \tag{69}$$

In fact the sine-gordon action may be now written in terms of Φ_D as

$$S = \frac{g^2}{32\pi^2}\int d^3x[\partial_\mu\Phi_D^\dagger\partial_\mu\Phi + M^2((\Phi_D)^2 + (\Phi_D^\dagger)^2)] \tag{70}$$

upto an irrelevant constant. The field Φ_D is not a conventional scalar field, since $\Phi_D^\dagger\Phi = 1$. This non-linear Z_2 sigma model can be generalized to a linear sigma model by the addition of the term $\lambda\int d^3x(\Phi_D^\dagger\Phi - 1)^2$ to the action (70). The action then exactly has the form conjectured in [4].

The sine gordon theory thus is itself a dual representation of the original Yang-Mills theory. The action (70) has the global Z_2 symmetry $\Phi_D \to \Phi_D^\dagger$ which is spontaneously broken leading to magnetic disorder and confinement. This is simply the symmetry $\chi \to -\chi$ of the sine-gordon model. It is clear from the action (70) that the dimensionless coupling constant is $\frac{\sqrt{M}}{g}$. This is inversely related to the gauge coupling g as expected in a dual formulation. The dual theory is weakly coupled when $\frac{M}{g^2}$ is small. In this limit the minima of the potential ($\cos\,\chi$) break the Z_2 symmetry spontaneously.

Finally we note that the above construction of the disorder operators can be easily extended to $SU(N)$ gauge theories following the treatment in [7].

VI. CONCLUSIONS

We have argued that in 2+1 dimensional pure Yang-Mills theory Debye screening in a gas of regularized and dressed Wu-Yang monopoles provides a consistent picture of quark confinement. We have used the results of [9] that in a three dimensional Coulomb gas the charge density field always clusters, leading to Debye screening even for arbitrarily low temperatures. Our line of argument has been self-consistent in nature, because Debye screening in turn implies a screened magnetic field and hence the stability operator around a single dressed monopole is expected to have no negative eigenvalues. The mass gap thus obtained is non-perturbative and determines the monopolesize self-consistently. A related issue is that the mean configuration $\bar{\rho}(x)$ in the presence of a single monopole source is in general non-classical and hence the associated scalar potential $\bar{\chi}(x)$ does not satisfy a classical equation. Hence the explicit evaluation of the Wilson loops is not as easily done. However on general grounds the existence of a mass gap leads to qualitative conclusions that are similar to the case of the Yang-Mills-Higgs system. Finally we have obtained a representation of the disorder operators of the theory in terms of the sine-gordon field which leads to a dual representation of the gauge theory.

VII. APPENDIX I

In this appendix we state the main results of [9] on the 3-dim. Coulomb gas. Theorem 2.1 in Brydges [9], adapted to our notation states that:

Given any $c_1 > 0$, there exits $c_2 > 0$ such that for $\frac{1}{c_1} \leq g^2\lambda$ and $z \leq \frac{1}{2}c_2^2 g^6$ (z is the fugacity) the correlation functions of the density operator exits and clusters exponentially, i.e. there exist strictly positive constants $M(z, g^2, \lambda)$, $c' = c'(n')$ such that for $n_1 < n'$

$$\begin{aligned} &|\langle \prod_{i=1}^{n_1} \rho(x_i) \prod_{j=n_1+1}^{n'} \rho(x_j + a)\rangle - \langle \prod_{i=1}^{n_1} \rho(x_i)\rangle \langle \prod_{j=n_1+1}^{n'} \rho(x_j)\rangle| \\ &\leq c' \exp\left(- \underset{2\leq n_1<j\leq n'}{inf} |x_i - x_j + a| \cdot M\right) \end{aligned} \tag{71}$$

$M(z, g^2, \lambda)$ is the mass gap whose inverse is the Debye screening length. In the limit $g^2\lambda \to \infty$ one has the classical Debye-Huckel limit $\frac{M}{g^2} \to 0$. If we apply (71) to the 2-point function of the density, for separation of the order λ, the monopole core size, which is the lattice spacing for the Coulomb gas we get

$$|\langle \Delta\rho(0)\Delta\rho(\lambda)\rangle| \leq c' e^{-M\lambda} \tag{72}$$

where $\Delta\rho(x) = \rho(x) - \langle \rho(x)\rangle$. (72) says that the fluctuations of $\rho(x)$ are bounded and finite.

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