

Spacelike boundaries from the $c = 1$ Matrix Model

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Abstract

We find classical solutions of two dimensional noncritical string theory which give rise to geometries with *spacelike* boundaries, similar to spacetimes with cosmological event horizons. In the $c = 1$ matrix model, these solutions have a representation as simple time dependent configurations. We obtain the causal structure of the resulting spacetimes. Using the macroscopic loop transform, we probe the form of the tachyon condensate in the asymptotic regions.

1 Introduction

The $c = 1$ matrix model [1], interpreted as a theory of D0 branes [2, 3], is the holographic description of two dimensional string theory. It provides an example of open-closed duality with an explicit dictionary, since the density of matrix eigenvalues is directly related to the string field [4]. The availability of such a model has inspired recent work aimed at understanding the physics of time dependent backgrounds. While classical time dependent solutions of the matrix model have been known for quite some time [5, 6, 7], only recently has it been suggested [8] that these backgrounds should be regarded as “matrix cosmologies” and fruitfully utilized to understand conceptual issues related to quantum cosmology such as particle production [9, 10, 11] and thermality [12].

The most closely studied class of time dependent solutions [9]-[12] is one in which the Liouville wall accelerates toward the weakly coupled region, and approaches \mathcal{I}^+ on an asymptotically null trajectory. In the fermi sea picture this was described by the edge of the fermi sea of eigenvalues moving away to infinity.

In this paper, we discuss a different class of solutions where the edge of the fermi sea *disappears* after some time, and the left and the right seas merge. Remarkably, we will find that in spacetime this is accompanied by the appearance of a future \mathcal{I}^+ which is *spacelike* rather than *null*. A time reversed version similarly leads to a spacelike \mathcal{I}^- while a time-symmetric version renders both \mathcal{I}^+ and \mathcal{I}^- spacelike.

While the causal structure of the spacetime is determined, computing the conformal factor of the metric is a much more delicate task which we do not attempt. The metric, as will be discussed, cannot be computed from classical information contained in the effective action for the collective field. A comparison of the computation of a *quantum effect* from the effective action for string theory with that from the matrix model, e.g. the computation of the outgoing stress-energy tensor of particles produced in the nontrivial background, might shed some light on this question [13], but is well beyond the scope of this paper.

It is well known that the bulk spacetime which naturally follows from the matrix model is related to the spacetime of perturbative string theory by a non-local transform. At the linearized level and in momentum-space these are given simply by momentum dependent leg pole factors [14, 15]. This non-locality implies a fuzziness of our causal diagrams at the *string scale* (at least in the asymptotic region). This fuzziness is present in any description of spacetime in string theory. This does not, therefore, modify our conclusions about the causal structure which is in any case a concept at distance scales larger than the string length.¹

We complete our analysis of the spacetime background by computing the form of the tachyon

¹In the nonlinear theory the transform is both non-local and nonlinear [16]. This nonlinearity is not relevant for the considerations of this paper.

condensate in the bulk of the weakly coupled region. To this end, we compute the macroscopic loop transform of our solutions, effectively probing the string spacetime with the end of the FZZT brane [17, 18]. We find that the asymptotic behavior of the tachyon profile so determined matches our expectations.

Appearance of spacelike \mathcal{I}^\pm is associated with the existence of cosmological horizons, and is reminiscent of de Sitter spacetimes. Perhaps an in-depth study of these scenarios could shed some light on the quantum mechanics of de Sitter.

This paper is organized as follows: in section 2 we describe our class of time dependent solutions to the matrix model and obtain the spacetime causal structure for these solutions, represented by the Penrose diagrams. In section 3, we discuss the non-locality in the matrix model-to-spacetime map and some of its consequences, introducing both the leg pole transform and the macroscopic loop transform. In 4.1 we discuss the macroscopic loop transform in detail and motivate its use for our purpose. Finally, in 4.2, we apply the loop transform to obtain information about the tachyon condensate.

2 Moving Fermi sea solutions

The classical limit of the matrix model is described by motion of an incompressible Fermi fluid in phase space under an equation of motion imposed by a Hamiltonian

$$H = \frac{1}{2}(p^2 - x^2) . \quad (1)$$

We will use $\alpha' = 1$ when discussing bosonic string theory, and $\alpha' = \frac{1}{2}$ when discussing 0B theory, which allows us to use the same matrix Hamiltonian for both cases.

The static Fermi sea profile, given by $(x-p)(x+p) = 2\mu$, corresponds to a flat linear dilaton background with a tachyon wall whose position is specified by the value of string coupling at the wall, $g_s \sim \mu^{-1}$. As is well known, the effective field η describing small fluctuations about the profile is massless and one can define two coordinate patches in the two regions $x^2 > 2\mu$, each with coordinates σ and τ , by the relations $x = \pm 2\mu \cosh \sigma$ (the \pm refer to the two sides of the potential) and $\tau = t$, in which the quadratic action is simply

$$S = \frac{1}{2} \int d\sigma d\tau ((\partial_\tau \eta)^2 - (\partial_\sigma \eta)^2) . \quad (2)$$

Up to string scale non-localities to be described later, the string theory spacetime is closely related to the spacetime defined by σ and τ . The metric inferred from (2) is of the form $\exp(\rho) \eta_{\mu\nu}$ in the σ, τ coordinates. The conformal factor ρ cannot be determined from this analysis, but the conformal structure can: it is simply half of flat space. At $\sigma = 0$, there is a reflecting boundary condition, since $x(-\sigma) = x(\sigma)$. This corresponds to the tachyon wall in

spacetime. Interactions of the η fields are cubic at the classical level when expressed in terms of η and its canonical conjugate Π_η and the coupling constant is $g(\sigma) = 1/(2\mu \sinh^2 \sigma)$, which shows that the theory is strongly nonlinear near the mirror.

The static Fermi sea is, of course, not the only solution of the equations of motion. A class of nontrivial solutions can be obtained by acting with various W_∞ transformations on this static profile [10]. In phase space, one class of transformations is given by

$$(x \pm p) \rightarrow (x \pm p) + \lambda_\pm e^{\pm rt} (x \mp p)^{r-1}, \quad (3)$$

where r is a non-negative integer and λ_\pm are finite parameters. This leads to a class of time dependent Fermi surfaces

$$x^2 - p^2 + \lambda_- e^{-rt} (x + p)^r + \lambda_+ e^{rt} (x - p)^r + \lambda_+ \lambda_- (x^2 - p^2)^{r-1} = 2\mu. \quad (4)$$

For $r = 1, 2$ the profile of the Fermi surface in phase space is quadratic (ie, it is intersected by a constant x line at most twice) and therefore corresponds to a classical solution of the collective field theory [19, 21]. Non-quadratic profiles generically signify large quantum fluctuations [20, 21].

The case of $r = 1$ was studied in some detail in [9]-[12]. Here, we will focus on $r = 2$.

Special coordinates, defined up to a conformal transformation as the coordinates for which the quadratic action for small fluctuations is of the form given in equation (2), can be found for non-static Fermi sea profiles as well [22, 23]. We will refer to these special coordinates as Alexandrov coordinates, and use them to define the causal structure of the theory. The philosophy here is that the collective field of the matrix model corresponds to an effective massless field in spacetime, and can therefore be used as a spacetime probe. However, since in two spacetime dimensions the kinetic term for such a field does not depend on the conformal factor of the metric, this probe does not allow us to determine this factor.

2.1 The closing hyperbola solution

We begin with a special case of equation (4), with $r = 2$, $\lambda_- = 0$, and $\lambda_+ < 0$. By choosing the origin of t , we may choose $\lambda_+ = -1$. Hence, consider the following profile (in this example, we will focus on just the right branch of the hyperbola):

$$(x - p)(x + p + e^{2t}(p - x)) = 2\mu. \quad (5)$$

This represents a hyperbola which starts out near the static configuration, but eventually ‘closes’ and escapes to infinity. The Alexandrov coordinates are given by the following coordinate transformation

$$x = \sqrt{2\mu} \frac{\cosh \sigma}{\sqrt{1 - e^{2\tau}}}, \quad (6)$$

$$t = \tau - \frac{1}{2} \ln(1 - e^{2\tau}) , \quad (7)$$

where the coordinate patch $\sigma > 0$ and $\tau < 0$ is enough to cover the *entire* evolution of the Fermi surface. Penrose diagram of this spacetime, which exhibits its causal structure, is shown in figure 1(b). Though this example is very simple, it is nontrivial, since \mathcal{I}^+ is spacelike.

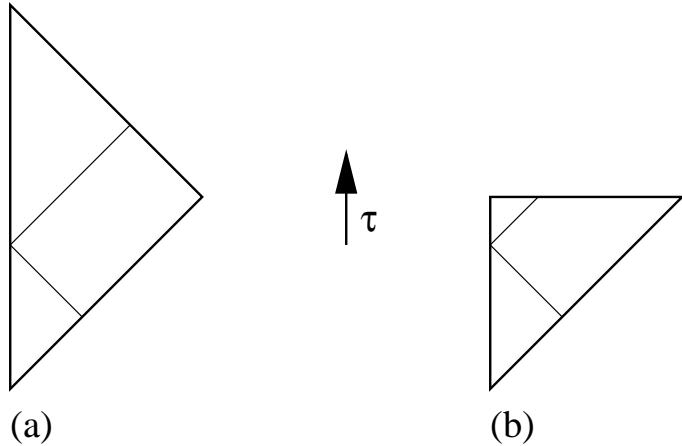


Figure 1: Causal structure of (a) Static flat space and (b) space resulting from the solution in equation (5). A sample null trajectory is shown ending on \mathcal{I}^+ .

This space-time is in fact geodesically incomplete and normally one would of course extend this to full (half) Minkowski space. However, in terms of the original matrix model time, this would mean that one has to extend beyond $t = \infty$, which does not make sense. The underlying matrix model therefore forces us to have this space-like boundary, perhaps suggesting that the spacetime effective theory is strongly coupled there.

The causal structure should be compared to the spacetime traced out by trajectories of points on the Fermi surface. These are a one-parameter family of curves

$$x_0(t) = \sqrt{2\mu} [e^t \cosh \tau_0 + \frac{1}{2} e^{\tau_0 - t}] , \quad (8)$$

where τ_0 is a real parameter. This is an *exact* solution of the equations of motion and therefore gives the motion of the center of an infinitesimal pulse is localized at x_0 .

Interestingly these curves are *null rays* in Alexandrov coordinates. The ray is given by $\tau + \sigma = \tau_0$ for an incoming pulse and $\tau - \sigma = \tau_0$ for the reflected pulse, as may be verified by substituting (8) in (7). Null rays are the trajectories of centers of wave packets at the *linearized* level since the fluctuations are massless scalars. It might appear strange this this continues to be the case in the full nonlinear theory. However this simply follows from the equation of motion. The equation $\ddot{x} = x$ written out in Alexandrov coordinates (7) becomes

$$\cosh \sigma [e^{4\tau} - 2e^{2\tau}] + \dot{\sigma}^2 \cosh \sigma [2e^{2\tau} - e^{4\tau} - 1] + \ddot{\sigma} \sinh \sigma [2e^{2\tau} - e^{4\tau} - 1] = -\cosh \sigma , \quad (9)$$

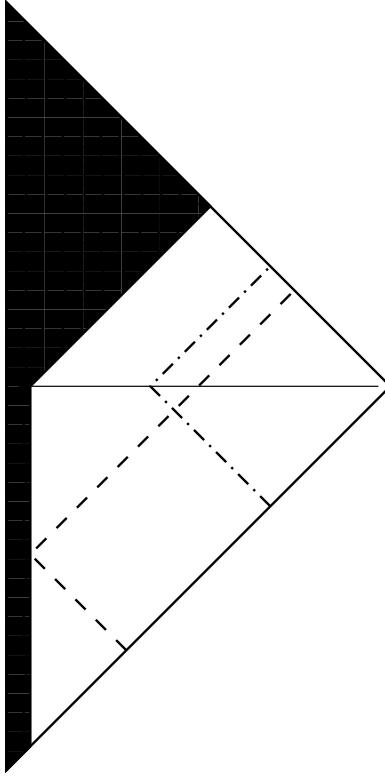


Figure 2: Trajectories of massless particles in fermion coordinates ($t, q \equiv \log x$) corresponding to spacetime structure in figure 1(b).

where dot denotes a derivative with respect to the Alexandrov time τ . It is clear from this that null rays $\dot{\sigma} = \pm 1$ automatically solve this equation.

In figure 2 we show these trajectories in roughly the fermion coordinates, t and $q \equiv \log x$. Notice that this is *not* a Penrose diagram, as 45 degree lines are not necessarily null.

2.2 The opening hyperbola solution

We will now focus on the following solution:

$$(x - p)(x + p + e^{2t}(x - p)) = 2\mu . \quad (10)$$

In terms of equation (4) this corresponds to $r = 2$ with $\lambda_- = 0$ and $\lambda_+ > 0$. We have further chosen the zero of time t to set $\lambda_+ = 1$. The solution represents two branches of a hyperbola which approach the static solution at $t \rightarrow -\infty$ and which then ‘open up’ and spill over the top of the potential at $t = 0$, see figure 3.

The configuration is qualitatively different for $t < 0$ and $t > 0$. For $t < 0$, each branch of the hyperbola is on its own side of the potential, and each branch intersects a vertical line

ξ

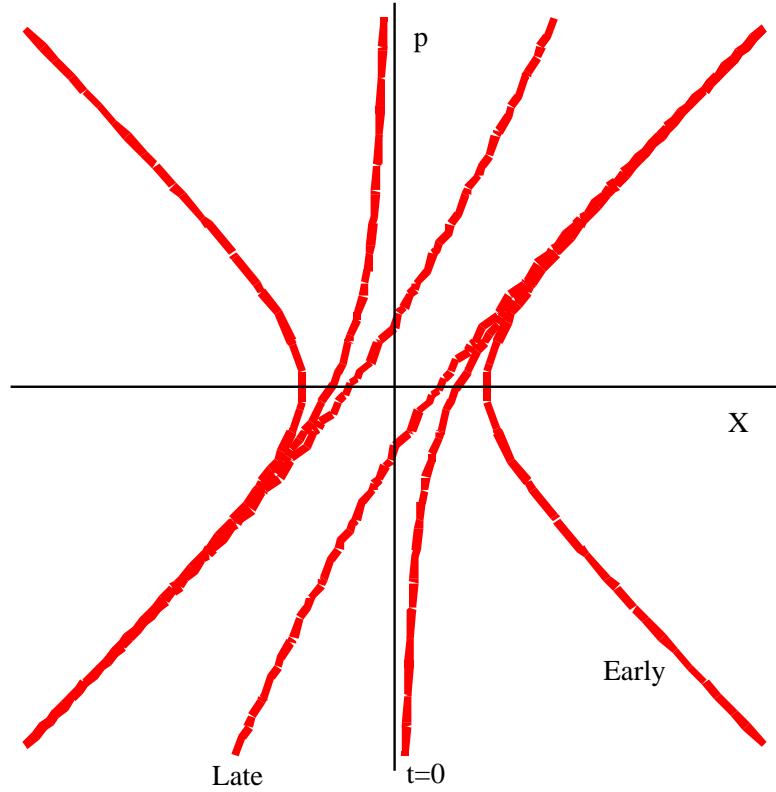


Figure 3: Time evolution of the Fermi sea given by equation (10).

twice. The eigenvalue density $\varphi_0(x, t)$ may be easily found to be

$$\varphi_0(x, t) = \frac{\sqrt{x^2 - 2\mu(1 - e^{2t})}}{1 - e^{2t}}. \quad (11)$$

This clearly shows that for $t < 0$ the density of eigenvalues vanishes in the region

$$|x| < \sqrt{2\mu(1 - e^{2t})}, \quad (12)$$

whereas for $t > 0$ we will see there is no such cut.

For $t < 0$, the techniques for defining Alexandrov coordinates used in [22] can be applied. The result is (\pm correspond to the right and left branch)

$$\begin{aligned} x &= \pm\sqrt{2\mu} \frac{\cosh \sigma}{\sqrt{1 + e^{2\tau}}}, \\ t &= \tau - \frac{1}{2} \ln(1 + e^{2\tau}) \end{aligned} \quad (13)$$

These Alexandrov coordinates consist of two patches (corresponding to the two branches of hyperbola), each covering the region $\sigma > 0$ and $-\infty < \tau < \infty$. The cut in eigenvalue distribution is at (in fermion coordinates x)

$$x_{end}^\pm = \pm \sqrt{2\mu(1 + e^{2\tau})^{-1}}. \quad (14)$$

As $t \rightarrow 0, \tau \rightarrow \infty$, $x_{end}^\pm \rightarrow 0$, and the left branch and right branch cuts meet. This corresponds to the hyperbola becoming ‘vertical’ (see figure 3).

At $t = 0$, the mirror disappears, and the eigenvalue density as a function of x becomes infinite. For $t > 0$, then, the effective field must be defined a little differently. The two branches of the solution are now ‘horizontal’, that is, $p(x)$ is a single-valued function of x on each branch. We parametrize the two branches via

$$\begin{aligned} x_\pm(\omega, t) &= \pm \sqrt{2\mu} \left(\cosh \omega - \frac{1}{2} e^{2t-\omega} \right), \\ p_\pm(\omega, t) &= \pm \sqrt{2\mu} \left(\sinh \omega - \frac{1}{2} e^{2t-\omega} \right). \end{aligned} \quad (15)$$

and define the collective field as $\varphi_0 = p_-(x) - p_+(x)$. This is

$$\varphi_0(x, t) = -\frac{\sqrt{x^2 + 2\mu(e^{2t} - 1)}}{e^{2t} - 1}. \quad (16)$$

Notice that this is the same expression as for $t < 0$, though the interpretation is different. (11) is the actual eigenvalue density, while (16) represent the ‘eigenvalue density minus infinity’, or the negative of the density of the complement of the eigenvalue distribution.

With these definitions, an approach similar to that of Alexandrov can now be followed, defining $\tilde{\omega}(\omega, t)$ such that $x_-(\tilde{\omega}, t) = x_+(\omega, t)$, and then the Alexandrov coordinates $\tau = -t + (\omega + \tilde{\omega})/2$ and $\sigma = (\omega - \tilde{\omega})/2$.

$$\begin{aligned} x &= \sqrt{2\mu} \frac{\sinh \sigma}{\sqrt{e^{-2\tau} - 1}}, \\ t &= -\tau - \frac{1}{2} \ln \left(e^{-2\tau} - 1 \right), \end{aligned} \quad (17)$$

is then the desired coordinate change. This patch of Alexandrov coordinates has $\tau < 0$ and $-\infty < \sigma < \infty$.

The crucial fact about this coordinate transformation is that the endpoint of the fermion time evolution, $t = \infty$, corresponds to $\tau = 0$. The underlying fermion dynamics instructs us to truncate the spacetime at this spacelike surface.

We need to make sure that these are the correct coordinate transformations. From the general formulae in [10] (equation (36)) it follows that the fluctuations are massless particles in a metric which is conformally equivalent to

$$ds^2 = \frac{1}{(1 - e^{2t})^2} \left(-dt^2 + \frac{[(1 - e^{2t})dx + e^{2t}x \, dt]^2}{x^2 - 2\mu(1 - e^{2t})} \right). \quad (18)$$

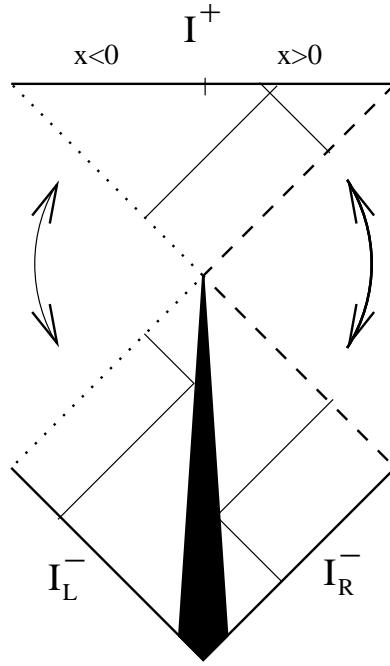


Figure 4: Causal structure obtained for the solution (10).

In this metric, $t = 0$ is a coordinate singularity. The transformations (13) and (17) render the metric Minkowskian

$$ds^2 = -d\tau^2 + d\sigma^2 \quad (19)$$

in each patch.

The trajectory of a point on the Fermi surface $x(t)$ may be easily determined and turns out to be

$$x(t) = \pm\sqrt{2\mu} \left(-e^t \sinh \tau_0 + \frac{1}{2} e^{-t} e^{\tau_0} \right). \quad (20)$$

Once again this is the *exact* trajectory. It is a null ray, as may be seen by expressing this in Alexandrov coordinates. For $t < 0$ the incoming null rays are given by $\tau + \sigma = \tau_0$ while the reflected ray is given by $\tau + \sigma = -\tau_0$. In the $t > 0$ region the trajectory (20) corresponds to a null ray $\tau + \sigma = -\tau_0$.

A particle which starts early on $\mathcal{I}_{R/L}^-$ (in our conventions this means $\tau_0 < 0$) will end up on the same side of the potential as the one on which it started. A particle which starts out later will cross the $x = 0$ line and end up on the other side of the potential. Every particle is reflected from the “mirror” at $\sigma = 0$ (independent of the value of τ_0), but this does not imply a reflection in x space. In fact for trajectories with $\tau_0 > 0$, $x(t)$ is monotonic.

By following particle trajectories, we can glue the three coordinate patches together to describe the causal structure of the entire spacetime, as is shown in figure 4. The lower, diamond-like patch corresponds to $t < 0$ while the upper triangle is the region $t > 0$. The

dashed and the dotted null lines are identified as shown. The trajectory on the right side of the diagram corresponds to an incoming pulse with $\tau_0 < 0$ while the trajectory which starts on the left side is an incoming pulse on the other side with $\tau_0 > 0$. The causal structure is quite intricate, and spacetime ends with a spacelike boundary \mathcal{I}^+ which is $\tau = 0$ of the $t > 0$ patch.

Once again one would normally extend the space-time beyond this space-like boundary, but the fundamental description in terms of the matrix model makes this extension meaningless.

These null trajectories are also shown in figure 5, in fermion coordinates. Once again, this is not a Penrose diagram.

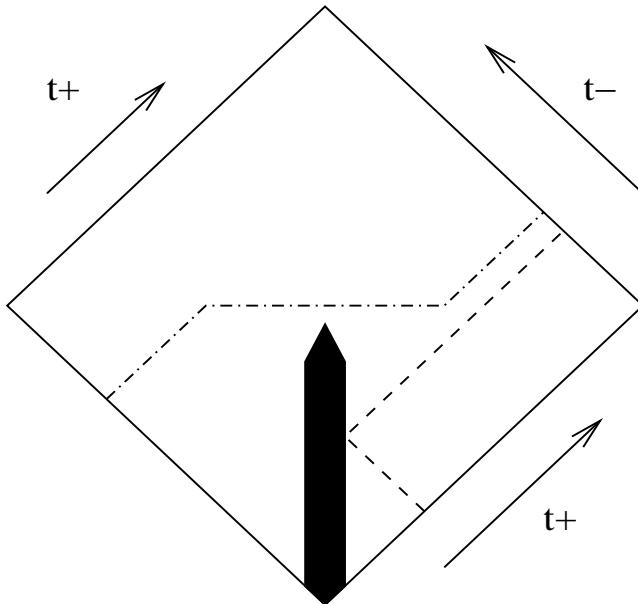


Figure 5: Diagram in fermion t^\pm coordinates, for solution (10), with trajectories of massless particles shown.

The diagrams in both figure 4 and in figure 5 have to be folded along the central vertical line since in the type 0B interpretation [24, 25] both sides of the potential correspond to the same spacetime region. It is therefore probably best to interpret figures 4 and 5 as representing a double cover of spacetime, with symmetric and antisymmetric fluctuations corresponding to the two spacetime fields of 0B theory.

For simplicity, we have so far restricted our attention to solutions which asymptotically approach the ground state in the past ($\lambda_- = 0$). This lead us to a spacelike \mathcal{I}^+ and a null \mathcal{I}^- . It is evident that a parallel discussion for the time reversed version would yield a spacelike \mathcal{I}^- and null \mathcal{I}^+ . It is also possible to take both $\lambda_\pm \neq 0$ in equation (4). We will not work out the details of the spacetime structure, but general features are obvious: in this case both \mathcal{I}^+ and \mathcal{I}^- are spacelike.

3 Mapping to string theory spacetime

It is well known that the complete dictionary between the matrix model and string theory in spacetime involves a non-local transformation. In momentum space these are the leg-pole factors. The S-matrix obtained from the matrix model has to be multiplied by these additional momentum dependent phases to match results from the string theory world-sheet. In space, these momentum dependent factors result in a non-local transform, which relates the matrix model effective field η to the spacetime string field $S(\phi, t) = e^{-\Phi} T(\phi, t)$, where T is the tachyon. Since this transform is a statement about the S-matrix, it has meaning only on \mathcal{I}^\pm . On \mathcal{I}^+ it is given by

$$S(t - \phi) = \int dv K(v) \eta(t - \phi - v) . \quad (21)$$

In this section we wish to investigate whether this non-locality modifies our causal structure.

3.1 Leg Pole Kernels

For fluctuations around the usual ground state the leg-pole kernel is known both for the bosonic and the 0B theory. In the bosonic string theory the kernel K is given by [15]

$$\begin{aligned} K_{bos}(v) &= \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{i\omega v} (\pi/2)^{-i\omega/4} \frac{\Gamma(-i\omega)}{\Gamma(i\omega)} = -\frac{z}{2} J_1(z) = \frac{d}{dv} J_0(z) , \\ z(v) &\equiv 2(2/\pi)^{1/8} e^{v/2} . \end{aligned} \quad (22)$$

We will denote the Fourier transform of the kernel by $L_B(\omega)$,

$$L_B(\omega) = (\pi/2)^{-i\omega/4} \Gamma(-i\omega)/\Gamma(i\omega) . \quad (23)$$

The asymptotic behavior of the kernel is

$$K_{bos}(v) \sim e^v , \quad v \rightarrow -\infty \quad (24)$$

and

$$K_{bos}(v) \sim e^{v/4} \cos(z + \pi/4) , \quad v \rightarrow +\infty . \quad (25)$$

K_{bos} decays exponentially for $v \rightarrow -\infty$ and grows while oscillating wildly for $v \rightarrow +\infty$.

The precise form for this leg pole transform is different in the 0B theory. Here we have two massless scalar fields in spacetime : the tachyon T_{NSNS} and the axion C_{RR} living in the *same* spacetime. There are two fields in the matrix model as well — these are the fluctuations of the Fermi sea on the two sides of the potential. In terms of the collective field fluctuation $\eta(x, t)$, written as a function of the “fermion” coordinate x , one can define symmetric and antisymmetric combinations $\eta_{S,A}(x, t)$ which may be thought to live on half of x space, which we will take to be $x > 0$

$$\eta_{S,A}(x, t) = \frac{1}{2} [\eta(x, t) \pm \eta(-x, t)] . \quad (26)$$

For on shell fields the Fourier transforms of $\eta_{S,A}$ are related to the spacetime fields by [24, 25]

$$T_{NSNS}(\omega) = L_{NSNS}(\omega) \eta_S(\omega) , \quad L_{NSNS}(\omega) = (\pi/2)^{-i\omega/8} \frac{\Gamma(i\omega/2)}{\Gamma(-i\omega/2)} , \quad (27)$$

$$C_{RR}(\omega) = L_{RR}(\omega) \eta_A(\omega) , \quad L_{RR}(\omega) = (\pi/2)^{-i\omega/8} \frac{\Gamma\left(\frac{1}{2} + \frac{i\omega}{2}\right)}{\Gamma\left(\frac{1}{2} - \frac{i\omega}{2}\right)} . \quad (28)$$

This implies that the corresponding kernels are given by

$$K_{NSNS}(v) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{i\omega v} (\pi/2)^{-i\omega/8} \frac{\Gamma\left(\frac{i\omega}{2}\right)}{\Gamma\left(-\frac{i\omega}{2}\right)} = -z J_1(z) \quad (29)$$

and

$$K_{RR}(v) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{i\omega v} (\pi/2)^{-i\omega/8} \frac{\Gamma\left(\frac{1}{2} + \frac{i\omega}{2}\right)}{\Gamma\left(\frac{1}{2} - \frac{i\omega}{2}\right)} = z J_0(z) , \quad (30)$$

where

$$z(v) \equiv 2(2/\pi)^{1/8} e^v . \quad (31)$$

The asymptotic behavior for $v \rightarrow -\infty$ is in this case

$$\begin{aligned} K_{NSNS}(v) &\sim e^{2v} , \\ K_{RR}(v) &\sim e^v . \end{aligned} \quad (32)$$

Recall that our conventions are $\alpha' = 1$ in bosonic theory, and $\alpha' = \frac{1}{2}$ in 0B theory. We are therefore using string units; the formulae above show that the scale of the relative non-locality between matrix model quantities and spacetime string fields is of the order of the string length.

3.2 Macroscopic loops

Unfortunately, the leg pole transform gives no insight into what is happening in the bulk of space, as it describes only the null infinities. In order to achieve some amount of insight into the bulk, we will resort to the macroscopic loop transform. As discussed in detail in the next section, the macroscopic loop will define the notion of spacetime as perceived by the FZZT branes.

In the bosonic theory the macroscopic loop is defined by

$$W(l, t) \equiv \text{tr}\left(e^{-lM(t)}\right) = \int_0^{\infty} dx \varphi(x, t) e^{-lx} , \quad (33)$$

where φ denotes the eigenvalue density and $l = e^{-\phi}/\sqrt{2}$. This is again a non-local transformation on the collective field which is of course different from the leg pole transform. However, the two are approximately the same in the asymptotic regimes \mathcal{I}^{\pm} . Rewriting the macroscopic

loop transform for a small oscillation η on top of whatever background is under consideration, we get

$$w(t, \phi) = \int dx \exp(-\frac{1}{\sqrt{2}}e^{-\phi}x) \partial_x \eta = \int dq \exp(-\sqrt{\mu}e^{-\phi} \cosh q) \partial_q \eta . \quad (34)$$

Consider the outgoing modes, with $\phi \rightarrow +\infty$ and a finite support for the wave-packet $w(t - \phi)$. Then

$$w(t - \phi) = \int dy [e^y \exp(-e^y)] \eta(t - \phi - y - \log(\frac{\sqrt{\mu}}{2})) . \quad (35)$$

To compare with the leg pole transform we need to shift the origin of time by an amount $\log(\frac{\sqrt{\mu}}{2})$. Comparing with the kernel which appears in square brackets in (35) with the leg pole kernel K , we notice that the two kernels have the same exponential tail for large negative argument. The macroscopic loop transform approximates the exact leg pole expressions here. The two transforms start to differ when the argument is close to zero: for positive values of the argument K has wild oscillations while kernel in (35) decays very rapidly. The Fourier transform of the macroscopic loop kernel is

$$\int dy e^{-i\omega y} \sqrt{\mu} e^y \exp(-\frac{1}{2}\sqrt{\mu}e^y) \sim \Gamma(i\omega) . \quad (36)$$

Comparing with (23) we see that the two kernels have precisely the same poles, but the residue is the same only for the first pole, which determines long range behavior.

In the 0B theory, the macroscopic loop transform needs to be modified appropriately [24, 25]. The NSNS loop is defined by

$$W_{NSNS}(l, t) = \text{Tr} e^{-l(M(t)^2 - \mu)} = \int_{-\infty}^{\infty} dx e^{-l(x^2 - \mu)} \varphi(x, t) , \quad (37)$$

where now $l = e^{-2\phi}$. The fluctuation of this quantity is given in terms of the symmetric field η_S introduced above

$$w_{NSNS}(\phi, t) = 2 \int_0^{\infty} d\sigma e^{-e^{-2\phi}\mu \cosh(2\sigma)} \partial_{\sigma} \eta_S . \quad (38)$$

The kernel of this transformation is consistent with the e^{2v} fall-off of the NSNS leg-pole transform.

In the RR sector there are two macroscopic loops (we are ignoring here the GSO projection)

$$W_{RR}^{\pm}(l, t) = \sqrt{2l} \text{Tr} (M(t) e^{-l(M(t)^2 \mp \mu)}) = \int_{-\infty}^{\infty} dx \sqrt{2l} x e^{-l(x^2 \mp \mu)} \varphi(x, t) . \quad (39)$$

The fluctuation of W_{RR}^+ is given by

$$w_{RR}^+(\phi, t) = 2 \int_0^{\infty} d\sigma \sqrt{2\mu} e^{-\phi} \cosh \sigma e^{-e^{-2\phi}\mu \cosh(2\sigma)} \partial_{\sigma} \eta_A . \quad (40)$$

The kernel falls off like e^{ϕ} , again in agreement with (32). Similar statements can be made about w_{RR}^- .

3.3 Implications for time-dependent solutions

The leg pole factors were derived by comparing the S-matrix of perturbative fluctuations around the ground state obtained from matrix model with the S-matrix obtained from world-sheet string theory. In general time dependent backgrounds the world-sheet formulation is not known at the moment and a general principle which determines these transforms is lacking. In the absence of such a principle we assume that the generic features of the correct transform would be the same as around the ground state.

In the previous sections we obtained the causal structure of the spacetime generated by our time dependent solution in Alexandrov coordinates, which are locally related to the matrix model coordinates (x, t) . Since the spacetime perceived by perturbative strings is different from this spacetime and related to it by a non-local transform, sharp null rays in the Penrose diagrams would correspond spread out pulses in the perturbative string spacetime. Assuming that the scale of non-locality introduced by a leg pole transform is still the string scale, one has to conclude that the diagrams are smeared out, but only by an amount which is of the order of the string length. Probes which have low energies will continue to perceive the spacetime exactly as depicted.

This may be checked by an explicit calculation of the macroscopic loop, whose arguments provide spacetime as perceived by FZZT branes. Consider for example a fluctuation which has the form

$$\eta(x, t) = \frac{1}{\sigma\sqrt{\pi}} \exp \left[-\frac{(x - x_0(t))^2}{\sigma^2} \right] \quad (41)$$

for some trajectory $x_0(t)$ of its center. Then the corresponding NSNS macroscopic loop is

$$W(l) = \frac{2lx_0(t)}{(l\sigma^2 + 1)^{3/2}} \exp \left[-\frac{lx_0^2(t)}{l\sigma^2 + 1} \right] \quad (42)$$

In the asymptotic region $l \rightarrow 0$ it is clear that $W(l)$ peaks at $l = \frac{1}{x_0^2(t)}$ with a width in l space of the same order. This means that in the ϕ space the width is of order unity. Recalling that in our discussion all lengths are measured in units of the string scale we therefore see that the width in physical ϕ space (which agrees with the space of string theory in the asymptotic region) is of the order of the string length². In the 0B interpretation one has to fold over the Penrose diagram in (4) along $x = 0$ with a proper identification of the null rays as appropriate mixtures of the tachyon and the axion field. Our discussion shows that the causal structure Penrose diagrams in figures 1 and 4 reflect quite accurately the string spacetime $\phi - \tau$ in the region of large ϕ .

²The same result holds for the bosonic theory as well.

4 Tachyon condensates

Consider perturbing the Liouville world-sheet action by adding a tachyon condensate:

$$S = S_{\text{Liouville}} + \frac{1}{4\pi} \int dz^2 T(\phi, t) . \quad (43)$$

We are interested in the relationship between the tachyon condensate and the perturbed solutions of the matrix model from section 2. One way to probe such a perturbation of the world-sheet action is via a one point function: when the one point function is computed in perturbation theory for a small perturbation of the world-sheet action, the first nontrivial term is proportional to the tachyon condensate. At the same time, the one point function on the string world-sheet corresponds to a wave functional, and so it should satisfy a minisuperspace wave equation. It was shown in [17] that the Laplace transform of the FZZT disc one point function satisfies the minisuperspace wave equation exactly, and not just in the classical (minisuperspace) limit. In this section, we will propose to use the Laplace transform of the one point function, which is just the macroscopic loop introduced above, to obtain a candidate for a world-sheet perturbation $T(\phi, t)$.

4.1 Macroscopic Loops and FZZT branes

We will review here the salient facts from [24, 25, 26].

The macroscopic loop is related via a Laplace transform to a disc one-point function with an FZZT brane boundary. The FZZT branes [17, 18] are labeled by a boundary cosmological constant μ_B . They should be thought of as D1-branes extending from the weak coupling region toward the strong coupling region, and dissolving at a point determined by μ_B . These branes provide quasi-local probes of the geometry of spacetime generated by the matrix model [27, 28]. The non-locality can be removed by taking a Laplace transform, at which point we arrive at the macroscopic loop, which is defined by cutting a fixed-length hole in the world-sheet, and thus holding the boundary at a constant dilaton value.

Consider therefore a FZZT brane with a boundary cosmological constant μ_B in the bulk background parametrized by the bulk cosmological constant μ . In bosonic theory, up to a normalization constant, the one point function is [17, 18]³

$$\langle V_{ik} \rangle_{\text{FZZT}} \sim \frac{\cos(\pi sk)}{ik} \Gamma(1 + ibk) \Gamma(1 + ib^{-1}k) , \quad (44)$$

where the parameter s is related to μ_B via

$$\cosh^2(\pi bs) = \frac{\mu_B^2}{\mu} \sin(\pi b^2) . \quad (45)$$

³We are quoting these formulae for a general quantum improvement term $Q = b + 1/b$, in order to regularize the expressions.

For $b = 1$, we can rewrite (44) as

$$\langle V_{ik} \rangle_{FZZT} \sim \cos(\pi sk) \Gamma(1 + ik) \Gamma(ik) . \quad (46)$$

This quantity is related via a Laplace transform to the macroscopic loop defined in (33). The relationship is as follows: consider a perturbation of the eigenvalue density $\varphi(x, t) = \sqrt{x^2 - 2\mu} + \partial_x \eta$. The macroscopic loop of the perturbation alone is then ($x = \sqrt{2\mu} \cosh \sigma$)

$$w(l, t) = \int_0^\infty d\sigma \partial_\sigma \eta(\sigma, t) e^{-l\sqrt{2\mu} \cosh \sigma} . \quad (47)$$

From the discussion of Alexandrov coordinates in the previous section we know that $\eta(\sigma, t = \tau)$ satisfies the standard massless Klein-Gordon equation $[\partial_\tau^2 - \partial_\sigma^2]\eta = 0$ at the linearized level. Consider therefore a mode of this field given by $\eta = e^{\pm ikt} \sin(k\sigma)$. Substituting this into (47), we obtain

$$w(l, k) = \int_0^\infty d\sigma e^{-l\sqrt{2\mu} \cosh \sigma} k \cos(k\sigma) = k K_{ik}(\sqrt{2\mu}l) . \quad (48)$$

Taking a Laplace transform of this result, we obtain

$$\int_0^\infty \frac{dl}{l} e^{-\sqrt{2\mu}l \cosh(\pi s)} w(l, k) = \frac{\pi \cos(\pi ks)}{\sinh(\pi k)} . \quad (49)$$

To relate this to the worldsheet theory, we need to multiply by the leg-pole transform in equation (23), obtaining

$$-i \cos(\pi ks) \Gamma(1 + ik) \Gamma(ik) , \quad (50)$$

which agrees with (46). Furthermore, the fluctuation of a loop transform (47) satisfies the minisuperspace equation for a tachyon field, given by

$$[\partial_t^2 - (l\partial_l)^2 + 2\mu l^2]w(l, k) = 0 . \quad (51)$$

In the 0B theory, the FZZT-brane one point functions have been computed in [29], and are given by

$$\langle V_{ik}^{NSNS} \rangle_{FZZT, \pm} \sim \frac{\cos(\pi ks)}{ik} \Gamma(1 + ikb/2) \Gamma(1 + ib^{-1}k/2) , \quad (52)$$

$$\langle V_{ik}^{RR} \rangle_{FZZT, +} \sim \frac{\cos(\pi ks)}{ik} \Gamma(1/2 + ikb/2) \Gamma(1/2 + ib^{-1}k/2) , \quad (53)$$

$$\langle V_{ik}^{RR} \rangle_{FZZT, -} \sim \frac{\sin(\pi ks)}{ik} \Gamma(1/2 + ikb/2) \Gamma(1/2 + ib^{-1}k/2) , \quad (54)$$

where the \pm refer to the two different FZZT boundary conditions possible in 0B theory.

The macroscopic loops in the 0B theory have been defined in (37) and (39). In both cases, the loop parameter l is now related to the spacetime coordinate ϕ by $l = e^{-2\phi}$.

Repeating the steps for bosonic string, we compute the fluctuation w_{NSNS} of the macroscopic loop W_{NSNS} due to a mode $\eta = \sin(k\sigma)$ with $x = \sqrt{2\mu} \cosh \sigma$, to obtain

$$w_{NSNS}(l, k) = k K_{ik/2}(\mu l) . \quad (55)$$

This quantity has to be multiplied by the leg pole factor for NSNS fields given in (27). Computing the Laplace transform we get

$$\frac{\Gamma\left(\frac{ik}{2}\right)}{\Gamma\left(\frac{-ik}{2}\right)} \int \frac{dl}{l} e^{-\mu l \cosh(2\pi s)} w_{NSNS} = -i \cos(\pi s k) \Gamma(1 + ik/2) \Gamma(ik/2) , \quad (56)$$

which is seen to agree with (52). It is easy to check that w_{NSNS} satisfies the minisuperspace equation of motion [24]

$$[\partial_t^2 - 4(l\partial_l)^2 + 4\mu^2 l^2] w_{NSNS} = 0 . \quad (57)$$

For the RR field, there are two transforms defined in (39). The fluctuation of these quantities are

$$w_{RR}^\pm = \int_{-\infty}^{\infty} dx \sqrt{2l} x e^{-l(x^2 \mp \mu)} \partial_x \eta(x, t) . \quad (58)$$

These two functions satisfy the equations of motion [24]

$$\begin{aligned} [\partial_t^2 - 4(l\partial_l)^2 \pm 4\mu l + 4\mu^2 l^2] w_{RR}^\pm &= 0 , \\ [\pm 2(l\partial_l) + 2\mu l] w_{RR}^\pm &= i\partial_t W_{RR}^\mp . \end{aligned} \quad (59)$$

Again, we can recover the appropriate one-point functions (53) and (54) by computing a Laplace transform of the fluctuation w . We must use $x = \sqrt{2\mu} \cosh \sigma$ for the “+” case and $x = \sqrt{2\mu} \sinh \sigma$ for the “-” case, and the fluctuation must be given by $\eta = \sin k\sigma$ and $\cos k\sigma$ respectively. We then obtain

$$w_{RR}^+ = \frac{k}{2} \sqrt{\mu l} \left(K_{1/2+ik/2} + K_{1/2-ik/2} \right) (\mu l) \quad (60)$$

$$w_{RR}^- = \frac{-ik}{2} \sqrt{\mu l} \left(K_{1/2+ik/2} - K_{1/2-ik/2} \right) (\mu l) . \quad (61)$$

Multiplying by the leg pole factor L_{RR} (28) and computing the Laplace transforms (there is a factor of $\sqrt{l\mu}$ coming from fermionic modes), we obtain

$$\frac{\Gamma\left(\frac{1}{2} + \frac{ik}{2}\right)}{\Gamma\left(\frac{1}{2} - \frac{ik}{2}\right)} \int \frac{dl}{l} \sqrt{\mu l} \begin{bmatrix} \cosh \\ \sinh \end{bmatrix} (\pi s) e^{-\mu l \cosh(2\pi s)} w_{RR}^\pm = -i \begin{bmatrix} \cos \\ \sin \end{bmatrix} (\pi s k) (\Gamma(1/2 + ik/2))^2 \quad (62)$$

which is in agreement with (53) and (54) for $b = 1$.

Notice that the difference between W_{RR}^+ and W_{RR}^- is small in the asymptotic region, since for $\phi \rightarrow \infty$, $l \rightarrow 0$, $e^{\pm\mu l} \rightarrow 1$.

4.2 The tachyon condensate

We will now use the above expressions for the macroscopic loop transform to compute perturbations of the Liouville action corresponding to our time dependent solutions from section 2.

We begin by analyzing, in bosonic string theory, the simpler solution whose fermi sea is given by equation (5). The eigenvalue density is given by

$$\varphi = \frac{\sqrt{x^2 - 2\mu(1 + e^{2t})}}{1 + e^{2t}}. \quad (63)$$

Applying the bosonic loop transform (33), we obtain

$$\begin{aligned} W(\phi, t) &= \int_{\sqrt{2\mu(1+e^{2t})}}^{\infty} dx \exp[-e^{-\phi}x] \frac{\sqrt{x^2 - 2\mu(1 + e^{2t})}}{1 + e^{2t}} \\ &= \sqrt{2\mu(1 + e^{2t})^{-1}} e^{\phi} K_1\left(\sqrt{2\mu(1 + e^{2t})} e^{-\phi}\right). \end{aligned} \quad (64)$$

In order to obtain the spacetime tachyon field from this expression, it is necessary to subtract the background $\mu = 0$ static solution given by $\varphi = |x|$, with transform

$$\int_0^{\infty} dx x e^{-e^{-\phi}x} = e^{2\phi}. \quad (65)$$

The tachyon field, after being dressed with the dilaton is

$$T(\phi, t) = 1 - \sqrt{2\mu(1 + e^{2t})^{-1}} e^{-\phi} K_1\left(\sqrt{2\mu(1 + e^{2t})} e^{-\phi}\right). \quad (66)$$

Let's look at some asymptotic behaviors of T . First, early time $t \rightarrow -\infty$. This should correspond to the static Liouville background. We obtain

$$T(\phi, t \rightarrow -\infty) = 1 - \sqrt{2\mu} e^{-\phi} K_1(\sqrt{2\mu} e^{-\phi}). \quad (67)$$

Using the fact that for small x ,

$$K_1(x) \sim \frac{1}{x} + \frac{x}{2} \left(\ln \frac{x}{2} - \text{const} \right), \quad (68)$$

we get that for large $\phi \gg 0$,

$$T(\phi \rightarrow +\infty, t \rightarrow -\infty) = \mu e^{-2\phi} (\phi + \text{const}), \quad (69)$$

which is precisely the expected Liouville potential.

In the region where $1 + e^{2t} \ll e^{2\phi}$, we get

$$T(\phi \gg 0, t < \phi) = \frac{e^{2t}}{1 + e^{2t}} + \mu e^{-2\phi} \left(\phi + \text{const} - \ln \sqrt{1 + e^{2t}} \right). \quad (70)$$

Notice that the time dependent terms became important around $t = 0$, which is the point shown in figure 2 where the ‘null’ trajectory suddenly turns around.

In order to get a feeling for how this looks in the conformally flat coordinates, lets make the association

$$e^{2\phi} \sim \frac{e^{2\sigma}}{1 - e^{2\tau}} \quad (71)$$

$$\frac{e^{2t}}{1 + e^{2t}} = e^{2\tau} \quad (72)$$

which follows from the approximate locality of the loop transform and equation (7). We obtain the first few terms for the tachyon condensate, in the regime where $\exp(\sigma) \gg 1$

$$T(\sigma \gg 0, \tau < 0) = e^{2\tau} + e^{-2\sigma}(1 - e^{2\tau})(\sigma + \text{const}) . \quad (73)$$

This has a few interesting features. First, we see that there is a *spacelike* condensation of the tachyon; the first term, independent of σ starts out zero in the past and becomes important around $\tau = 0$, just before the world ends on the spacelike boundary. Second, we see that the Liouville potential is still there. Third, we notice that the timelike and spacelike CFTs are coupled through the last term.

The analysis of the solution given by (10) is more complicated. Since the fermi sea spills over the top of the potential, it is necessary to analyze this in the 0B string theory, using the more complicated loop transforms (37) and (39).

The eigenvalue density for our solution is

$$\varphi_0 = \frac{\sqrt{x^2 + 2\mu(e^{2t} - 1)}}{1 - e^{2t}} . \quad (74)$$

Since the solution is symmetric under $x \rightarrow -x$, the RR loop transform W_{RR} vanishes. For the NSNS field, the integrals can be computed using the following formulae

$$2 \int_1^\infty dz e^{-\alpha x^2} \sqrt{z^2 - 1} = \frac{e^{-\alpha/2}}{2} (K_1(\alpha/2) - K_0(\alpha/2)) \quad (75)$$

$$2 \int_0^\infty dz e^{-\alpha x^2} \sqrt{z^2 + 1} = \frac{e^{\alpha/2}}{2} (K_1(\alpha/2) + K_0(\alpha/2)) \quad (76)$$

and the answer is,

$$W_{NSNS} = \mu e^{\mu l e^{2t}} \Re \left(K_1(\mu l(1 - e^{2t})) - K_0(\mu l(1 - e^{2t})) \right) , \quad (77)$$

where we have used $\Re(K_1(-|x|)) = -K_1(|x|)$ and $\Re(K_0(-|x|)) = K_0(|x|)$ to write a single formula encompassing $t < 0$ and $t > 0$. The asymptotic behavior for small l is

$$\Re(K_1(x) - K_0(x)) \sim \frac{1}{x} + \ln \frac{|x|}{2} - \text{const} . \quad (78)$$

Since the loop transform equations for 0B depend on μ , we cannot subtract from (77) the vacuum $\mu = 0$ expression as we did before. We must instead subtract the loop transform of the background with a cosmological constant,

$$\int_{-\infty}^{\infty} dx \sqrt{x^2 - 2\mu} e^{-lx^2} = \mu (K_1(\mu l) - K_0(\mu l)) . \quad (79)$$

Subtracting, then, (79) from (77), multiplying by e^Φ , and retaining only the leading non-vanishing terms as $\phi \rightarrow \infty$, we obtain that the tachyon condensate is

$$\delta T(\phi \gg 0, t < 0) \sim \frac{e^{2t}}{1 - e^{2t}} = e^{2\tau} , \quad (80)$$

$$\delta T(\phi \gg 0, t > 0) \sim \frac{-e^{2t}}{e^{2t} - 1} = -e^{-2\tau} . \quad (81)$$

Notice that the static Liouville term is not present; it has been subtracted off in (79).

The properties of this condensate are somewhat similar to the “closing hyperbola” solution. In particular there is a spacelike condensation of tachyons. However, the condensate is infinite at $t = 0$, corresponding to the junction of the coordinate patches shown in figure 4. This is a sign that something quite singular and strongly coupled must be happening at this point.

5 Discussion

The most interesting feature of our solutions is that \mathcal{I}^\pm are spacelike. Spacetimes with spacelike \mathcal{I}^\pm , e.g. de Sitter spacetime, have particle and event horizons perceived by timelike geodesics. For two dimensional string theory perturbative fluctuations are massless. However there are various kinds of D-branes which are massive and it would be interesting to investigate whether such D-brane probes perceive horizons. This could shed some light on origins of thermality associated with cosmological horizons.

It must be emphasized that the conformal structures are given in figures 1 and 4. The metric there is conformally flat. The non-local transform to macroscopic loops lead to a string length fuzziness in this diagram, but that is a feature of any such diagram in string theory. On the other hand, the tachyon condensate is naturally given in fermion coordinates, such as were used in figures 2 and 5, where the metric is very nontrivial (these are nonconformal deformations of the Penrose diagrams). The tachyon condensate computed in section 4.2 seems in agreement with our expectations.

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7 Note Added

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